See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/257516547

Fiscal and monetary policy interactions: A game theory approach

Article *in* Annals of Operations Research · July 2013 DOI: 10.1007/s10479-013-1379-3

ATIONS	;	READS 2,241		
uthor	rs:			
R	Helton Saulo		Leandro Chaves Rego	
Y	University of Brasília		Vniversidade Federal do Ceará	
	89 PUBLICATIONS 819 CITATIONS		69 PUBLICATIONS 921 CITATIONS	
	SEE PROFILE		SEE PROFILE	
	Jose Angelo Divino			
e.	Universidade Católica de Brasília			
	80 PUBLICATIONS 483 CITATIONS			
	SEE PROFILE			

Choques monetários e tecnológicos e as flutuações cíclicas na economia Brasileira View project

A New estimator for the covariance of the PLS coefficients with applications to chemical data View project

All content following this page was uploaded by Helton Saulo on 25 February 2016.

Fiscal and monetary policy interactions: a game theory approach

Helton Saulo · Leandro C. Rêgo · Jose A. Divino

Published online: 23 April 2013 © Springer Science+Business Media New York 2013

Abstract The interaction between fiscal and monetary policy is analyzed by means of a game theory approach. The coordination between these two policies is essential, since decisions taken by one institution may have disastrous effects on the other one, resulting in welfare loss for the society. We derived optimal monetary and fiscal policies in context of three coordination schemes: when each institution independently minimizes its welfare loss as a Nash equilibrium of a normal form game; when an institution moves first and the other follows, in a mechanism known as the Stackelberg solution; and, when institutions behave cooperatively, seeking common goals. In the Brazilian case, a numerical exercise shows that the smallest welfare loss is obtained under a Stackelberg solution which has the monetary policy as leader and the fiscal policy as follower. Under the optimal policy, there is evidence of a strong distaste for inflation by the Brazilian society.

Keywords Fiscal policy \cdot Monetary policy \cdot Nash equilibrium \cdot Stackelberg equilibrium \cdot Cooperative solution

1 Introduction

The macroeconomic analysis has experienced wide changes in recent years. The rule-based policymaking approach has taken the scene in both fiscal and monetary policies all over the

H. Saulo (🖂)

L.C. Rêgo Department of Statistics, Federal University of Pernambuco, 50740-540, Recife, PE, Brazil e-mail: leandro@de.ufpe.br

J.A. Divino Department of Economics, Catholic University of Brasilia, 70790-160, Brasilia, DF, Brazil e-mail: jangelo@ucb.br

Department of Economics, Federal University of Rio Grande do Sul, 90040-000, Porto Alegre, RS, Brazil e-mail: heltonsaulo@gmail.com

world. Woodford (2003), for instance, calls our attention to the adoption of interest rate rules by the monetary policy aiming at inflation stabilization. The relevance of commitment with explicit rules is strengthened by the adoption of inflation targeting regimes by many central banks worldwide, including Central Bank of Brazil, Bank of England, Reserve Bank of New Zealand, Swedish Riksbank, among others. However, independent movements by the monetary authority might result in conflicting interests with the fiscal authority. To resolve this dispute, one might make use of the game theory framework.

Our goal is to analyze the interaction between fiscal and monetary policy, taking into account different policy regimes, resulting from alternative forms of interactions between the policy authorities in a game theory environment. We allow for the policymakers to simultaneously set their instruments without cooperation as in a normal form game, engage in a Stackelberg leadership scheme, and simultaneously set their instruments in a cooperative game to pursue common objectives. Whenever possible, the models are solved analytically. Otherwise, we provide numerical approximation for the solutions using the Brazilian economy as a reference.

Backus and Driffill (1985) and Tabellini (1985), for instance, used the theory of repeated games to demonstrate that, under certain conditions, equilibria with low inflation could also appear under discretionary policymaking.¹ The argument is that when the monetary authority fights against inflation by relying on good reputation, it influences private-sector expectations on future inflation. Thus, even under discretion, the policymakers need to demonstrate a certain amount of credibility. In this scenario there appeared, for instance, the Taylor rule (Taylor 1993), establishing a reaction function for the nominal interest rate in response to variations in inflation and the output gap. Later on, Taylor's empirical rule was rationalized by optimizing behavior of individuals and firms in a New Keynesian framework.

Coordination between fiscal and monetary policies has to take care of conflicting interests, since each policymaker is primarily concerned with his own objectives. In this scenario, the induced economic spillovers and externalities become very important. This issue is addressed by Engwerda (1998), Engwerda et al. (1999, 2002), who modeled dynamic games among monetary and fiscal policymakers.

Dixit (2001) built several models of the EMU (European Monetary Union) and the European Central Bank (ECB) in order to analyze the interactions of some countries' monetary and fiscal policies. He found that the voting mechanism² achieved moderate and stable inflation. In case of a repeated game, the ECM should foresee eventual member drawback and overcome this perturbation. Dixit (2001) emphasizes the dangerous role played by unconstrained national fiscal policies, which can undermine the ECB's monetary policy commitment.

Van Aarle et al. (2002) implemented a monetary and fiscal policy framework in the EMU area according to the Engwerda et al. (2002) model. The idea was to study various interactions, spillovers, and externalities involving macroeconomic policies under alternative policy regimes. Numerical examples were provided for some types of coalitions. It is interesting to highlight that, in the simulations, full cooperation did not induce a Pareto improvement for the ECB.

The traditional three-equation Taylor-rule New Keynesian model was extended by Kirsanova et al. (2005) to include the fiscal policy and analyze policy coordination. The idea was to amend the set-up to a five-equation system in order to describe the role of the fiscal

¹Macroeconomic policy characterized by absence of commitment.

²This is a mechanism of decision-making in the European Central Bank (ECB).

policy, which might give feed back on the level of debt and helps the monetary authority to stabilize inflation. Policy interactions were considered in three scenarios: (i) non-cooperative policies, (ii) partially cooperative policies, and (iii) benevolent policies. The results suggested that, if the authorities are benevolent and cooperative, the monetary authority will bear all the burden of the stabilization. In addition, the Nash equilibrium will produce large welfare losses when the monetary authority is benevolent and the fiscal authority discounts too much the future or aims for an excessive level of output.

Lambertini and Rovelli (2003) also studied monetary and fiscal policy coordination using a game theory approach. Particularly, they argue that each policy maker prefers to be the follower in a Stackelberg situation. Moreover, when compared to the Nash equilibrium, both Stackelberg solutions are preferable. Due to implementation issues, they also claim that fiscal authorities would naturally behave as leaders in a strategic game with monetary authorities. Favero (2004), on the other hand, shows that the strategic complementarity or substitutability between fiscal and monetary policy might depend on the type of shock hitting the economy. In addition, countercyclical fiscal policy might be welfare-reducing if fiscal and monetary policy rules are inertial and not coordinated.

Our major contribution states that, in the Brazilian case, the monetary leadership under the Stackelberg solution yields the smallest welfare loss for the society. The monetary leadership might be associated to the existence of a monetary dominance in the Brazilian economy during the recent period, as empirically suggested by Tanner and Ramos (2002), Fialho and Portugal (2005), and Gadelha and Divino (2008). Under the optimal policy, a sensitivity analysis performed by varying the relative weights placed by the monetary and fiscal authorities on their target variables revealed a strong distaste for inflation by the Brazilian society. In addition, impulse response functions indicated strong reactions of the monetary authority to inflationary pressures. There is also an inflationary effect coming from fiscal shocks, which contributes to reinforce the key role played by the monetary authority to stabilize the economy.

The remainder of the paper is organized as follows. The next section details the baseline macroeconomic model and discusses some elements of game theory. The third section introduces the fiscal and monetary policy games. The fourth section discusses the numerical approach used to approximate some of the solutions. The numerical results are presented and analyzed in the fifth section. Finally, the sixth section is dedicated to the concluding remarks.

2 The baseline model

The New Keynesian framework has been largely used to analyze optimal monetary and fiscal policy rules. The system of equations is a linear approximation, in logarithmic form, of a dynamic stochastic general equilibrium (DSGE) model with sticky prices. The DSGE approach attempts to explain aggregate economic fluctuations, such as economic growth and the effects of monetary and fiscal policies, on the basis of macroeconomic models derived from microeconomic principles. The model is forward-looking and consists of an aggregate supply equation, also known as the New Keynesian Phillips curve, an aggregate demand equation, also called the IS curve. Additionally, there is an intertemporal budget constraint, with which the government should comply, and the optimal monetary and fiscal policy rules. These two policy rules will be derived later on. The aggregate demand function, represented by the intertemporal IS curve, results from the first-order conditions of the individual's optimization problem. The IS curve can be modeled taking into account the primary deficit, as in Nordhaus (1994), the public debt as in Kirsanova et al. (2005) and Bénassy (2007), or even the level of government expenditures as in Muscatelli et al. (2004). In this paper, we amend the IS curve proposed by Woodford (2003) in order to capture the effects of the public debt on aggregate demand. Thus, the set-up considers the following closed economy IS curve in log-linearized form³

$$\hat{x}_{t} = E_{t}\hat{x}_{t+1} - \sigma(\hat{i}_{t} - E_{t}\pi_{t+1}) + \alpha\hat{b}_{t} + \hat{r}_{t}^{n},$$
(1)

where $\hat{x}_t = (\hat{Y}_t - \hat{Y}_t^n)$ is the output gap (difference between actual and potential output), \hat{i}_t is the nominal interest rate, \hat{r}_t^n is a demand shock, E_t represents the time *t* expected value of the next period inflation rate π_{t+1} and output gap \hat{x}_{t+1} , b_t is the real stock of government debt, $\sigma > 0$ is the intertemporal elasticity of substitution in private spending, and α measures the sensitivity of the output gap with respect to the debt. Notice that the aggregate demand relationship depends also on future expected values and not just current ones.

On the aggregate supply curve (Phillips curve), firms face a decision to choose a price that solves their profit maximization problem. The assumption of price rigidity (Calvo 1983), according to which a fraction $0 < \vartheta < 1$ of prices remains fixed during each period, allows the derivation of the following (log-linearized) aggregate supply:

$$\pi_t = \kappa \hat{x}_t + \beta E_t \pi_{t+1} + \nu_t, \tag{2}$$

where the current inflation rate (π_t) depends on the expected E_t inflation rate at t + 1, and the current output gap \hat{x} . We allow a supply shock v_t , as in Woodford (2003), to have a trade-off between inflation versus output gap stabilization. The parameter $\kappa > 0$ measures the sensitivity of inflation with respect to the output gap and β , where $0 < \beta < 1$ is the intertemporal discount factor.

