Model

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1 Social Planer's Problem

In this economy, the social planner's problem is to choose the consumption C_t, the labor h_t, the rate of capacity utilization u_t, the investment I_t and the capital stock K_{t+1}, facing the law of motion of capital

$$K_{t+1} = (1 - \delta_0 - \delta_1(u_t - 1) - \frac{\delta_2}{2}(u_t - 1)^2)K_t + \Omega_t[I_t - \frac{1}{2\delta_0\eta}(\frac{I_t}{K_t} - \tau)^2 K_t] \tag{1}$$

and constrained by resource

$$Y_t = C_t + I_t A_t \tag{2}$$

to maximize agent's utility

$$E_0 \sum_{t=0}^{\infty} \left\{ \frac{[(C_t - \theta_c C_{t-1})(l_t - \theta_l l_{t-1})^{\chi}]^{(1-\gamma)} - 1}{1-\gamma} \right\}. \tag{3}$$

l_t is the leisure. The total time endowment is normalized to unit:

$$h_t + l_t = 1.$$

The production function is:

$$Y_t = Z_t(u_t K_t)^{\alpha}(X_t h_t)^{1-\alpha}.$$

The first-order conditions with respect to C_t, h_t, u_t, I_t and K_{t+1} are:

$$\begin{aligned} \Lambda_t = {} & (C_t - \theta_c C_{t-1})^{-\sigma}(1 - h_t - \theta_l(1 - h_{t-1}))^{\chi(1-\sigma)} \\ & - \theta_c \beta E_t[(C_{t+1} - \theta_c C_t)^{-\sigma}(1 - h_{t+1} - \theta_l(1 - h_t))^{\chi(1-\sigma)}]. \end{aligned} \tag{4}$$

$$\begin{aligned} (1-\alpha)\Lambda_t Z_t X_t(\frac{u_t K_t}{X_t h_t})^{\alpha} = {} & \chi(C_t - \theta_c C_{t-1})^{1-\sigma}(1 - h_t - \theta_l(1 - h_{t-1}))^{\chi(1-\sigma)-1} \\ & - \theta_l \beta \chi E_t[(C_{t+1} - \theta_c C_t)^{1-\sigma}(1 - h_{t+1} - \theta_l(1 - h_t))^{\chi(1-\sigma)-1}]. \end{aligned} \tag{5}$$

$$\alpha Z_t(\frac{u_t K_t}{X_t h_t})^{\alpha-1} = Q_t(\delta_1 + \delta_2(u_t - 1)). \tag{6}$$

$$Q_t \Omega_t[1 - \frac{1}{\eta\delta_0}(\frac{I_t}{K_t} - \tau)] = A_t. \tag{7}$$

$$
\begin{aligned}
Q_t\Lambda_t = \beta E_t\Lambda_{t+1}\{&Q_{t+1}[(1-\delta_0-\delta_1(1-u_{t+1})-\frac{\delta_2}{2}(1-u_{t+1})^2)\\
&+\Omega_{t+1}(-\frac{1}{2\eta\delta_0}(\frac{I_{t+1}}{K_{t+1}}-\tau)^2+\frac{1}{\eta\delta_0}(\frac{I_{t+1}}{K_{t+1}}-\tau)\frac{I_{t+1}}{K_{t+1}})]\\
&+Z_{t+1}u_{t+1}\alpha(\frac{u_{t+1}K_{t+1}}{X_{t+1}h_{t+1}})^{\alpha-1}\}.
\end{aligned}
\tag{8}
$$

Λ_t and $Q_t\Lambda_t$ are the Langrangin multiplier, X_t and A_t are non-stationary shocks, z_t and Ω_t are stationary shocks.

