RBC model

(Without price and wage stickiness)

Household's problem

Utility function

$$\boldsymbol{U(C_t, N_t) = \frac{C_t^{1-\sigma_c}}{1-\sigma_c} - \frac{N_t^{1+\phi}}{1+\phi}}$$

And

$$\boldsymbol{N_t = \left[S_t^{\frac{\theta_n+1}{\theta_n}} + U_t^{\frac{\theta_n+1}{\theta_n}}\right]^{\frac{\theta_n}{\theta_n+1}}}$$

- **Household**

$$\max_{S_t, U_t, K_{t+1}, B_{t+1}} U = E_0 \sum_{t=0}^{\infty} \beta^t \left\{\frac{C_t^{1-\sigma_c}}{1-\sigma_c} - \frac{N_t^{1+\phi}}{1+\phi}\right\}$$

Where

$$N_t = \left[S_t^{\frac{\theta_n+1}{\theta_n}} + U_t^{\frac{\theta_n+1}{\theta_n}}\right]^{\frac{\theta_n}{\theta_n+1}}$$

$$s.t.$$

$$P_t C_t + P_t K_{t+1} \leq Q_t K_t + W_{s,t} S_t + W_{u,t} U_t + P_t (1-\delta) K_t$$

F.O.C.s

对 C_t:

$$C_t^{-\sigma_c} = \lambda_t P_t \ \ (1)$$

对 S_t:

$$U_N N_s = -\lambda_t W_{s,t} \ \ (2)$$

$$-N_t^{\phi} \cdot \left[S_t^{\frac{\theta_n+1}{\theta_n}} + U_t^{\frac{\theta_n+1}{\theta_n}}\right]^{-\frac{1}{\theta_n+1}} S_t^{\frac{1}{\theta_n}} = -\lambda_t W_{s,t}$$

As $S_t^{\frac{\theta_n+1}{\theta_n}} + U_t^{\frac{\theta_n+1}{\theta_n}} = N_t^{\frac{\theta_n+1}{\theta_n}}$

$$N_t^{\phi} \cdot N_t^{\frac{-1}{\theta_n}} S_t^{\frac{1}{\theta_n}} = \lambda_t W_{s,t}$$

对 U_t:

$$U_U N_u = -\lambda_t W_{u,t} \ \ (3)$$

$$-N_t^{\phi} \cdot \left[S_t^{\frac{\theta_n+1}{\theta_n}} + U_t^{\frac{\theta_n+1}{\theta_n}}\right]^{-\frac{1}{\theta_n+1}} U_t^{\frac{1}{\theta_n}} = -\lambda_t W_{u,t}$$

$$N_t^{\phi} \cdot N_t^{\frac{-1}{\theta_n}} U_t^{\frac{1}{\theta_n}} = \lambda_t W_{u,t}$$

对 K_{t+1}:

$$\lambda_{t+1}\big(Q_{t+1} + P_{t+1}(1-\delta)\big) = \lambda_t P_t \ \ (4)$$

Rearrange:

Combine (1)(2)

$$C_t^{-\sigma_c} = \lambda_t P_t \ \ (1)$$

$$N_t^{\phi} \cdot N_t^{\frac{-1}{\theta_n}} S_t^{\frac{1}{\theta_n}} = \lambda_t W_{s,t} \ \ (2)$$

$$\rightarrow$$

$$N_t^{\phi} \cdot N_t^{\frac{-1}{\theta_n}} \cdot S_t^{\frac{1}{\theta_n}} = C_t^{-\sigma_c} \cdot \frac{W_{s,t}}{P_t}$$

$$\rightarrow$$

$$\frac{W_{s,t}}{P_t} = N_t^{\phi} \cdot N_t^{-\frac{1}{\theta}} \cdot S_t^{\frac{1}{\theta}} \cdot C_t^{\sigma_c}$$