The debt in Eq. (1) also needs to be modeled. Here, the real stock of debt \hat{b}_t is treated as in Kirsanova et al. (2005). Thus, the period *t* real stock of debt, \hat{b}_t , depends on the stock of debt in the previous period, \hat{b}_{t-1} , flows of interest payments, government spending, and revenues, such that:

$$\hat{b}_{t} = (1 + i^{*})\hat{b}_{t-1} + \overline{b}\hat{i}_{t} + \hat{g}_{t} - \varpi \,\hat{x}_{t} + \eta_{t}, \qquad (3)$$

where i^* is the equilibrium interest rate, \overline{b} accounts for the steady state value of the debt, \hat{i}_t is the interest rate, \hat{g}_t represents the government spending, $\overline{\sigma}$ is the tax rate, \hat{x}_t denotes the output gap, and η_t stands for the debt shock.

The monetary policy and fiscal policy variables are interest rate and government spending, respectively. Through Eq. (1), one can see that the aggregate demand monetary policy transmission takes place when an increase (decrease) in the interest rate is greater than the expected increase (decrease) in the inflation rate at t + 1. The reduction (rise) in the aggregate demand of the economy lowers (increases) inflation via Eq. (2). On the other hand, Eq. (3) establishes that an increase (decrease) in the government spending raises (lowers) the level of debt, which in turn increases (decreases) the level of activity of the economy through Eq. (1). Everything else constant, the ultimate result is an increase (decrease) in the inflation rate by Eq. (2). Notice also that a high inflation rate has corrosive effects on the income coming from public bonds, as argued by Kirsanova et al. (2005).

Equations (1), (3), and (2) define the basic equilibrium conditions of the model. The optimization problems of Sect. 3 closes the model by deriving optimal rules for both monetary policy (interest rate rule) and fiscal policy (government spending rule).

³The hat notation is used to denote deviations from the steady state in logarithm form.

2.1 Game theoretic approach

Monetary and fiscal authorities interact with each other in order to minimize their respective loss functions. We use game theory models to analyze such interactions. The games analyzed here have two individual players, namely monetary authority (central bank) and fiscal authority (treasury). Each player has his own instrument, represented by the interest rate (i) and government spending (g). If players act independently of each other, then we have a non-cooperative game, while if they coordinate their actions, we have a cooperative solution.

We consider three different scenarios where the interactions between fiscal and monetary authorities take place.

2.1.1 Normal form game

When monetary and fiscal policymakers set their instruments simultaneously and noncooperatively, we model this situation using a normal form game. In general, a normal form game is described by the set of players in the game, a set of actions for each player in the game, and for each player a utility function that assigns a real value to every possible way the game can be played; the higher the value of the utility function, the better the outcome of the game for the player. In our case, there are two players: the fiscal and monetary authorities. The fiscal authority chooses a level of government spending (g) and the monetary authority chooses the level of interest rate (i). Both players try to minimize their loss function. We consider the most well-known solution in game theory, the so-called Nash equilibrium of the game. Intuitively, a pair of interest rate and government spending is a Nash equilibrium if none of the players can unilaterally deviate from the equilibrium and obtain some gain.

2.1.2 Extensive form game

When the fiscal and monetary authorities move sequentially, we model this situation using an extensive form game. In general, an extensive form game is described by a game tree where in each node of the tree a player chooses one of the available actions which are described by the branches of the tree. We consider only games with perfect information where each player knows the history of the game each time he moves. A utility value for each player is associated to each final node of the tree. In economics, if there are only two players and each one of them moves only once, the player who moves first is known as the Stackelberg leader and the player who moves last is known as the follower. We use the solution concept which in game theory is known as subgame perfect equilibrium. Intuitively, in such solution concept the game is solved from the end of the tree to the beginning. We consider that each player (fiscal and monetary authorities) acts as the Stackelberg leader and anticipates the response from the other player.

2.1.3 Cooperative game

When monetary and fiscal policymakers set their instruments simultaneously but in a cooperative way in order to pursue a common objective of maximizing social welfare, we model the situation as a cooperative game.

The cooperation mechanism between the fiscal and monetary authorities occurs indirectly when both authorities associate a positive weight on their instrumental variables. This mechanism permits a direct adjustment to ongoing actions taken by the other authority. The authorities face a common optimization problem, i.e. they try to minimize a common loss function.

3 Fiscal and monetary policy games

In this section, we derive the optimal reaction functions for different regimes of coordination. The monetary and fiscal authorities minimize their loss functions subject to the equilibrium conditions of the economy. The authorities solve each optimization problem and commit themselves to the optimal policy rules, having no incentive to deviate from them. The rules present relevant properties of time consistency and timelessness. The former characteristic is due to commitment, and the latter hinges on the fact that the monetary and fiscal authorities need only be committed to determine policies at the later dates by rules that are optimal from a similar perspective. As stressed by Damjanovic et al. (2005), the timeless optimal policy is the policy that these authorities would have decided upon for the current period had such a decision been taking infinitely far in the past.

The technical tools considered throughout this paper follow the general linear-quadratic policy approach introduced by Giannoni and Woodford (2002a), with applications in Giannoni and Woodford (2002b). This approach is widely used in the monetary policy literature. Particularly, Giannoni and Woodford (2002a) justify the use of this approach since (a) the policy rule should be consistent with the desired equilibrium, which is a determinate equilibrium under commitment; (b) the policy rule should be time-invariant and refer only to the evolution of target variables which represent the authority's stabilization goals; and (c) the derived policy rule should continue to be optimal no matter what the statistical properties of the exogenous disturbances hitting the economy are. Appendix A provides more details on this technique.

3.1 A normal form game between fiscal and monetary policymakers

The monetary authority, represented by the central bank, tries to minimize a current period quadratic loss function, with positive weights γ_{π} , γ_{x} , and γ_{i} on deviations of inflation from the target (zero), output gap, and deviations of the interest rate from the equilibrium rate (i^{*}) , such that:

$$L_t^M = \gamma_\pi \pi_t^2 + \gamma_x \hat{x}_t^2 + \gamma_i \left(\hat{i}_t - i^*\right)^2,$$

subject to the equilibrium conditions of the economy.

Thus, the monetary authority's problem can be written as:

$$\min E_0 \left\{ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\gamma_\pi \pi_t^2 + \gamma_x \hat{x}_t^2 + \gamma_i \left(\hat{i}_t - i^* \right)^2 \right) \right\},$$

subject to
(1) and (2). (4)

Notice that the equilibrium conditions can be represented by Eqs. (1), (2) and (3). However, under the current solution, there is no interaction between fiscal and monetary policies, and the monetary authority takes as given the fiscal variables, which are exogenous to his choices. Therefore, Eq. (3) is excluded because it defines the dynamics of the debt and the

fiscal side of the economy.⁴ A similar reasoning can be applied to equivalent policy problems discussed ahead. The Lagrangian for this problem is:⁵

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \frac{1}{2} \gamma_{\pi} \pi_t^2 + \frac{1}{2} \gamma_x \hat{x}_t^2 + \frac{1}{2} \gamma_i (\hat{i}_t - i^*)^2 \\ + \Lambda_{1,t} (\hat{x}_t - \hat{x}_{t+1} + \sigma (\hat{i}_t - \pi_{t+1}) - \alpha \hat{b}_t - \hat{r}_t^n) \\ + \Lambda_{2,t} (\pi_t - \kappa \hat{x}_t - \beta \pi_{t+1} - \nu_t) \end{bmatrix} \right\},$$
(5)

where $\Lambda_{1,t}$ and $\Lambda_{2,t}$ are the Lagrange multipliers associated to the constraints in period *t*. See Appendix B for an explanation on how to derive the results. The first order conditions yield the following equations:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = \gamma_\pi \pi_t - \beta^{-1} \sigma \Lambda_{1,t-1} + \Lambda_{2,t} - \Lambda_{2,t-1} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{x}_t} = \gamma_x \hat{x}_t + \Lambda_{1,t} - \beta^{-1} \Lambda_{1,t-1} - \kappa \Lambda_{2,t} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial (\hat{i}_t - i^*)} = \gamma_i (\hat{i}_t - i^*) + \sigma \Lambda_{1,t} = 0.$$
(6)

Isolating and substituting the Lagrange multipliers we obtain the following optimal nominal interest rate rule:⁶

$$\hat{i}_{t} = -\Gamma_{0}i^{*} + \Gamma_{i,1}\hat{i}_{t-1} - \Gamma_{i,2}\hat{i}_{t-2} + \Gamma_{\pi,0}\pi_{t} + \Gamma_{x,0}\hat{x}_{t} - \Gamma_{x,1}\hat{x}_{t-1},$$
(7)

where the coefficients are $\Gamma_0 = \frac{\sigma_k}{\beta}$, $\Gamma_{i,1} = (\frac{\sigma_k}{\beta} + \frac{1}{\beta} + 1)$, $\Gamma_{i,2} = \frac{1}{\beta}$, $\Gamma_{\pi,0} = \frac{\gamma_\pi \sigma_k}{\gamma_i}$, $\Gamma_{x,0} = \frac{\gamma_x \sigma}{\gamma_i}$, and $\Gamma_{x,1} = \frac{\gamma_x \sigma}{\gamma_i}$.

The rule (7), which the central bank commits to follow, has contemporaneous and lagged responses to the output gap. Additionally, it encompasses a history dependence since it depends on past interest rates. The response is inversely related to the size of β . Therefore, the more importance consumers attach to future variables, the stronger the monetary policy leverage is. Notice that the steeper the slope of the Phillips curve, measured by κ , the stronger the interest rate response to inflation deviations from the target. On the other hand, an increase in the weight placed on interest rate deviations, γ_i , diminishes the interest rate reaction to inflation and output gap deviations. The elasticity of intertemporal substitution, σ , also plays an important role in the monetary authority reaction function. For instance, a higher value of σ implies stronger responses of the interest rate to deviations in both inflation rate and output gap.

The fiscal side resembles the monetary one, with the difference being that the fiscal authority (treasury) takes into account government spending. So, the period loss function assumes the following form:⁷

$$L_{t}^{F} = \rho_{\pi} \pi_{t}^{2} + \rho_{x} \hat{x}_{t}^{2} + \rho_{g} \hat{g}_{t}^{2},$$

⁴Equation (3) might bring on a multiplier effect into the other equations, which is neglected under the current solution.

⁵Note that the dating of the expectations operator captures the idea of the policy maker choosing a rule exante which will be followed in the future. As we have a solution under commitment, the Lagrangian is solved for expectations at time zero, which characterizes the time when the rule was defined, thereafter followed without deviations. Thus, we removed the expectations operator on both inflation and output gap at t + 1.