2 Balanced Growth Path

Y_t has the same trend as A_tI_t and $(K_t)^\alpha(X_t)^{1-\alpha}$, and I_t has the same trend as K_t, so we have:

$$\text{trend of Y} = (\text{trend of Y}/A)^\alpha \times X^{1-\alpha},$$

so the trend of Y is $A_t^{\alpha/(1-\alpha)}X_t$, which is denoted as X_t^Y. The trend component of capital is $X_t^K = A_t^{1/(\alpha-1)}X_t$. Detrending (**??**), (**??**), (**??**)-(**??**) yields the balanced growth path.

$$
\begin{aligned}
(\Lambda_t/(X_t^Y)^{-\sigma}) = &(C_t/(X_t^Y)-\theta_cC_{t-1}/(X_t^Y))^{-\sigma}(1-h_t-\theta_l(1-h_{t-1}))^{\chi(1-\sigma)}\\
&-\theta_c\beta E_t[(C_{t+1}/(X_t^Y)-\theta_cC_t/(X_t^Y))^{-\sigma}(1-h_{t+1}-\theta_l(1-h_t))^{\chi(1-\sigma)}].
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
(1-\alpha)\Lambda_t/(X_t^Y)^{-\sigma}Z_t(\frac{u_tK_t}{h_t(X_t^K)})^\alpha = &\chi(C_t/(X_t^Y)-\theta_cC_{t-1}/(X_t^Y))^{1-\sigma}(1-h_t-\theta_l(1-h_{t-1}))^{\chi(1-\sigma)-1}\\
&-\theta_l\beta\chi E_t[(C_{t+1}/(X_t^Y)-\theta_cC_t/(X_t^Y))^{1-\sigma}(1-h_{t+1}-\theta_l(1-h_t))^{\chi(1-\sigma)-1}].
\end{aligned}
\tag{10}
$$

$$\alpha Z_t(\frac{u_tK_t}{X_t^Kh_t})^{\alpha-1} = Q_t/(A_t)(\delta_1+\delta_2(u_t-1)). \tag{11}$$

$$Q_t/(A_t)\Omega_t[1-\frac{1}{\eta\delta_0}(\frac{I_tX_t^K}{X_t^KK_t}-\tau)] = 1. \tag{12}$$

$$
\begin{aligned}
Q_t/(A_t)\frac{A_t}{A_{t+1}}\Lambda_t/(X_t^Y)^{-\sigma} = \beta E_t\Lambda_{t+1}/(X_{t+1}^Y)^{-\sigma}\frac{(X_{t+1}^Y)^{-\sigma}}{(X_t^Y)^{-\sigma}}\{&Q_{t+1}/(A_{t+1})[(1-\delta_0-\delta_1(1-u_{t+1})-\frac{\delta_2}{2}(1-u_{t+1})^2)\\
&+\Omega_{t+1}(-\frac{1}{2\eta\delta_0}(\frac{I_{t+1}}{K_{t+1}}-\tau)^2+\frac{1}{\eta\delta_0}(\frac{I_{t+1}}{K_{t+1}}-\tau)\frac{I_{t+1}}{K_{t+1}})]\\
&+Z_{t+1}u_{t+1}\alpha(\frac{u_{t+1}K_{t+1}}{X_{t+1}^Kh_{t+1}})^{\alpha-1}\}.
\end{aligned}
\tag{13}
$$

$$Y_t/X_t^Y = C_t/X_t^Y + I_tX_t^K \tag{14}$$

$$K_{t+1}/X_t^K = (1-\delta_0-\delta_1(u_t-1)-\frac{\delta_2}{2}(u_t-1)^2)K_t/X_t^K+\Omega_t[I_t/X_t^K-\frac{1}{2\delta_0\eta}(\frac{I_t}{K_t}-\tau)^2K_t/X_t^K] \quad (15)$$

Let $\mu_t^y = X_t^Y/X_{t-1}^Y$, $\mu_t^k = X_t^K/X_{t-1}^K$ and $\mu_t^a = A_t/A_{t-1}$. The lowercases of C_t, I_t, K_t, Λ_t and Q_t denote the detrended variable. $c_t = C_t/X_t^Y, i_t = I_t/X_t^K, k_t = K_t/X_{t-1}^K, \lambda_t = \Lambda_t/(X_t^Y)^{-\sigma}, q_t = Q_t/A_t$.