Combine (1)(3)

$$C_t^{-\sigma_c} = \lambda_t P_t \ \ (1)$$

$$N_t^{\phi} \cdot N_t^{\frac{-1}{\theta_n}} U_t^{\frac{1}{\theta_n}} = \lambda_t W_{u,t} \ \ (3)$$

$$\rightarrow$$

$$N_t^{\phi} \cdot\cdot N_t^{\frac{-1}{\theta_n}} \cdot U_t^{\frac{1}{\theta}} = C_t^{-\sigma_c} \cdot \frac{W_{u,t}}{P_t}$$

$$\rightarrow$$

$$\frac{W_{u,t}}{P_t} = N_t^{\phi} \cdot N_t^{-\frac{1}{\theta}} \cdot U_t^{\frac{1}{\theta}} \cdot C_t^{\sigma_c}$$

Combine (1)(4)

$$C_t^{-\sigma_c} = \lambda_t P_t \ \ (1)$$

$$\lambda_{t+1}\left(Q_{t+1} + P_{t+1}(1-\delta)\right) = \lambda_t P_t \ \ (4)$$

$$\rightarrow$$

$$\beta \mathrm{E}_t \left(\frac{C_{t+1}^{-\sigma_c}}{P_{t+1}}\right)\left(Q_{t+1} + P_{t+1}(1-\delta)\right) = C_t^{-\sigma_c}$$

$$\rightarrow$$

$$C_t^{-\sigma_c} = \beta E_t \left\{ C_{t+1}^{-\sigma_c} \left(\frac{Q_{t+1}}{P_{t+1}} + (1-\delta) \right) \right\}$$

- **Firm**

$$Y_t = A_t \left\{ \mu U_t^{\sigma} + (1-\mu)\left[\lambda K_t^{\rho} + (1-\lambda) S_t^{\rho}\right]^{\frac{\sigma}{\rho}} \right\}^{\frac{1}{\sigma}}$$

$$\max_{S_t, K_t, U_t} \pi_t = A_t \left\{ \mu U_t^{\sigma} + (1-\mu)\left[\lambda K_t^{\rho} + (1-\lambda) S_t^{\rho}\right]^{\frac{\sigma}{\rho}} \right\}^{\frac{1}{\sigma}} - \frac{\mathrm{Q_t}}{P_t} K_t - \frac{\mathrm{W_{s,t}}}{P_t} S_t - \frac{\mathrm{W_{u,t}}}{P_t} U_t$$

F.O.C.

对 S_t:

$$\frac{\mathrm{W_{s,t}}}{P_t} = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1-\sigma}{\sigma}} \cdot (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma-\rho}{\rho}} \cdot (1-\lambda)S_t^{\rho-1}$$

对 U_t:

$$\frac{\mathrm{W_{u,t}}}{P_t} = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1-\sigma}{\sigma}} \cdot \mu U_t^{\sigma-1}$$

对 $\mathrm{K_t}$:

$$\frac{\mathrm{Q_t}}{P_t} = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1-\sigma}{\sigma}} \cdot (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma-\rho}{\rho}} \cdot \lambda K_t^{\rho-1}$$

- Market clearing

$$Y_t = C_t + X_t$$

$$K_{t+1} = (1-\delta)K_t + X_t$$

$$\log(A_{t+1}) = \rho_a \log(A_t) + e_t$$

- Summary Solving for the steady state

Households

$$1 = \beta(q + 1 - \delta) \quad (1''')$$

$$C + K = qK + w_s S + w_u U + (1-\delta)K \quad (2''')$$

$$N^\phi \cdot N^{-\frac{1}{\theta}} \cdot S^{\frac{1}{\theta}} \cdot C^{\sigma_c} = \frac{W_{s,t}}{P} \quad (3''')$$

$$N^\phi \cdot N^{-\frac{1}{\theta}} \cdot U^{\frac{1}{\theta}} \cdot C^{\sigma_c} = \frac{W_{u,t}}{P} \quad (4''')$$

Firm

$$\frac{\mathrm{W_{s,t}}}{P_t} = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1-\sigma}{\sigma}} \cdot (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma-\rho}{\rho}} \cdot (1-\lambda)S_t^{\rho-1} \quad (5''')$$

$$\frac{\mathrm{W_{u,t}}}{P_t} = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1-\sigma}{\sigma}} \cdot \mu U_t^{\sigma-1} \quad (6''')$$

$$\frac{\mathrm{Q_t}}{P_t} = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1-\sigma}{\sigma}} \cdot (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma-\rho}{\rho}} \cdot \lambda K_t^{\rho-1} \quad (7''')$$

$$Y_t = A_t\left\{\mu U_t^\sigma + (1-\mu)\left[\lambda K_t^\rho + (1-\lambda)S_t^\rho\right]^{\frac{\sigma}{\rho}}\right\}^{\frac{1}{\sigma}} \quad (8''')$$