⁶This solution coincides with that proposed by Woodford (2003).

⁷Kirsanova et al. (2005) and Dixit and Lambertini (2000) use a similar loss function.

where ρ_{π} , ρ_x , and ρ_g are positive weights placed on deviations of inflation rate, output gap, and government spending, respectively. The debt does not enter the loss function. The reason relies on the fact that if the fiscal policy feeds back on debt with a large coefficient, then it tends to be welfare-reducing, since the economy will exhibit cycles and increase the volatility of both inflation and output (Kirsanova et al. 2005).

The fiscal authority's problem is to solve:

$$\min E_0 \left\{ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\rho_\pi \pi_t^2 + \rho_x \hat{x}_t^2 + \rho_g \hat{g}_t^2 \right) \right\},$$

subject to
(1), (2) and (3). (8)

The fiscal authority's Lagrangian is

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \frac{1}{2} \rho_\pi \pi_t^2 + \frac{1}{2} \rho_x \hat{x}_t^2 + \frac{1}{2} \rho_g \hat{g}_t^2 \\ + \Lambda_{1,t} (\hat{x}_t - \hat{x}_{t+1} + \sigma(\hat{i}_t - \pi_{t+1}) - \alpha \hat{b}_t - \hat{r}_t^n) \\ + \Lambda_{2,t} (\pi_t - \kappa \hat{x}_t - \beta \pi_{t+1} - \nu_t) \\ + \Lambda_{3,t} (\hat{b}_t - (1 + i^*) \hat{b}_{t-1} - \overline{b} \hat{i}_t - \hat{g}_t + \overline{\sigma} \hat{x}_t - \eta_t) \end{bmatrix} \right\}.$$
(9)

The associated first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \pi_{t}} = \rho_{\pi} \pi_{t} - \beta^{-1} \sigma \Lambda_{1,t-1} + \Lambda_{2,t} - \Lambda_{2,t-1} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{x}_{t}} = \rho_{x} \hat{x}_{t} + \Lambda_{1,t} - \beta^{-1} \Lambda_{1,t-1} - \kappa \Lambda_{2,t} + \varpi \Lambda_{3,t} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{g}_{t}} = \rho_{g} \hat{g}_{t} - \Lambda_{3,t} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{b}_{t}} = -\alpha \Lambda_{1,t} + \Lambda_{3,t} - (1+i^{*})\beta E_{t}(\Lambda_{3,t+1}) = 0.$$
(10)

Isolating and substituting for the Lagrangian multipliers, we have the optimal nominal government spending rule:

$$\hat{g}_{t} = -\Theta_{\pi,0}\pi_{t} + \Theta_{g,1}\hat{g}_{t-1} - \Theta_{g,2}\hat{g}_{t-2} + \Theta_{g,+1}E_{t}\hat{g}_{t+1} - \Theta_{x,0}\hat{x}_{t} + \Theta_{x,1}\hat{x}_{t-1}, \quad (11)$$

where the coefficients are: $\Theta_{\pi,0} = \frac{\rho_{\pi}\alpha\kappa}{\rho_g B}$, $\Theta_{g,1} = \frac{A}{B}$, $\Theta_{g,2} = \frac{1}{\beta B}$, $\Theta_{g,+1} = (1+i^*)\frac{\beta}{B}$, $\Theta_{x,0} = \frac{\rho_x\alpha}{\rho_g B}$, $\Theta_{x,1} = \frac{\rho_x\alpha}{\rho_g B}$. Additionally, $A = (\beta^{-1}\sigma\kappa + \frac{1}{\beta} + 1 + (1+i^*))$, $B = ((1+i^*)(\sigma\kappa\alpha + 1+\beta) + \sigma\alpha + 1)$.

According to Eq. (11), which the fiscal authority commits to follow, fiscal policy feeds back on current inflation, current and past output gap, and lagged and expected government spending. The rule encompasses a forward and backward history dependence since the government spending responds to past and future government spending. Notice that increases in the weight placed on government spending ρ_g reduces the reaction to inflation and output gap deviations.

3.2 Stackelberg leadership

We now address the equilibrium which emerges when the fiscal (monetary) authority moves first, as a Stackelberg leader, anticipating the response from the monetary (fiscal) authority. The leader takes into account the follower's optimal policy, whereas the follower's optimal policy remains as a Nash equilibrium solution. Consider, first, the loss function for the fiscal authority acting as leader. We have the following problem:

$$\min E_0 \left\{ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\rho_\pi \pi_t^2 + \rho_x \hat{x}_t^2 + \rho_g \hat{g}_t^2 \right) \right\},$$

subject to
(1), (2), (3) and (7). (12)

The corresponding Lagrangian might be written as:

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \frac{1}{2} \rho_{\pi} \pi_t^2 + \frac{1}{2} \rho_x \hat{x}_t^2 + \frac{1}{2} \rho_g \hat{g}_t^2 \\ + \Lambda_{1,t} (\hat{x}_t - \hat{x}_{t+1} + \sigma(\hat{i}_t - \pi_{t+1}) - \alpha \hat{b}_t - \hat{r}_t^n) \\ + \Lambda_{2,t} (\pi_t - \kappa \hat{x}_t - \beta \pi_{t+1} - \nu_t) \\ + \Lambda_{3,t} (\hat{b}_t - (1 + i^*) \hat{b}_{t-1} - \overline{b} \hat{i}_t - \hat{g}_t + \varpi \hat{x}_t - \eta_t) \\ + \Lambda_{4,t} (\hat{i}_t + \Gamma_0 i^* - \Gamma_{i,1} \hat{i}_{t-1} + \Gamma_{i,2} \hat{i}_{t-2} - \Gamma_{\pi,0} \pi_t \\ - \Gamma_{x,0} \hat{x}_t + \Gamma_{x,1} \hat{x}_{t-1}) \end{bmatrix} \right\}.$$
 (13)

The associated first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \pi_{t}} = \rho_{\pi} \pi_{t} - \beta^{-1} \sigma \Lambda_{1,t-1} + \Lambda_{2,t} - \Lambda_{2,t-1} - \Gamma_{\pi,0} \Lambda_{4,t} = 0,
\frac{\partial \mathcal{L}}{\partial \hat{x}_{t}} = \rho_{x} \hat{x}_{t} + \Lambda_{1,t} - \beta^{-1} \Lambda_{1,t-1} - \kappa \Lambda_{2,t} + \varpi \Lambda_{3,t} - \Gamma_{\pi,0} \Lambda_{4,t} + \beta \Gamma_{x,1} E_{t} (\Lambda_{4,t+1}) = 0,
\frac{\partial \mathcal{L}}{\partial \hat{g}_{t}} = \rho_{g} \hat{g}_{t} - \Lambda_{3,t} = 0,
\frac{\partial \mathcal{L}}{\partial \hat{b}_{t}} = -\alpha \Lambda_{1,t} + \Lambda_{3,t} - (1 + i^{*}) \beta E_{t} (\Lambda_{3,t+1}) = 0.$$
(14)

This optimization problem cannot be solved analytically. Thus, we implement the numerical solution proposed by Juillard and Pelgrin (2007), where a timeless-perspective solution is derived according to Woodford (2003). The next section will provide further details on such a problem.

On the other hand, when acting as a leader, the monetary authority aims to minimize:

$$\min E_0 \left\{ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\gamma_\pi \pi_t^2 + \gamma_x \hat{x}_t^2 + \gamma_t \left(\hat{i}_t - i^* \right)^2 \right) \right\},$$

subject to (15)

(1), (2) and (11).

The corresponding Lagrangian is given by:

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \frac{1}{2} \gamma_{\pi} \pi_t^2 + \frac{1}{2} \gamma_x \hat{x}_t^2 + \frac{1}{2} \gamma_i (\hat{i}_t - i^*)^2 \\ + \Lambda_{1,t} (\hat{x}_t - \hat{x}_{t+1} + \sigma (\hat{i}_t - \pi_{t+1}) - \alpha \hat{b}_t - \hat{r}_t^n) \\ + \Lambda_{2,t} (\pi_t - \kappa \hat{x}_t - \beta \pi_{t+1} - \nu_t) \\ + \Lambda_{3,t} (\hat{g}_t + \Theta_{\pi,0} \pi_t - \Theta_{g,1} \hat{g}_{t-1} + \Theta_{g,2} \hat{g}_{t-2} \\ - \Theta_{g,+1} E_t \hat{g}_{t+1} + \Theta_{x,0} \hat{x}_t - \Theta_{x,1} \hat{x}_{t-1}) \end{bmatrix} \right\}.$$
(16)

The implied first order conditions are:

Deringer

$$\frac{\partial \mathcal{L}}{\partial \pi_{t}} = \gamma_{\pi} \pi_{t} - \beta^{-1} \sigma \Lambda_{1,t-1} + \Lambda_{2,t} - \Lambda_{2,t-1} + \Theta_{\pi,0} \Lambda_{3,t} = 0,
\frac{\partial \mathcal{L}}{\partial \hat{x}_{t}} = \gamma_{x} \hat{x}_{t} + \Lambda_{1,t} - \beta^{-1} \Lambda_{1,t-1} - \kappa \Lambda_{2,t} + \Theta_{x,0} \Lambda_{3,t} - \beta \Theta_{x,1} E_{t} (\Lambda_{3,t+1}) = 0, \quad (17)
\frac{\partial \mathcal{L}}{\partial (\hat{i}_{t} - i^{*})} = \gamma_{i} (\hat{i}_{t} - i^{*}) + \sigma \Lambda_{1,t} = 0.$$

Likewise, there is no analytical solution for this problem. The numerical solution, based on Juillard and Pelgrin (2007), is discussed in the next section.

3.3 Cooperation between policymakers

Here we analyze the outcome which emerges when the fiscal and monetary policymakers cooperate with each other in pursuing a common objective. This means that the fiscal (monetary) authority takes into account the monetary (fiscal) reaction function. Under cooperation, both fiscal and monetary authorities face a common optimization problem:

$$\min E_0 \left\{ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(\xi_\pi \pi_t^2 + \xi_x \hat{x}_t^2 + \xi_i \left(\hat{i}_t - i^* \right)^2 + \xi_g \hat{g}_t^2 \right) \right\},$$

subject to (18)

(1), (2) and (3),

where $\xi_{\pi} = \gamma_{\pi} + \rho_{\pi}$, $\xi_x = \gamma_x + \rho_x$, $\xi_i = \gamma_i$ and $\xi_g = \rho_g$. That is, the positive weights placed on the deviations of inflation and output gap are the sum of the weights placed by each authority on those variables, while the weights on interest rate and government spending deviations remain unchanged.