$$\begin{aligned}\lambda_t &= (c_t-\theta_c c_{t-1}/(\mu_t^y))^{-\sigma}(1-h_t-\theta_l(1-h_{t-1}))^{\chi(1-\sigma)}\\ &\quad -\theta_c\beta E_t[(c_{t+1}\mu_{t+1}^y-\theta_c c_t)^{-\sigma}(1-h_{t+1}-\theta_l(1-h_t))^{\chi(1-\sigma)}].\end{aligned} \quad (16)$$

$$\begin{aligned}(1-\alpha)\lambda_t Z_t(\frac{u_tk_t}{h_t(\mu_t^k)})^\alpha &= \chi(c_t-\theta_c c_{t-1}/(\mu_t^y))^{1-\sigma}(1-h_t-\theta_l(1-h_{t-1}))^{\chi(1-\sigma)-1}\\ &\quad -\theta_l\beta\chi E_t[(c_{t+1}\mu_{t+1}^y-\theta_c c_t)^{1-\sigma}(1-h_{t+1}-\theta_l(1-h_t))^{\chi(1-\sigma)-1}].\end{aligned} \quad (17)$$

$$\alpha Z_t(\frac{u_tk_t}{\mu_t^k h_t})^{\alpha-1} = q_t(\delta_1+\delta_2(u_t-1)). \quad (18)$$

$$q_t\Omega_t[1-\frac{1}{\eta\delta_0}(\frac{i_t\mu_t^k}{k_t}-\tau)] = 1. \quad (19)$$

$$\begin{aligned}q_t\lambda_t &= \beta E_t\lambda_{t+1}\mu_t^a(\mu_{t+1}^y)^{-\sigma}\{q_{t+1}[(1-\delta_0-\delta_1(1-u_{t+1})-\frac{\delta_2}{2}(1-u_{t+1})^2)\\ &\quad +\Omega_{t+1}(-\frac{1}{2\eta\delta_0}(\frac{i_{t+1}\mu_{t+1}^k}{k_{t+1}}-\tau)^2+\frac{1}{\eta\delta_0}(\frac{i_{t+1}\mu_{t+1}^k}{k_{t+1}}-\tau)\frac{i_{t+1}\mu_{t+1}^k}{k_{t+1}})]\\ &\quad +Z_{t+1}u_{t+1}\alpha(\frac{u_{t+1}k_{t+1}}{\mu_{t+1}^k h_{t+1}})^{\alpha-1}\}.\end{aligned} \quad (20)$$

$$y_t = c_t + i_t \quad (21)$$

$$k_{t+1} = (1-\delta_0-\delta_1(u_t-1)-\frac{\delta_2}{2}(u_t-1)^2)k_t/\mu_t^k+\Omega_t[i_t-\frac{1}{2\delta_0\eta}(\frac{i_t\mu_t^k}{k_t}-\tau)^2k_t/\mu_t^k] \quad (22)$$

3 Steady State

(τ is equal to steady state level of $i\mu^k/k$).

$$\lambda = ((\mu^y)^\sigma-\theta_c\beta)[(c\mu^y-\theta_c c)^{-\sigma}(1-h-\theta_l(1-h))^{\chi(1-\sigma)}]. \quad (23)$$

$$(1-\alpha)\lambda Z(\frac{uk}{h(\mu^k)})^\alpha = \chi((\mu^y)^{\sigma-1}-\theta_l\beta)[(c\mu^y-\theta_c c)^{1-\sigma}(1-h-\theta_l(1-h))^{\chi(1-\sigma)-1}]. \quad (24)$$

$$\alpha Z(\frac{uk}{\mu^k h})^{\alpha-1} = q(\delta_1 + \delta_2(u-1)). \tag{25}$$

$$q\Omega = 1. \tag{26}$$

$$q\lambda = \beta\lambda\mu^a(\mu^y)^{-\sigma}\{q[(1-\delta_0-\delta_1(1-u)-\frac{\delta_2}{2}(1-u)^2)] + Zu\alpha(\frac{uk}{\mu^k h})^{\alpha-1}\}. \tag{27}$$

$$y = Z(\frac{uk}{\mu^k})^{\alpha} h^{1-\alpha} = c + i \tag{28}$$

$$k = (1-\delta_0-\delta_1(u-1)-\frac{\delta_2}{2}(u-1)^2)k/\mu^k + \Omega i \tag{29}$$