Market clearing

$$Y = C + \delta K \ (9''')$$

- **Log-linearized system**
  households

$$\widehat{w_{s,t}} = \left(\phi - \frac{1}{\theta_n}\right)\hat{n}_t + \sigma_c \hat{c}_t + \frac{1}{\theta_n}\hat{s}_t$$

$$\widehat{w_{u,t}} = \left(\phi - \frac{1}{\theta_n}\right)\hat{n}_t + \sigma_c \hat{c}_t + \frac{1}{\theta_n}\hat{u}_t$$

$$-\sigma_c \hat{c}_t = -\sigma_c E_t \hat{c}_{t+1} + \beta q \ E_t(\widehat{q_{t+1}})$$

$$\widehat{n_t} = N^{-\frac{\theta_n+1}{\theta_n}}\left(S^{\frac{\theta_n+1}{\theta_n}}\widehat{s_t} + U^{\frac{\theta_n+1}{\theta_n}}\widehat{u_t}\right)$$

Firm

$$\widehat{y_t} = \widehat{A_t} + \left\{\{Z\}^{-1} \cdot (1-\mu)[\lambda \mathrm{K}^{\rho} + (1-\lambda)\mathrm{S}^{\rho}]^{\frac{\sigma}{\rho}-1}\right\}\left(\lambda K^{\rho}\widehat{k_t} + (1-\lambda)S^{\rho}\widehat{s_t}\right) + \{Z\}^{-1} \cdot \mu U^{\sigma}\widehat{u_t}$$

$$\widehat{w_{s,t}} = \widehat{A_t} + \left\{(1-\sigma)\cdot Z^{-1}\cdot(1-\mu)[\lambda K^{\rho} + (1-\lambda)S^{\rho}]^{\frac{\sigma}{\rho}-1} + (\sigma-\rho)[\lambda K^{\rho} + (1-\lambda)S^{\rho}]^{-1}\right\}\left(\lambda K^{\rho}\widehat{k_t} + (1-\lambda)S^{\rho}\widehat{s_t}\right) + (\rho-1)\widehat{s_t} + \{(1-\sigma)\cdot Z^{-1}\cdot \mu U^{\sigma}\}\cdot\widehat{u_t}$$

$$\widehat{w_{u,t}} = \widehat{A_t} + \left\{(1-\sigma)\cdot Z^{-1}\cdot(1-\mu)[\lambda K^{\rho} + (1-\lambda)S^{\rho}]^{\frac{\sigma}{\rho}-1}\right\}\cdot\left(\lambda K^{\rho}\hat{k}_t + (1-\lambda)S^{\rho}\hat{s}_t\right) + \{(1-\sigma)\cdot Z^{-1}\cdot\mu U^{\sigma} + (\sigma-1)\}\hat{u}_t$$

$$\widehat{q_t} = \widehat{A_t} + \left\{(1-\sigma)\cdot Z^{-1}\cdot(1-\mu)[\lambda K^{\rho} + (1-\lambda)S^{\rho}]^{\frac{\sigma}{\rho}-1} + (\sigma-\rho)[\lambda K^{\rho} + (1-\lambda)S^{\rho}]^{-1}\right\}\left(\lambda K^{\rho}\hat{k}_t + (1-\lambda)S^{\rho}\widehat{s_t}\right) + (\rho-1)\widehat{k_t} + \{(1-\sigma)\cdot Z^{-1}\cdot\mu U^{\sigma}\}\widehat{u_t}$$

Other equations

$$Y\hat{y}_t = C\hat{c}_t + X\widehat{x_t}$$

$$K\hat{k}_{t+1} = (1-\delta)K\hat{k}_t + X\widehat{x_t}$$

$$\widehat{A_t} = \rho_A \widehat{A_{t-1}} + e_t$$