The Lagrangian for this problem is given by:

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \frac{1}{2} \xi_{\pi} \pi_t^2 + \frac{1}{2} \xi_x \hat{x}_t^2 + \frac{1}{2} \xi_i (\hat{t}_t - i^*)^2 + \frac{1}{2} \xi_g \hat{g}_t^2 \\ + \Lambda_{1,t} (\hat{x}_t - \hat{x}_{t+1} + \sigma (\hat{t}_t - \pi_{t+1}) - \alpha \hat{b}_t - \hat{r}_t^n) \\ + \Lambda_{2,t} (\pi_t - \kappa \hat{x}_t - \beta \pi_{t+1} - \nu_t) \\ + \Lambda_{3,t} (\hat{b}_t - (1 + i^*) \hat{b}_{t-1} - \overline{b} \hat{t}_t - \hat{g}_t + \overline{\omega} \hat{x}_t - \eta_t) \end{bmatrix} \right\}, \quad (19)$$

with the following first order conditions:

$$\frac{\partial \mathcal{L}}{\partial \pi_{t}} = \xi_{\pi} \pi_{t} - \beta^{-1} \sigma \Lambda_{1,t-1} + \Lambda_{2,t} - \Lambda_{2,t-1} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{x}_{t}} = \xi_{x} \hat{x}_{t} + \Lambda_{1,t} - \beta^{-1} \Lambda_{1,t-1} - \kappa \Lambda_{2,t} + \varpi \Lambda_{3,t} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial (\hat{i}_{t} - i^{*})} = \xi_{i} (\hat{i}_{t} - i^{*}) + \sigma \Lambda_{1,t} - \overline{b} \Lambda_{3,t} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \hat{g}_{t}} = \xi_{g} \hat{g}_{t} - \Lambda_{3,t} = 0.$$
(20)

The resulting optimal nominal interest rate rule is:

$$\hat{i}_{t} = -\Gamma_{0}i^{*} + \Gamma_{i,1}\hat{i}_{t-1} - \Gamma_{i,2}\hat{i}_{t-2} + \Gamma_{\pi,0}\pi_{t} + \Gamma_{x,0}\hat{x}_{t} - \Gamma_{x,1}\hat{x}_{t-1} + \Gamma_{g,0}\hat{g}_{t} - \Gamma_{g,1}\hat{g}_{t-1} + \Gamma_{g,2}\hat{g}_{t-2},$$
(21)

🖄 Springer

where the coefficients are: $\Gamma_0 = \frac{\sigma\kappa}{\beta}$, $\Gamma_{i,1} = (\frac{\sigma\kappa}{\beta} + \frac{1}{\beta} + 1)$, $\Gamma_{i,2} = \frac{1}{\beta}$, $\Gamma_{\pi,0} = \frac{\xi_{\pi}\sigma\kappa}{\xi_i}$, $\Gamma_{x,0} = \frac{\xi_{x}\sigma}{\xi_i}$, and $\Gamma_{x,1} = \frac{\xi_{\pi}\sigma}{\xi_i}$, $\Gamma_{g,0}(\frac{\overline{b}\xi_g}{\xi_i} + \frac{\sigma\omega\xi_g}{\xi_i})$, $\Gamma_{g,1} = (\frac{\sigma\kappa\overline{b}\xi_g}{\beta\xi_i} + \frac{\overline{b}\xi_g}{\beta\xi_i} + \frac{\overline{b}\xi_g}{\xi_i} + \frac{\sigma\omega\xi_g}{\xi_i})$, $\Gamma_{g,2} = (\frac{\overline{b}\xi_g}{\beta\xi_i})$. The resulting optimal government spending rule is given by:

$$\hat{g}_{t} = -\Theta_{\pi,0}\pi_{t} + \Theta_{i,0}(\hat{i}_{t} - i^{*}) - \Theta_{i,1}(\hat{i}_{t-1} - i^{*}) + \Theta_{i,2}(\hat{i}_{t-2} - i^{*}) - \Theta_{x,0}\hat{x}_{t} + \Theta_{x,1}\hat{x}_{t-1} + \Theta_{g,1}\hat{g}_{t-1} - \Theta_{g,2}\hat{g}_{t-2},$$
(22)

where the coefficients are defined as: $\Theta_{\pi,0} = \frac{\kappa \xi_{\pi}}{C}$, $\Theta_{i,0} = \frac{\xi_i}{\sigma C}$, $\Theta_{i,1} = (\frac{\kappa \xi_i}{\beta C} + \frac{\xi_i}{\beta \sigma C} + \frac{\xi_i}{\sigma C})$, $\Theta_{i,2} = \frac{\xi_i}{\beta \sigma C}$, $\Theta_{x,0} = \Theta_{x,1} = \frac{\xi_x}{C}$, $\Theta_{g,1} = (\frac{\kappa b \xi_g}{\beta C} + \frac{b \xi_g}{\beta \sigma C} + \frac{b \xi_g}{\sigma C} + \frac{\omega \xi_g}{C})$, $\Theta_{g,2} = \frac{b \xi_g}{\beta \sigma C}$. Finally, $C = (\frac{b \xi_g}{\sigma} + \omega \xi_g)$.

 $(\frac{\overline{b}\xi_g}{\sigma} + \overline{\omega}\xi_g)$. The above rules resemble the ones obtained in the normal form game. However, in both equations, there are cross responses to the other authority policy instrument. That is, the optimal nominal interest rule responds to current and lagged government spendings, while the optimal government spending rule reacts to current and lagged interest rates. The cooperation occurs via those cross responses. One can also notice that, in the fiscal rule, there is no response to future government spending, possibly because cooperation eliminated the forward looking feature of that policy rule.

4 Numerical approach

4.1 Simulation of the Nash equilibrium

The model's equilibrium is described by ten equations, being five endogenous and five exogenous processes. The endogenous variables are $(\hat{x}_t, \pi_t, \hat{b}_t, \hat{i}_t, \hat{g}_t)$ while the exogenous ones are $(\hat{r}_t^n, v_t, \eta_t, \Xi_t, O_t)$. Following the definitions from previous sections, the set of equations characterizing the equilibrium can be represented as:

IS curve: $\hat{x}_t = E_t \hat{x}_{t+1} - \sigma(\hat{i}_t - E_t \pi_{t+1}) + \alpha \hat{b}_t + \hat{r}_t^n$ Phillips curve: $\pi_t = \kappa \hat{x}_t + \beta E_t \pi_{t+1} + v_t$ Public debt: $\hat{b}_t = (1 + i^*) \hat{b}_{t-1} + b \hat{i}_t + \hat{g}_t - \varpi \hat{x}_t + \eta_t$ Monetary rule: $\hat{i}_t = -\Gamma_0 i^* + \Gamma_{i,1} \hat{i}_{t-1} - \Gamma_{i,2} \hat{i}_{t-2} + \Gamma_{\pi,0} \pi_t + \Gamma_{x,0} \hat{x}_t - \Gamma_{x,1} \hat{x}_{t-1} + \mathcal{E}_t$ Fiscal rule: $\hat{g}_t = -\Theta_{\pi,0} \pi_t + \Theta_{g,1} \hat{g}_{t-1} - \Theta_{g,2} \hat{g}_{t-2} + \Theta_{g,+1} E_t \hat{g}_{t+1} - \Theta_{x,0} \hat{x}_t + \Theta_{x,1} \hat{x}_{t-1} + O_t$ Demand shock: $\hat{r}_t^n = \chi_r \hat{r}_{t-1}^n + \varepsilon_r$ Supply shock: $v_t = \chi_v v_{t-1} + \varepsilon_v$ Debt shock: $\eta_t = \chi_\eta \eta_{t-1} + \varepsilon_\eta$ Monetary policy shock: $\mathcal{E}_t = \chi_{\mathcal{E}} \mathcal{E}_{t-1} + \varepsilon_{\mathcal{E}}$ Fiscal policy shock: $O_t = \chi_0 O_{t-1} + \varepsilon_0$

As usual, the exogenous processes are assumed to follow AR(1) stationary processes. The AR(1) process reflects, relatively well, the persistence that exists in many macroeconomic time series. Moreover, each ε_j is independent and identically distributed with zero mean and variance σ_j^2 . Based on Brazilian data, we set $\chi_j = 0.9$ in order to capture the high persistence of those shocks and $\sigma_j^2 = 0.04$, $\forall j$. At this point, we do not allow for nonzero correlations among the shocks. The DYNARE for MATLAB was used to solve for the rational expectations model.⁸

⁸See Laffargue (1990), Boucekkine (1995), Juillard (1996), Collard and Juillard (2001a, 2001b) for details on how the model can be solved.

4.2 Simulation of the Stackelberg solution

The Stackelberg solution, as stressed earlier, is numerically solved under commitment following Juillard and Pelgrin (2007) and Woodford (2003). The structural equations, which represent constraints on possible equilibrium outcomes under Stackelberg leadership, are represented by a system of the form:

$$\begin{bmatrix} Z_{t+1} \\ E_t Z_{t+1} \end{bmatrix} = A \begin{bmatrix} Z_t \\ Z_t \end{bmatrix} + Bu_t + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0}_{n_z \times 1} \end{bmatrix},$$
(23)

where z_t is a $n_z \times 1$ vector of non-predetermined (forward looking) variables, Z_t is a $n_Z \times 1$ vector of predetermined (backward looking) variables, u_t is a $k \times 1$ vector of policy instruments, and ε_{t+1} is a $n_Z \times 1$ vector of zero-mean uncorrelated shocks. The number of rows of each matrix is, then, $n = n_z + n_Z$. The matrices A and B are functions of structural parameters.

The above representation allows us to deal with all structural equations in matrix format. The intertemporal loss function can be written as:

$$\frac{1}{2}E_1 \sum_{t=1}^{\infty} \beta^{t-1} y_t' W y_t \tag{24}$$

where, for the sake of simplicity, we can define the vector $x_t = (Z_t, z_t)$. Also, $y_t = (x'_t, u'_t)'$. The matrix *W* is

$$\begin{pmatrix} W_{xx} & W_{xu} \\ W'_{xu} & W_{uu} \end{pmatrix}$$
(25)

where the matrices W_{xx} and W_{uu} are, without loss of generality, assumed to be symmetric.