4 Calibration

Steady state h is set to 0.2, u is set to 1, μ^a is set to 0.9982, μ^y is set to 1.0044, Ω is set to 1, Z is set to 1. We have

$$q = 1,$$
$$\delta_1 = \alpha(\frac{k}{\mu^k h})^{\alpha-1},$$
$$k = \left(\frac{\beta\mu^a(\mu^y)^{-\sigma}\alpha(h\mu^k)^{1-\alpha}}{1-\beta(1-\delta_0)\mu^a(\mu^y)^{-\sigma}}\right)^{1/(1-\alpha)}$$
$$i = k(1-(1-\delta_0)/\mu^k)$$
$$c = (\frac{k}{\mu^k})^{\alpha} h^{1-\alpha} - i$$

Combine (**??**) and (**??**), we get

$$\chi = \frac{(1-\alpha)(k/(h\mu^k))^{\alpha}(1-h-\theta_l(1-h))((\mu^y)^{\sigma}-\theta_c\beta)}{((\mu^y)^{\sigma-1}-\theta_l\beta)(c\mu^y-\theta_c c)}.$$

α, β and δ_0 is given, χ is calibrated by steady state value, and all other steady state value can be analytically solved given parameter values.

However, χ is an parameter to be estimated, that is unreasonable to be calibrated in this way. I provide another guess about the equation form, which is actually used in related literature, that normalizes h with a scale parameter ψ. The scaled version of equation (**??**) and (**??**) and corresponding s.s. are

$$\begin{aligned}\lambda_t = (c_t - \theta_c\frac{c_{t-1}}{\mu_t^y})^{-\sigma}(1-\psi h_t - \theta_l(1-\psi h_{t-1}))^{\chi(1-\sigma)} \\ - \theta_c\beta E_t[(c_{t+1}\mu_{t+1}^y - \theta_c c_t)^{-\sigma}(1-\psi h_{t+1} - \theta_l(1-\psi h_t))^{\chi(1-\sigma)}]\end{aligned} \tag{30}$$

$$(1-\alpha)\lambda_t z_t(\frac{u_t k_t}{\mu_t^k})^{\alpha} h_t^{-\alpha} = \psi\chi(c_t - \theta_c\frac{c_{t-1}}{\mu_t^y})^{1-\sigma}(1-\psi h_t - \theta_l(1-\psi h_{t-1}))^{\chi(1-\sigma)-1} - \theta_l\beta\chi\psi E_t[(c_{t+1}\mu_{t+1}^y - \theta_c c_t)^{1-\sigma}(1-\psi h_{t+1} - \theta_l(1-\psi h_t))^{\chi(1-\sigma)-1}] \tag{31}$$

$$\lambda = ((\mu^y)^{\sigma} - \theta_c\beta)[(c\mu^y - \theta_c c)^{-\sigma}(1-\psi h - \theta_l(1-\psi h))^{\chi(1-\sigma)}]. \tag{32}$$

$$(1-\alpha)\lambda Z(\frac{uk}{h(\mu^k)})^{\alpha} = \psi\chi((\mu^y)^{\sigma-1} - \theta_l\beta)[(c\mu^y - \theta_c c)^{1-\sigma}(1-\psi h - \theta_l(1-\psi h))^{\chi(1-\sigma)-1}]. \tag{33}$$

$$(1-\alpha)Z(\frac{uk}{h(\mu^k)})^{\alpha} = \psi\chi\frac{((\mu^y)^{\sigma-1} - \theta_l\beta)}{((\mu^y)^{\sigma} - \theta_c\beta)}[(c\mu^y - \theta_c c)(1-\psi h - \theta_l(1-\psi h))^{-1}].$$

$$\psi = \frac{(1-\alpha)Z(\frac{uk}{h(\mu^k)})^{\alpha}(1-\theta_l)}{\chi\frac{((\mu^y)^{\sigma-1}-\theta_l\beta)}{((\mu^y)^{\sigma}-\theta_c\beta)}(c\mu^y - \theta_c c) + h(1-\theta_l)(1-\alpha)Z(\frac{uk}{h(\mu^k)})^{\alpha}}$$

5 Full system

Here I do not include the measurement equations.