The Lagrangian that follows from (23) and (24) is given by:

$$\mathcal{L} = E_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[x_t' W_{xx} x_t + 2x_t' W_{xu} u_t + u_t' W_{uu} u_t + 2\rho_{t+1}' (Ax_t + Bu_t + \xi_{t+1} - x_{t+1}) \Big]$$
(26)

where $\xi_{t+1} = (\varepsilon_{t+1}, z_{t+1} - Ez_{t+1})$ and ρ_{t+1} is a vector of multipliers. Subsequently, we should take the first order conditions with respect to ρ_{t+1} , x_t , and u_t , such that

$$\begin{bmatrix} I_n & \mathbf{0}_{nxk} & \mathbf{0}_{nxn} \\ \mathbf{0}_{nxn} & \mathbf{0}_{nxk} & \beta A' \\ \mathbf{0}_{kxn} & \mathbf{0}_{nxk} & -B' \end{bmatrix} \begin{bmatrix} x_{t+1} \\ u_{t+1} \\ E_t \rho_{t+1} \end{bmatrix} = \begin{bmatrix} A & B & \mathbf{0}_{nxn} \\ -\beta W_{xx} & -\beta W_{xu} & I_n \\ W'_{xu} & W_{uu} & \mathbf{0}_{kxn} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \\ \rho_t \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{xx1} \\ \mathbf{0}_{kx1} \end{bmatrix}$$
(27)

where $\rho_0 = 0$ and x_0 given.

Klein (2000) shows how to solve the above via generalized Schur decomposition. However, the optimal Ramsey policy, as previously described, is time inconsistent. The time inconsistency comes from the fact that, in the current period, the policymaker sets the optimal policy after the private agents' expectations realization, such that he might have an incentive to re-optimize in the future. The method proposed by Woodford (2003) to overcome this issue was numerically implemented by Juillard and Pelgrin (2007). They generalize Woodford's timeless perspective solution by computing initial values for the Lagrange multipliers. A timeless perspective hinges on the fact that the equilibrium evolution from time $t = t_0$ onward is optimal, subject to the constraint that the economy's initial evolution be the one associated with the policy in case (Woodford 2003). That is, the policymaker renounces the possibility of setting the Lagrange multipliers to zero if he reoptimizes later on (Juillard and Pelgrin 2007).

Considering now a timeless perspective solution, the Lagrange multipliers can be defined as

$$\rho_{t+1} = g_{\rho}(x_t, u_t, \rho_t, \xi_{t+1}).$$

Further, inserting the above restriction and adopting a timeless perspective policy, namely, that the choice of the Lagrange multipliers ρ_0 is governed by the same rule from time $t = t_0$ onwards, we have the Lagrangian for the timeless perspective policy

$$\mathcal{L} = E_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[x_t' W_{xx} x_t + 2x_t' W_{xu} u_t + u_t' W_{uu} u_t + 2\rho_{t+1}' (Ax_t + Bu_t + \xi_{t+1} - x_{t+1}) \Big] + \beta^{-1} \rho_0' (y_0 - \overline{y}_0)$$
(28)

where $y_0 = (x'_0, 0)$. The first order conditions remain as before:

$$\begin{bmatrix} I_n & \mathbf{0}_{nxk} & \mathbf{0}_{nxn} \\ \mathbf{0}_{nxn} & \mathbf{0}_{nxk} & \beta A' \\ \mathbf{0}_{kxn} & \mathbf{0}_{nxk} & -B' \end{bmatrix} \begin{bmatrix} x_{t+1} \\ u_{t+1} \\ E_t \rho_{t+1} \end{bmatrix} = \begin{bmatrix} A & B & \mathbf{0}_{nxn} \\ -\beta W_{xx} & -\beta W_{xu} & I_n \\ W'_{xu} & W_{uu} & \mathbf{0}_{kxn} \end{bmatrix} \begin{bmatrix} x_t \\ u_t \\ \rho_t \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{xx1} \\ \mathbf{0}_{kx1} \end{bmatrix}$$
(29)

where $\rho_0 \neq 0$ and x_0 given.

The non-predetermined variables of y_0 are selected such that (i) the function of the predetermined variables exists in the initial period, and (ii) there is a solution for the optimization problem under the following condition $y_0 = \overline{y}_0$, for t > 0. The Stackelberg problem is, then, solved with the help of MATLAB.

4.3 Simulation of the cooperative solution

The cooperative case is similar to the one obtained in the normal form game. The major differences are in the optimal policy rules (fiscal and monetary). Additional variables enter those equations modifying thus the optimal responses to other variables.

The equilibrium of the model is described by ten equations, where five are endogenous and the other five are exogenous processes. The endogenous ones are represented by $(\hat{x}_t, \pi_t, \hat{b}_t, \hat{i}_t, \hat{g}_t)$ while the exogenous ones are $(\hat{r}_t^n, v_t, \eta_t, \Xi_t, O_t)$. Next, we describe each one of those equations.

IS curve: $\hat{x}_t = E_t \hat{x}_{t+1} - \sigma(\hat{i}_t - E_t \pi_{t+1}) + \alpha \hat{b}_t + \hat{r}_t^n$ Phillips curve: $\pi_t = \kappa \hat{x}_t + \beta E_t \pi_{t+1} + \nu_t$ Public debt: $\hat{b}_t = (1 + i^*) \hat{b}_{t-1} + \overline{b} \hat{i}_t + \hat{g}_t - \overline{\sigma} \hat{x}_t + \eta_t$ Monetary rule: $\hat{i}_t = -\Gamma_0 i^* + \Gamma_{i,1} \hat{i}_{t-1} - \Gamma_{i,2} \hat{i}_{t-2} + \Gamma_{\pi,0} \pi_t + \Gamma_{x,0} \hat{x}_t - \Gamma_{x,1} \hat{x}_{t-1} + \Gamma_{g,0} \hat{g}_t - \Gamma_{g,1} \hat{g}_{t-1} + \Gamma_{g,2} \hat{g}_{t-2} + \Xi_t$ Fiscal rule: $\hat{g}_t = -\Theta_{\pi,0} \pi_t + \Theta_{i,0} (\hat{i}_t - i^*) - \Theta_{i,1} (\hat{i}_{t-1} - i^*) + \Theta_{i,2} (\hat{i}_{t-2} - i^*) - \Theta_{x,0} \hat{x}_t + \Theta_{x,1} \hat{x}_{t-1} + \Theta_{g,1} \hat{g}_{t-1} - \Theta_{g,2} \hat{g}_{t-2} + O_t$ Demand shock: $\hat{r}_t^n = \chi_r \hat{r}_{t-1}^n + \varepsilon_r$ Supply shock: $\nu_t = \chi_v \nu_{t-1} + \varepsilon_v$ Debt shock: $\eta_t = \chi_\eta \eta_{t-1} + \varepsilon_\eta$ Monetary policy shock: $\Xi_t = \chi_{\Xi} \Xi_{t-1} + \varepsilon_{\Xi}$ Fiscal policy shock: $O_t = \chi_0 O_{t-1} + \varepsilon_0$

Parameter	Definition	Value	Reference
σ	Intertemporal elasticity of substitution in private consumption	5.00	Nunes and Portugal (2009)
α	Sensitivity of output gap to the debt	0.20	Pires (2008)
κ	Sensitivity of inflation rate to the output gap	0.50	Gouvea (2007), Walsh (2003)
β	Sensitivity of agents to the inflation rate	0.99	Cavallari (2003), Pires (2008)
<i>i</i> *	Natural rate of interest	0.07	Barcelos Neto and Portugal (2009)
\overline{b}	Steady state debt value	0.20	Kirsanova et al. (2005), Nunes and Portugal (2009)
ω	Tax rate	0.26	Kirsanova et al. (2005), Nunes and Portugal (2009)

Table 1 Calibration of the parameters

In the cooperative solution, as in the Nash equilibrium one, the exogenous processes are assumed to follow stationary AR(1) representations, where each ε_j is independent and identically distributed with zero mean and constant variance σ_j^2 . The same calibration described in Table 1 is applied here. The simulation was carried on in Dynare for MATLAB.

5 Numerical results

In order to evaluate the performance of the alternative regime of coordination, we simulate the models encompassing the Phillips curve, IS curve, government budget constraint, and optimal monetary and fiscal rules. Additionally, we provide an overview on the social losses generated by the distinct monetary and fiscal policy arrangements, and compute impulse response functions. The calibration exercise is meant for the Brazilian economy in the period after the implementation of the Real Plan.⁹ Following most of the literature, we assume that each period corresponds to one quarter of a year. The calibrated parameters, along with the respective sources, are reported in Table 1.

One of the major goals of the simulation is to obtain variances of the variables under the optimal trajectories, allowing for the computation of the expected social loss associated to each regime of coordination. As a robustness check, we calculate and plot social losses generated by alternative monetary and fiscal policy decisions, i.e. by varying the weights placed on the target variables. We also compute impulse response functions to analyze how the dynamics of the model behave under shocks of demand, supply, debt, monetary policy, and fiscal policy. Therefore, the analysis will focus on efficient aspects for macroeconomic stabilization.

5.1 Social loss analysis

The social loss is defined as the sum of the authorities' expected individual losses, which can be easily obtained by computing the unconditional variance.¹⁰ Taking, for instance, the

⁹The Real Plan was edited in June 1994.

¹⁰See Woodford (2003) for details.

σ	κ	$L^M = \pi_t^2$ $L^F = 0.5\pi$	$+ 0.5\hat{x}_t^2 + 0$ $\tau_t^2 + \hat{x}_t^2 + 0$	$0.05(\hat{i}_t - i^*)^2$ $0.3\hat{g}_t^2$			L^M	L^F	L ^S
		Variance o	f						
		π_t	\hat{x}_t	\hat{b}_t	î _t	ĝt			
0.50	0.10	12.7452	0.7057	19.1851	27.5664	3.4317	12.9905	4.2008	17.1913
	0.50	5.2119	0.2723	7.9440	11.3445	1.4225	5.3084	1.7033	7.0117
	0.90	2.7197	0.1369	4.1761	5.9437	0.7486	2.7688	0.8842	3.6500
2.50	0.10	9.6978	2.1892	211.2660	16.3928	11.3685	10.2861	5.6368	15.9229
	0.50	2.3218	0.4530	52.7042	4.0432	2.6607	2.4452	1.2729	3.7181
	0.90	0.9891	0.1819	22.8264	1.7428	1.1210	1.0389	0.5301	1.5690
5.00	0.10	8.1027	2.9793	414.0482	11.6572	18.6392	8.8767	6.6826	15.5593
	0.50	1.5190	0.4980	81.3970	2.2583	3.4490	1.6492	1.1882	2.8374
	0.90	0.6059	0.1908	33.0275	0.9111	1.3681	0.6559	0.4654	1.1213

 Table 2
 Loss values for different coefficients under the Nash solution

monetary authority period loss function, $L_t^M = \gamma_\pi \pi_t^2 + \gamma_x \hat{x}_t^2 + \gamma_i (\hat{i}_t - i^*)^2$, it is easy to calculate the expected loss for the monetary authority, given by:¹¹

$$L^{M} = \gamma_{\pi}^{2} \operatorname{var}(\pi_{t}) + \gamma_{x}^{2} \operatorname{var}(\hat{x}_{t}) + \gamma_{i}^{2} \operatorname{var}(\hat{i}_{t} - i^{*}).$$
(30)

Thus, the social loss is given by $L^S = L^M + L^F$.

The welfare criterion defines a function which depends upon both monetary and fiscal social losses. We make use of that criterion to analyze the cooperative solution, which occurs indirectly when both authorities associate a positive weight on their instrumental variables. The mechanism permits a direct adjustment to ongoing actions taken by the other authority. Basically, the problem is to maximize a social utility (welfare) or, on the other hand, to minimize the social loss function L^S , which is defined by $L^S = L^M + L^F$, that is, the sum of the authorities' individual losses.

The results reported in Tables 2, 3, 4 and 5 show the variance of each time series and the losses of each authority for different values of σ and κ . The former parameter is the intertemporal elasticity of substitution in private consumption and the latter one measures the sensitivity of the inflation rate to the output gap in the Phillips curve. These parameters came from Eqs. (1) and (2). The reason for choosing these parameters is that they play a crucial role in both structural equations and policy rules.

According to Table 2, keeping κ unchanged, the increases in σ tend to reduce the loss for the monetary but not for the fiscal authority. A high intertemporal elasticity of substitution in private spending means a preference for future consumption, namely, the agents are willing to postpone consumption. Under a higher interest rate, aggregate demand experiences a shrinkage, reducing the output gap and inflation. The monetary policy is more effective, leading to smaller monetary loss under a higher σ . The fiscal policy also experiences a similar decrease in loss, but not for all parameter combinations.

Turning now to the parameter κ , when it increases for a given σ , both fiscal and monetary losses decrease. The idea behind a rise in parameter κ is a steeper Phillips curve. Thus, a higher value of κ tends to increase the sensitivity of inflation to the output gap, yielding a negative effect on the loss. It is interesting to notice that when $\sigma = 5.00$ and $\kappa = 0.90$, we

¹¹In order to simplify the notation, we will not distinguish between social loss and expected social loss.

σ	κ	$L^M = \pi_t^2$ $L^F = 0.5$	$x^2 + 0.5\hat{x}_t^2 + 0$ $\pi_t^2 + \hat{x}_t^2 + 0.$	$0.05(\hat{i}_t - i^*)^2$ $3\hat{g}_t^2$			L^M	L^F	L ^S
		Variance of							
		π_t	\hat{x}_t	\hat{b}_t	\hat{i}_t	ĝt			
0.50	0.10	0.0355	0.0077	0.0318	0.0612	0.0034	0.0376	0.0169	0.0545
	0.50	0.0219	0.0066	0.0183	0.0365	0.0036	0.0236	0.0124	0.0360
	0.90	0.0149	0.0052	0.0120	0.0241	0.0035	0.0163	0.0092	0.0255
2.50	0.10	0.0311	0.0677	0.1868	0.0213	0.0087	0.0481	0.0763	0.1244
	0.50	0.0142	0.0311	0.0238	0.0086	0.0034	0.0220	0.0350	0.0570
	0.90	0.0080	0.0173	0.0071	0.0047	0.0017	0.0123	0.0195	0.0318
5.00	0.10	0.0296	0.1147	0.1973	0.0095	0.0072	0.0583	0.1227	0.1810
	0.50	0.0121	0.0427	0.0148	0.0038	0.0020	0.0228	0.0459	0.0687
	0.90	0.0064	0.0217	0.0057	0.0021	0.0010	0.0118	0.0234	0.0352

 Table 3 Loss values for different coefficients under the Stackelberg solution: Fiscal leadership

Table 4 Loss values for different coefficients under the Stackelberg solution: Monetary leadership

σ	к	$L^M = \pi_t^2$ $L^F = 0.5$	$x^2 + 0.5\hat{x}_t^2 + 0$ $\pi_t^2 + \hat{x}_t^2 + 0$	$\frac{1.05(\hat{i}_t - i^*)^2}{3\hat{g}_t^2}$			L^M	L^F	L^S
		Variance of							
		π_t	\hat{x}_t	\hat{b}_t	\hat{i}_t	ĝt			
0.50	0.10	0.0325	0.0051	1.1690	0.0001	0.0486	0.0338	0.0176	0.0514
	0.50	0.0143	0.0119	0.3080	0.0018	0.0337	0.0173	0.0185	0.0358
	0.90	0.0073	0.0103	0.1555	0.0050	0.0196	0.0099	0.0139	0.0238
2.50	0.10	0.0325	0.0050	1.1446	0.0013	0.0491	0.0338	0.0175	0.0513
	0.50	0.0132	0.0138	0.2759	0.0009	0.0353	0.0167	0.0203	0.0370
	0.90	0.0055	0.0132	0.1352	0.0001	0.0212	0.0088	0.0165	0.0253
5.00	0.10	0.0326	0.0050	1.1389	0.0015	0.0490	0.0339	0.0176	0.0515
	0.50	0.0133	0.0137	0.2690	0.0018	0.0353	0.0167	0.0202	0.0369
	0.90	0.0055	0.0133	0.1305	0.0006	0.0213	0.0088	0.0166	0.0254

obtain the lowest loss ($L^{S} = 1.1213$), meaning that social welfare is maximized under that parameter combination.

The results reported in Table 3 resembles the Nash equilibrium case when we consider variations in κ . On the other hand, variations in σ do not have clear effects, given that there are decreases and increases in the monetary loss depending on the value of κ . The combination of $\sigma = 0.50$ and $\kappa = 0.90$ provides the lowest loss ($L^S = 0.0255$). In addition, the losses for the fiscal leadership are lower than the losses for the Nash equilibrium, suggesting that the former is more efficient.

Table 4 shows losses similar to what was observed under the fiscal leadership. However, increases in σ have lower impacts on the fiscal loss. Once again, the pair of values $\sigma = 0.50$ and $\kappa = 0.90$ provides the lowest loss ($L^{S} = 0.0238$). Comparing all tables, that value is the global minimum, which was obtained under a monetary leadership solution.

The coordination scheme presented in Table 5 has characteristics similar to the Nash equilibrium outcome. Thereby, the same analysis can be employed here. The combination of

σ	κ	$L^{M} = \pi_{t}^{2} + 0.5\hat{x}_{t}^{2} + 0.05(\hat{i}_{t} - i^{*})^{2}$					L^M	L^F	L^S
		$\frac{L^2 = 0.5\pi_t^2 + x_t^2 + 0.5g_t^2}{12}$							
		Variance of							
		π_t	\hat{x}_t	\hat{b}_t	\hat{i}_t	\hat{g}_t			
0.50	0.10	12.1107	0.7355	29.9229	30.4118	4.0704	12.3706	4.1296	16.5002
	0.50	4.4734	0.2630	11.2059	11.3251	1.5229	4.5675	1.5184	6.0859
	0.90	2.2562	0.1300	5.7000	5.7413	0.7742	2.3030	0.7637	3.0667
2.50	0.10	7.8141	1.6191	364.3679	15.3638	11.8087	8.2573	4.6354	12.8927
	0.50	0.1146	0.1094	0.5639	0.1531	0.4110	0.1423	0.1750	0.3173
	0.90	1.6526	0.3492	72.2684	3.1152	2.4879	1.7477	0.9862	2.7339
5.00	0.10	5.6878	1.9734	769.0163	9.0333	17.8931	6.2037	5.0057	11.2094
	0.50	1.0476	0.3632	112.2076	1.4691	3.0751	1.1421	0.9019	2.0440
	0.90	0.4451	0.1524	44.5084	0.6046	1.2813	0.4847	0.3790	0.8637

Table 5 Loss values for different coefficients under the cooperative solution



Fig. 1 Social losses for different weights under the Nash solution

 $\sigma = 2.50$ and $\kappa = 0.50$ leads to the minimum value for the loss function ($L^{S} = 0.3173$). This performance, however, is well above the smallest loss obtained under a monetary leadership in the Stackelberg game.

According to the smallest social loss criterion, the policy regimes might be ordered as (1) monetary leadership, (2) fiscal leadership, (3) cooperative solution, and (4) Nash equilibrium solution. Thus, when the monetary authority moves first as a Stackelberg leader we get the best scheme of coordination between the authorities. In addition, both Stackelberg solutions are superior to the remaining ones. Finally, comparing the cooperative and the Nash equilibrium solutions, we can note that the former regime is more efficient in minimizing the social loss.

5.2 Sensitivity analysis

As a robustness check, we evaluated social losses generated by the three mechanisms of coordination discussed in the previous section. In each case, it is assumed that the economy is hit by a supply shock and the weights placed in output gap, inflation, and government spending vary from 0.10 to 1.50, and in interest rate from 0.05 to 1.00. The resulting losses are shown in Figs. 1, 2, 3 and 4.



Fig. 2 Social losses for different weights under the fiscal leadership solution



Fig. 3 Social losses for different weights under the monetary leadership solution

Under the Nash equilibrium solution, Fig. 1 shows that the monetary loss increases proportionately to the relative weights placed by the central bank on output gap and interest rate. This is due to the fact that the society dislikes inflation more than the other two variables. Notice that the distaste for interest rate fluctuations is the smallest, given that its impact on the social loss is the strongest. Differently, the fiscal loss directly increases with the size of the relative weights placed by the fiscal authority on government spending and inflation stabilization. The reason is because the output gap is an important variable for the fiscal policy. So, the society would prefer that the fiscal authority give more relative importance to output gap stabilization. Here, changing the relative weight on the government spending stabilization has the greater impact on the fiscal loss.

Furthermore, Fig. 2 reveals that under a fiscal leadership in the Stackelberg solution, the monetary loss is very sensitive to the relative weight attached by the central bank to output gap stabilization. On the other hand, the fiscal loss presents a high sensitivity to the weight on inflation stabilization. Note that Figs. 1 and 2 strengthen the different consequences on the loss under the Nash and Stackelberg (fiscal leadership) solutions.

Subsequently, Fig. 3 demonstrates that, under a monetary leadership in the Stackelberg solution, the monetary loss behaves similarly to the case under fiscal leadership when the central bank changes the relative weights on the output gap and interest rate. The fiscal loss under a monetary leadership is more sensitive to the relative weight placed on government spending.