FOC w.r.t. c_t,

$$\lambda_t = (c_t - \theta_c\frac{c_{t-1}}{\mu_t^y})^{-\sigma}(1-\psi h_t - \theta_l(1-\psi h_{t-1}))^{\chi(1-\sigma)} - \theta_c\beta E_t[(c_{t+1}\mu_{t+1}^y - \theta_c c_t)^{-\sigma}(1-\psi h_{t+1} - \theta_l(1-\psi h_t))^{\chi(1-\sigma)}]. \tag{34}$$

FOC w.r.t. h_t,

$$(1-\alpha)\lambda_t z_t(\frac{u_t k_t}{\mu_t^k})^{\alpha} h_t^{-\alpha} = \psi\chi(c_t - \theta_c\frac{c_{t-1}}{\mu_t^y})^{1-\sigma}(1-\psi h_t - \theta_l(1-\psi h_{t-1}))^{\chi(1-\sigma)-1} - \psi\theta_l\beta\chi E_t[(c_{t+1}\mu_{t+1}^y - \theta_c c_t)^{1-\sigma}(1-\psi h_{t+1} - \theta_l(1-\psi h_t))^{\chi(1-\sigma)-1}]. \tag{35}$$

FOC w.r.t. u_t,

$$\alpha Z_t(\frac{u_t k_t}{\mu_t^k h_t})^{\alpha-1} = q_t(\delta_1 + \delta_2(u_t - 1)). \tag{36}$$

FOC w.r.t. i_t,

$$q_t\Omega_t[1 - \frac{1}{\eta\delta_0}(\frac{i_t\mu_t^k}{k_t} - \tau)] = 1. \tag{37}$$

FOC w.r.t. k_{t+1},

$$\begin{aligned} q_t\lambda_t = \beta\mu^a_{t+1}(\mu^y_{t+1})^{-\sigma}\lambda_{t+1}\{q_{t+1}[(1-\delta_0-\delta_1(1-u_{t+1})-\frac{\delta_2}{2}(1-u_{t+1})^2) \\ +\Omega_{t+1}(-\frac{1}{2\eta\delta_0}(\frac{i_{t+1}\mu^k_{t+1}}{k_{t+1}}-\tau)^2+\frac{1}{\eta\delta_0}(\frac{i_{t+1}\mu^k_{t+1}}{k_{t+1}}-\tau)\frac{i_{t+1}\mu^k_{t+1}}{k_{t+1}})] \\ +Z_{t+1}u_{t+1}\alpha(\frac{u_{t+1}k_{t+1}}{\mu^k_{t+1}h_{t+1}})^{\alpha-1}\}. \end{aligned} \tag{38}$$

Resource constraint,

$$y_t = c_t + i_t. \tag{39}$$

Law of motion of capital,

$$k_t = [1-\delta_0-\delta_1(u_{t-1}-1)-\frac{\delta_2}{2}(u_{t-1}-1)^2]\frac{k_{t-1}}{\mu^k_{t-1}}+\Omega_{t-1}(i_{t-1}-\frac{1}{2\delta_0\eta}(\frac{i_{t-1}\mu^k_{t-1}}{k_{t-1}}-\tau)^2\frac{k_{t-1}}{\mu^k_{t-1}}) \tag{40}$$

Production function,

$$y_t = z_t(\frac{u_tk_t}{\mu^k_t})^\alpha h_t^{1-\alpha}. \tag{41}$$

rf_t's defination,

$$rf_t = \frac{1}{\beta}\frac{\lambda_t}{\lambda_{t+1}}(\mu^y_{t+1})^\sigma \tag{42}$$

v_t's defination,

$$v_t = \beta\frac{\lambda_{t+1}}{\lambda_t}(\mu^y_{t+1})^{1-\sigma}(v_{t+1}+\alpha y_{t+1}-i_{t+1}) \tag{43}$$