Fig. 4 Social losses for different weights under the cooperative solution



Fig. 5 Impulse responses to a supply shock under the monetary leadership solution

Finally, the losses shown in Fig. 4 under the full cooperative solution are quite similar to the Nash equilibrium case, but to a lower degree. In general, the Nash equilibrium and fully cooperative solutions display closer responses when the relative weights placed on target variables are changed.

5.3 Impulse response analysis

We limit our attention to the monetary leadership solution since it is the best scheme of coordination according to the loss analysis. The impulse responses to alternative exogenous shocks are presented in Figs. 5, 6 and 7. The responses of the variables are for one standard deviation supply, demand or fiscal shock under the monetary leadership solution.

Figure 5 shows the effects of a supply shock under monetary leadership. The immediate effect of the shock is a rise in inflation, which leads the monetary authority to increase the interest rate. A peak of that policy is reached in the ninth quarter, when the interest rate



Fig. 6 Impulse responses to a demand shock under the monetary leadership solution



Fig. 7 Impulse responses to a fiscal shock under the monetary leadership solution

starts going back to the equilibrium. Due to the strong response of the monetary policy, the output gap falls and pushes the government expenditure to a lower level. The debt response is hump-shaped because it follows movements in the interest rate, reaching a peak after six quarters.

The effects of a demand shock under monetary leadership are shown in Fig. 6. On impact, the shock pushes the output gap upwards, which in turn increases inflation and leads the monetary authority to raise the interest rate. The peak in the interest rate is reached in the fourth quarter and convergence to the equilibrium is faster than under a supply shock. The debt and government spending are less volatile when compared to the supply shock.

A positive fiscal shock, displayed in Fig. 7, increases government spending, debt, and the output gap. In addition, that shock is inflationary, given that there is a rise in inflation on impact. The response of the monetary policy is delayed, but the increase in the interest rate is sufficient to bring the economy back to equilibrium. Also, the debt converges slower than the government spending due to the effects of high interest rates.

6 Concluding remarks

This paper has applied the game theory approach to a conventional macroeconomic optimization problem to analyze the performance of alternative coordination schemes, represented by the Nash equilibrium solution, Stackelberg leadership, and the cooperative solution, in the interaction between fiscal and monetary policies. The comparisons among the distinct regimes were made in terms of social loss, sensitivity to selected parameters, and impulse response functions. Whenever possible, analytical solutions were derived for optimal monetary and fiscal rules. In the Stackelberg case, however, due to the complexity of the solution, only a numerical simulation was obtained.

The numerical approach provided evidence of relative superiority for the monetary leadership in the Stackelberg solution. Thus, when the monetary authority moves first, as a Stackelberg leader, taking into account the optimal fiscal policy obtained under the Nash equilibrium solution, one reaches the smallest social loss. This monetary leadership might be associated to the existence of a monetary dominance in the Brazilian economy during the recent period. This evidence is supported by empirical findings provided by Tanner and Ramos (2002), Fialho and Portugal (2005), Gadelha and Divino (2008), among others.

In particular, according to our results, the monetary leadership led to the lowest social loss. A sensitivity analysis executed by varying the relative weights placed by the monetary and fiscal authorities on their target variables showed that the Nash equilibrium and cooperative solutions yielded more uniform responses. On the other hand, the monetary leadership revealed a strong distaste for inflation by the Brazilian society. The impulse response functions, computed for the best coordination scheme, indicated strong reactions of the monetary authority to inflationary pressures. In addition, there is a clear inflationary effect coming from fiscal shocks. Under the Stackelberg solution, the time series presented low volatility and faster convergence to the equilibrium after the alternative exogenous shocks.

For future works, it would be interesting to analyze the performance of coordination regimes under commitment and discretion, to apply the framework to a bargain problem in a more complex environment and to extend the model to a block of countries, particularly in South America, involving a monetary integration with common fiscal targets.

Acknowledgements J.A. Divino and H. Saulo acknowledge CNPq for the financial support. L.C. Rêgo acknowledges financial support from FACEPE under grants APQ-0150-1.02/06 and APQ-0219-3.08/08, and from MCT/CNPq under grants 475634/2007-1 and 306358/2010-7.

Appendix A

In this appendix we describe the general linear-quadratic policy approach introduced by Giannoni and Woodford (2002a) with applications by Giannoni and Woodford (2002a), to

derive an optimal monetary policy rule. Note that this approach can easily be extended to the fiscal optimization problems discussed in this paper.

Woodford (2003, pp. 23–24) argues that standard dynamic programming methods are valid only for optimization problems that evolve in response to the current action of the controller. Hence, they do not apply to problems of monetary stabilization policy since the central bank's actions depend on both the sequence of instrument settings in the present time and the private-sector's expectations regarding future policies. A direct implementation of the maximum principle is not indicated, since we have discrete-time problems with conditional expectations on some variables which affect the solution under commitment.

A.1 General linear-quadratic policy problem

Giannoni and Woodford (2002a) deal with policy problems in which the constraints for the various state variables can be represented by a system of linear (or log-linear) equations, and in which a quadratic function of these variables can be used to represent the policymaker's objectives. In general, the optimal policy rules considered by the authors take the form

$$\phi_i i_t + \phi'_z \bar{z}_t + \phi'_z \bar{Z}_t + \phi'_s \bar{s}_t = \bar{\phi}, \qquad (31)$$

where i_t is the policy instrument, \bar{z}_t and \bar{Z}_t are the vectors of nonpredetermined and predetermined endogenous variables (e.g., the output gap forecast $E_t x_{t+k}$ may be an element of \bar{z}_t), \bar{s}_t is a vector of exogenous state variables, and ϕ_i , ϕ_z , ϕ_Z , and ϕ_s , are vectors of coefficients and $\bar{\phi}$ is a constant. As pointed out by Buiter (1982), a variable is nonpredetermined if and only if its current value is a function of current anticipations of future values of endogenous and/or exogenous variables. It is predetermined if its current value depends only on past values of endogenous and/or exogenous variables.

The discounted quadratic loss function is assumed to have the form

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} L_t,$$
(32)

where t_0 stands for the initial date at which a policy rule is adopted, $0 < \beta < 1$ denotes the discount factor, and L_t specifies the period loss, that is,

$$L_{t} = \frac{1}{2} (\tau_{t} - \tau^{*})' W (\tau_{t} - \tau^{*}).$$
(33)

where τ_t is a vector of target variables, τ^* is its corresponding vector of target values, and W is a symmetric, positive-definite matrix. The target variables are assumed to be linear functions

$$\tau_t = T y_t, \tag{34}$$

where $y_t \equiv [Z_t \ z_t \ i_t]^\top$, Z_t is a subset of the predetermined variables \overline{Z}_t , z_t is a subset of the vector of nonpredetermined endogenous variables \overline{z}_t , and T is a matrix of coefficients. It is assumed that Z_t encompasses all of the predetermined endogenous variables that constrain the possible equilibrium evolution of the variables Z_T and z_T for $T \ge t$. Also, s_t , i.e. the subset of exogenous states, encompasses all of the exogenous states which possess information on the possible future evolution of the variables Z_T and z_T for $T \ge t$.

The endogenous variables z_t and Z_t take the form

$$\hat{I}\begin{bmatrix} Z_{t+1}\\ E_t z_{t+1} \end{bmatrix} = A\begin{bmatrix} Z_t\\ z_t \end{bmatrix} + Bi_t + Cs_t,$$
(35)

where each matrix has $n = n_z + n_Z$ rows, n_z and n_Z denotes the number of nonpredetermined and predetermined endogenous variables, respectively. Note that we may partition the matrices as

$$\hat{I} = \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix},$$

where the upper and lower blocks have n_Z and n_z rows, respectively. The zero restrictions in the upper blocks refer to the fact that the first n_Z equations define the elements of Z_t as elements of z_{t-j} for some $j \ge 1$. It is assumed that A_{22} is non-singular in order to let the last n_z equations be solved for z_t as a function of Z_t , s_t , i_t , and $E_t z_{t+1}$. In addition, B_2 is not zero in all elements, resulting in an instrument with some effect.

Definition (Giannoni and Woodford 2002a) A policy rule that determines a unique nonexplosive rational expectations equilibrium is optimal from a timeless perspective if the equilibrium determined by the rule is such that (a) the nonpredetermined endogenous variables z_t can be expressed as a time-invariant function of a vector of predetermined variables \overline{Z}_t and a vector of exogenous variables \overline{s}_t ; that is, a relation of the form $z_t = f_0 + f_Z \overline{Z}_t + f_s \overline{s}_t$, applies for all dates $t \ge t_0$; and (b) the equilibrium evolution of the endogenous variables $\{y_t\}$ for all dates $t \ge t_0$ minimizes (32) among the set of all bounded processes, subject to the constraints implied by the economy's initial state Z_{t_0} , the requirements for rational expectations equilibrium (i.e., the structural equations (35)), and a set of additional constraints of the form

$$\tilde{E}z_{t_0} = \tilde{E}[f_0 + f_Z \bar{Z}_{t_0} + f_s \bar{s}_{t_0}],$$
(36)

on the initial behavior of the nonpredetermined endogenous variables.

According to Woodford (1999), the Lagrangian for the minimization problem can be written as

$$\mathcal{L}_{t_0} = E_{t_0} \left\{ \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Big[L(y_t) + \varphi'_{t+1} \tilde{A} y_t - \beta^{-1} \varphi'_t \tilde{I} y_t \Big] \right\},\tag{37}$$

where $\tilde{A} \equiv [A \ B]$ and $\tilde{I} \equiv [\hat{I} \ 0]$. Note that $L(y_t)$ denotes the period loss L_t expressed as a quadratic function of y_t and φ_{t+1} denotes the vector of Lagrange multipliers related to the constraints (35). Applying the law of iterated expectations, the conditional expectation can be eliminated from the term $E_t z_{t+1}$ in these constraints. Set

$$\varphi_{t+1} \equiv \begin{bmatrix} \xi_{t+1} \\ \Xi_t \end{bmatrix}$$

and insert the term

$$\varphi_{t_0}' \tilde{I} y_{t_0} = \xi_{t_0}' Z_{t_0} + \Xi_{t_0-1}' \tilde{E} z_{t_0}, \qquad (38)$$

into (37), where $\xi'_{t_0} Z_{t_0}$ represents the constraints imposed by the given initial values Z_{t_0} , and $\Xi'_{t_0-1}\tilde{E}z_{t_0}$ represents the constraints (36). Finally, differentiating the Lagrangian (37) with respect to the endogenous variables y_t , we yield the first-order conditions

$$\tilde{A}' E_t \varphi_{t+1} + T' W \big(\tau_t - \tau^* \big) - \beta^{-1} \tilde{I}' \varphi_t = 0,$$
(39)

for each $t \ge t_0$. Solving (39) under some assumptions (Giannoni and Woodford 2002a), it is possible to obtain a policy rule of the form expressed in (31).