μ^y_t's defination,

$$\mu^y_t = (\mu^a_t)^{(\frac{\alpha}{\alpha-1})}\mu^\chi_t \tag{44}$$

μ^k_t's defination,

$$\mu^k_t = (\mu^a_t)^{(\frac{1}{\alpha-1})}\mu^\chi_t \tag{45}$$

We further define four shock processes—LAT, ISP, TFP, MEI—as $\hat{\mu}^\chi = log(\frac{\mu^\chi}{\bar{\mu}^\chi})$, $\hat{\mu}^a = log(\frac{\mu^a}{\bar{\mu}^a})$, $z = log(Z)$ and $\omega = log(\Omega)$. Exogenous process of LAT,

$$\hat{\mu}^\chi_t = \rho^u_\chi log(\hat{\mu}^\chi_{t-1}) + (1-\rho^u_\chi)\mu^{\chi,lr}_{t-1} + e^u_{t,\chi} \tag{46}$$

$$\mu^{\chi,lr}_t = \rho^{lr}_\chi\mu^{\chi,lr}_{t-1} + e^{lr}_{t,\chi} \tag{47}$$

Exogenous process of ISP,

$$\hat{\mu}^a_t = \rho^u_a\hat{\mu}^a_{t-1} + (1-\rho^u_a)\mu^{a,lr}_{t-1} + e^u_{t,a} \tag{48}$$

$$\mu_t^{a,lr} = \rho_a^{lr} \mu_{t-1}^{a,lr} + e_{t,a}^{lr} \tag{49}$$

Exogenous process of TFP,

$$z_t = \rho_z^u (z_{t-1}) + (1 - \rho_z^u) z_{t-1}^{lr} + e_{t,z}^u \tag{50}$$

$$z_t^{lr} = \rho_z^{lr} z_{t-1}^{lr} + e_{t,z}^{lr} \tag{51}$$

Exogenous process of MEI,

$$\omega_t = \rho_\omega^u (\omega_{t-1}) + (1 - \rho_\omega^u) \omega_{t-1}^{lr} + e_{t,\omega}^u \tag{52}$$

$$\omega_t^{lr} = \rho_\omega^{lr} \omega_{t-1}^{lr} + e_{t,\omega}^{lr} \tag{53}$$

All $\rho < 1$.

6 Estimation

For estimation, I need to provide observation variables–y_t^{obs}, c_t^{obs}, i_t^{obs}, ai_t^{obs}, v_t^{obs}, $rfr_t^{obs} h_t^{obs}$. Their definition is

$$\begin{aligned} y_t^{obs} &= \Delta log(Y_t) \\ c_t^{obs} &= \Delta log(C_t) \\ i_t^{obs} &= \Delta log(I_t) \\ ai_t^{obs} &= \Delta log(A_t I_t) \\ v_t^{obs} &= \Delta log(v_t) \\ rf_t^{obs} &= log(rf_t) \\ h_t^{obs} &= log(h_t) \end{aligned}$$

If the lowercase variable in Section 5 denotes level variable (not log transformation),for example $y_t = Y_t / X_t^Y$, we should add following measurement equations:

$$\begin{aligned} y_t^{obs} &= log(y_t) - log(y_{t-1}) + log(\mu_t^y) + e_t^y \\ c_t^{obs} &= log(c_t) - log(c_{t-1}) + log(\mu_t^y) + e_t^c \\ i_t^{obs} &= log(i_t) - log(i_{t-1}) + log(\mu_t^k) + e_t^i \\ ai_t^{obs} &= log(i_t) - log(i_{t-1}) + log(\mu_t^k) + log(\mu_t^a) + e_t^{ai} \\ v_t^{obs} &= log(v_t) - log(v_{t-1}) + log(\mu_t^y) + e_t^v \\ rf_t^{obs} &= log(rf_t) + e_t^{rf} \\ h_t^{obs} &= log(h_t) + e_t^h \end{aligned}$$

In steady state, we have:

$$
\begin{aligned}
y^{obs} &= log(\mu^y) \\
c^{obs} &= log(\mu^y) \\
i^{obs} &= log(\mu^k) \\
ai^{obs} &= log(\mu^k) + log(\mu^a) \\
v^{obs} &= log(\mu^y) \\
rf^{obs} &= log(rf) \\
h^{obs} &= log(h)
\end{aligned}
$$