Appendix B

This appendix explains the solution method used to derive the optimal nominal interest rate rule given by (7). Note that a similar procedure can be used to derive the other optimal rules.

The monetary authority minimizes the constrained loss function given by:

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} \frac{1}{2} \gamma_{\pi} \pi_t^2 + \frac{1}{2} \gamma_x \hat{x}_t^2 + \frac{1}{2} \gamma_i (\hat{i}_t - i^*)^2 \\ + \Lambda_{1,t} (\hat{x}_t - E_t \hat{x}_{t+1} + \sigma (\hat{i}_t - E_t \pi_{t+1}) - \alpha \hat{b}_t - \hat{r}_t^n) \\ + \Lambda_{2,t} (\pi_t - \kappa \hat{x}_t - \beta E_t \pi_{t+1} - \nu_t) \end{bmatrix} \right\}, \quad (40)$$

where the constraints include Eqs. (1) and (2), and $\Lambda_{1,t}$ and $\Lambda_{2,t}$ are the Lagrange multipliers.

In order to write the first-order conditions, we need to differentiate this equation with respect to the instrument $(\hat{t}_t - i^*)$ and the state variables π_t and \hat{x}_t . Before moving forward we need to consider how to deal with the expectation terms within the constraint. Since this is a policy under commitment, the dating of the expectations operator captures the idea of the policymaker choosing an ex-ante rule which will be followed in the future. Hence, the expectations operator on inflation and the output gap at t + 1 are removed. For example, if the inflation rate which the policymaker sets influences both actual and expected inflation, then he may directly optimize over the two. The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = \beta^t \gamma_\pi \pi_t - \beta^{t-1} \sigma \Lambda_{1,t-1} + \beta^t \Lambda_{2,t} - \beta^{t-1} \Lambda_{2,t-1}(\beta) = 0, \tag{41}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{x}_t} = \beta^t \gamma_x \hat{x}_t + \beta^t \Lambda_{1,t} - \beta^{t-1} \Lambda_{1,t-1} - \beta^t \kappa \Lambda_{2,t} = 0,$$
(42)

$$\frac{\partial \mathcal{L}}{\partial (\hat{i}_t - i^*)} = \beta^t \gamma_i (\hat{i}_t - i^*) + \beta^t \sigma \Lambda_{1,t} = 0.$$
(43)

Isolating $\Lambda_{1,t}$ in (43) and inserting into (42), we obtain

$$\gamma_x \hat{x}_t - \frac{\gamma_i}{\sigma} (\hat{i}_t - i^*) + \frac{\gamma_i}{\beta \sigma} (\hat{i}_{t-1} - i^*) - \kappa \Lambda_{2,t} = 0, \tag{44}$$

where $\Lambda_{1,t} = -\frac{\gamma_i}{\sigma}(\hat{i}_t - i^*)$ and $\Lambda_{1,t-1} = -\frac{\gamma_i}{\sigma}(\hat{i}_{t-1} - i^*)$. Repeating the procedure for $\Lambda_{2,t}$, we can eliminate all the Lagrange multipliers in (41). Then, isolating \hat{i}_t we have

$$\hat{i}_{t} = -\Gamma_{0}i^{*} + \Gamma_{i,1}\hat{i}_{t-1} - \Gamma_{i,2}\hat{i}_{t-2} + \Gamma_{\pi,0}\pi_{t} + \Gamma_{x,0}\hat{x}_{t} - \Gamma_{x,1}\hat{x}_{t-1},$$
(45)

where $\Gamma_0 = \frac{\sigma_K}{\beta}$, $\Gamma_{i,1} = (\frac{\sigma_K}{\beta} + \frac{1}{\beta} + 1)$, $\Gamma_{i,2} = \frac{1}{\beta}$, $\Gamma_{\pi,0} = \frac{\gamma_\pi \sigma_K}{\gamma_i}$, $\Gamma_{x,0} = \frac{\gamma_x \sigma}{\gamma_i}$, and $\Gamma_{x,1} = \frac{\gamma_x \sigma}{\gamma_i}$.

References

- Backus, D., & Driffill, D. (1985). Rational expectations and policy credibility following a change in regime. *Review of Economic Studies*, 52, 211–221.
- Barcelos Neto, P. C. F., & Portugal, M. S. (2009). The natural rate of interest in Brazil between 1999 and 2005. *Revista Brasileira de Economia*, 63(2), 103–118.
- Boucekkine, R. (1995). An alternative methodology for solving nonlinear forward-looking variables. Journal of Economic Dynamics & Control, 19, 711–734.
- Buiter, W. H. (1982). Predetermined and non-predetermined variables in rational expectations models. *Economics Letters*, 10, 49–54.
- Bénassy, J. P. (2007). Money, interest and policy: dynamic general equilibrium in a non-Ricardian world. Cambridge: MIT Press.

- Calvo, G. A. (1983). Staggered prices in a utility maximizing framework. *Journal of Monetary Economics*, 12, 383–398.
- Cavallari, M. C. L. (2003). A coordenação das políticas fiscal e monetária ótimas. In Proceedings of the 31st Brazilian economics meeting.
- Collard, F., & Juillard, M. (2001a). Accuracy of stochastic perturbation methods: the case of asset pricing models. *Journal of Economic Dynamics & Control*, 25, 979–999.
- Collard, F., & Juillard, M. (2001b). A higher-order Taylor expansion approach to simulation of stochastic forward-looking models with and application to a non-linear Phillips curve. *Computational Economics*, 17, 127–139.
- Damjanovic, T., Damjanovic, V., & Nolan, C. (2005). Optimal monetary policy rules from a timeless perspective. Centre for Dynamic Macroeconomic analysis working papers series.
- Dixit, A. (2001). Games of monetary and fiscal interactions in the EMU. European Economic Review, 45, 589–613.
- Dixit, A., & Lambertini, L. (2000). Fiscal discretion destroys monetary commitment. Available at SSRN http://ssrn.com/abstract=232654.
- Engwerda, J. C. (1998). On the open-loop Nash equilibrium in LQ-games. Journal of Economic Dynamics & Control, 22, 1487–1506.
- Engwerda, J. C., van Aarle, B., & Plasmans, J. (1999). The (in)finite horizon open-loop Nash LQ-game: an application to the EMU. Annals of Operations Research, 88, 251–273.
- Engwerda, J. C., van Aarle, B., & Plasmans, J. (2002). Cooperative and non-cooperative fiscal stabilization policies in the EMU. *Journal of Economic Dynamics & Control*, 26, 451–481.
- Favero, C. A. (2004). Comments on fiscal and monetary policy interactions: empirical evidence on optimal policy using a structural new-Keynesian model. *Journal of Macroeconomics*, 26, 281–285.
- Fialho, M. L., & Portugal, M. S. (2005). Monetary and fiscal policy interactions in Brazil: an application to the fiscal theory of the price level. *Estudos Econômicos*, 35(4), 657–685.
- Gadelha, S. R. B., & Divino, J. A. (2008). Dominância fiscal ou dominância monetária no Brasil? uma análise de causalidade. *Brazilian Journal of Applied Economics*, 12(4), 659–675.
- Giannoni, M. P., & Woodford, M. (2002a). NBER working paper series: Vol. 9419. Optimal interest-rate rules: I. General theory.
- Giannoni, M. P., & Woodford, M. (2002b). NBER working paper series: Vol. 9420. Optimal interest-rate rules: II. Applications.
- Gouvea, S. (2007). Central Bank of Brazil working papers: Vol. 143. Price rigidity in Brazil: evidence from CPI micro data. Available at http://ideas.repec.org/p/bcb/wpaper/143.html.
- Juillard, M. (1996). CEPREMAP working papers (Couverture Orange): Vol. 9602. Dynare: a program for the resolution and simulation of dynamic models with forward variables through the use of relaxation algorithm. Available at http://ideas.repec.org/p/cpm/cepmap/9602.html.
- Juillard, M., & Pelgrin, F. (2007). Computing optimal policy in a timeless-perspective: an application to a small-open economy. Working papers, 07–32, Bank of Canada.
- Kirsanova, T., Stehn, S. J., & Vines, D. (2005). The interactions between fiscal policy and monetary policy. Oxford Review of Economic Policy, 21(4), 532–564.
- Klein, P. (2000). Using the generalized Schur form to solve a multivariate linear rational expectations model. Journal of Economic Dynamics & Control, 24(10), 1405–1423.
- Laffargue, J. (1990). Résolution d'un modèle macroéconomique avec anticipations rationnelles. Annales d'Économie et de Statistique, 17, 97–119.
- Lambertini, L., & Rovelli, R. (2003). Monetary and fiscal policy coordination and macroeconomic stabilization. A theoretical analysis. Available at SSRN http://dx.doi.org/10.2139/ssrn.380322.
- Muscatelli, V., Tirelli, P., & Trecroci, C. (2004). Fiscal and monetary policy interactions: empirical evidence and optimal policy using a structural new-Keynesian model. *Journal of Macroeconomics*, 26, 257–280.
- Nordhaus, W. (1994). Policy games: coordination and independence in monetary and fiscal polices. Brookings Papers on Economic Activity, 2, 139–216.
- Nunes, A. F. N., & Portugal, M. S. (2009). Active and passive fiscal and monetary policies: an analysis for Brazil after the inflation targeting regime. In *Proceedings of the 37th Brazilian economics meeting*.
- Pires, M. C. C. (2008). Interação entre política monetária e fiscal no Brasil em modelos robustos a pequenas amostras. Ph.D. Dissertation, Department of Economics, Brasília, University of Brasília.
- Tabellini, G. (1985). Rational expectations and policy credibility following a change in regime. Giornali degli Economisti e Annali di Economia, 44, 389–425.
- Tanner, E., & Ramos, A. M. (2002). Fiscal sustainability and monetary versus fiscal dominance: evidence from Brazil 1991–2000. IMF Working Paper, 02/5.
- Taylor, J. (1993). Discretion versus policy rules in practice. Carnegie-Rochester Conference Series on Public Policy, 39, 195–214.

- Van Aarle, B., Engwerda, J., & Plasmans, J. (2002). Monetary and fiscal policy interactions in the EMU: a dynamic game approach. Annals of Operations Research, 109, 229–264.
- Walsh, C. E. (2003). Monetary theory and policy. Cambridge: MIT Press.
- Woodford, M. (1999). NBER Working paper series: Vol. 7261. Optimal monetary policy inertia.
- Woodford, M. (2003). *Interest and prices: foundations of a theory of monetary policy*. Princeton: Princeton University Press.

