Forming 1st stage Lagrange for the problem (assuming K_A and K_B as given)

$$\mathcal{L} = N_A^{\alpha} K_A^{\beta} + N_B^{\alpha} K_B^{\beta} + \lambda\,[\,N - N_A - N_B]$$

First order conditions w.r.t. N_A , N_B and λ and solving for N_A and N_B

$$N_A = N \frac{K_B^{\frac{\beta}{(\alpha-1)}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)}$$

$$N_B = N \frac{K_A^{\frac{\beta}{(\alpha-1)}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)}$$

Dividing N_A by N_B, we get

$$\frac{N_A}{N_B} = \left(\frac{K_B}{K_A}\right)^{\frac{\beta}{(\alpha-1)}}$$

$$\frac{N_A}{N_B} = \left(\frac{K_A}{K_B}\right)^{\frac{\beta}{(1-\alpha)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Substituting expression of N_A and N_B into the given production function of two cities

$$Y = Y_A + Y_B$$

$$Y = N_A^{\alpha} K_A^{\beta} + N_B^{\alpha} K_B^{\beta}$$

$$Y = \left(N \frac{K_B^{\frac{\beta}{(\alpha-1)}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} K_A^{\beta} + \left(N \frac{K_A^{\frac{\beta}{(\alpha-1)}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} K_B^{\beta}$$

$$= \frac{N^{\alpha}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)^{\alpha}} \left(K_B^{\frac{\alpha\beta}{(\alpha-1)}} K_A^{\beta} + K_A^{\frac{\alpha\beta}{(\alpha-1)}} K_B^{\beta}\right)$$

$$= \frac{N^{\alpha}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)^{\alpha}} \left(\frac{K_A^{\beta}}{K_B^{\frac{\alpha\beta}{(1-\alpha)}}} + \frac{K_B^{\beta}}{K_A^{\frac{\alpha\beta}{(1-\alpha)}}}\right)$$

$$= \frac{N^{\alpha}}{\left(\frac{1}{K_A^{\frac{\beta}{(1-\alpha)}}} + \frac{1}{K_B^{\frac{\beta}{(1-\alpha)}}}\right)^{\alpha}} \left(\frac{K_A^{\beta}}{K_B^{\frac{\alpha\beta}{(1-\alpha)}}} + \frac{K_B^{\beta}}{K_A^{\frac{\alpha\beta}{(1-\alpha)}}}\right)$$

$$= N^{\alpha} \frac{(K_A\, K_B)^{\frac{\alpha\beta}{(1-\alpha)}}}{\left(K_B^{\frac{\beta}{(1-\alpha)}} + K_A^{\frac{\beta}{(1-\alpha)}}\right)^{\alpha}} \left(\frac{(K_A^{\beta} K_A^{\frac{\alpha\beta}{(1-\alpha)}} + K_B^{\beta} K_B^{\frac{\alpha\beta}{(1-\alpha)}}}{K_B^{\frac{\alpha\beta}{(1-\alpha)}} K_A^{\frac{\alpha\beta}{(1-\alpha)}}}\right)$$

$$= N^{\alpha} \frac{(K_A\, K_B)^{\frac{\alpha\beta}{(1-\alpha)}}}{\left(K_B^{\frac{\beta}{(1-\alpha)}} + K_A^{\frac{\beta}{(1-\alpha)}}\right)^{\alpha}} \left(\frac{K_A^{\frac{\beta}{(1-\alpha)}} + K_B^{\frac{\beta}{(1-\alpha)}}}{(K_A K_B)^{\frac{\alpha\beta}{(1-\alpha)}}}\right)$$

$$\mathrm{Y} = N^{\alpha} \left(K_B^{\frac{\beta}{(1-\alpha)}} + K_A^{\frac{\beta}{(1-\alpha)}}\right)^{1-\alpha}$$

To make this into output per capita form, we need to divide the expression by N

$$y = \frac{N^{\alpha}}{N} \left(K_B^{\frac{\beta}{(1-\alpha)}} \frac{N^{\frac{\beta}{(1-\alpha)}}}{N^{\frac{\beta}{(1-\alpha)}}} + K_A^{\frac{\beta}{(1-\alpha)}} \frac{N^{\frac{\beta}{(1-\alpha)}}}{N^{\frac{\beta}{(1-\alpha)}}}\right)^{1-\alpha}$$

$$= N^{\alpha-1} N^{\beta} \left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{1-\alpha}$$

$$y = N^{\alpha+\beta-1} \left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{1-\alpha} \quad \text{...(2 \& 3)}$$

We can also express $y = y_A + y_B$

As $Y_A = N_A^{\alpha} K_A^{\beta}$

Substituting value of $N_A = N \frac{K_B^{\frac{\beta}{(\alpha-1)}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)}$ in above, we get $Y_A = \left(N \frac{K_B^{\frac{\beta}{(\alpha-1)}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} K_A^{\beta}$

To convert this into per capita form, we get

$$\frac{Y_A}{N} = y_A = \frac{1}{N}\left(N\frac{K_B^{\frac{\beta}{(\alpha-1)}}\ \frac{N^{\frac{\beta}{(\alpha-1)}}}{N^{\frac{\beta}{(\alpha-1)}}}}{\left(K_A^{\frac{\beta}{(\alpha-1)}} + K_B^{\frac{\beta}{(\alpha-1)}}\right)\frac{N^{\frac{\beta}{(\alpha-1)}}}{N^{\frac{\beta}{(\alpha-1)}}}}\right)^{\alpha} K_A^{\beta}\frac{N^{\beta}}{N^{\beta}}$$

$$y_A = N^{\alpha+\beta-1}\left(\frac{k_B^{\frac{\beta}{(\alpha-1)}}}{\left(k_A^{\frac{\beta}{(\alpha-1)}} + k_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} k_A^{\beta}$$

Similarly,

$$y_b = N^{\alpha+\beta-1}\left(\frac{k_A^{\frac{\beta}{(\alpha-1)}}}{\left(k_A^{\frac{\beta}{(\alpha-1)}} + k_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} k_B^{\beta}$$

Second stage problem of social planner in which optimal allocation of capita between 2 cities will be decided. Assuming log-utility for the consumers with discount rate ρ and congestion cost function of k^{σ} (which is assumed to be in per capita form)

Solving through current-value Hamiltonian

$$H = e^{-\rho t}\ \ln c + \lambda\left[N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{1-\alpha} - (i_A + i_B) - (k_A^{\sigma} + k_{B)}^{\sigma} - c\right] +$$
$$\mu_A\,(i_A - \delta k_A) + \mu_B\,(i_B - \delta k_B)$$

Driving first order conditions w.r.t. control variable (consumption) and state variable (capital)

We get,

$$\frac{\partial H}{\partial c} = 0 \quad \rightarrow \quad \frac{e^{-\rho t}}{c} = \lambda \quad \text{...(4)}$$

$$\frac{\partial H}{\partial i_A} \leq 0 \quad \rightarrow \quad -\lambda + \mu_A \leq 0 \quad \text{...(5)}$$

$$\frac{\partial H}{\partial i_B} \leq 0 \quad \rightarrow \quad -\lambda + \mu_B \leq 0 \quad \text{..(6)}$$

$$\frac{\partial H}{\partial k_A} = -\dot{\mu}_A \quad \rightarrow \quad \lambda\left[\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_A^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_A^{\sigma-1}\right] - \mu_A\delta = -\dot{\mu}_A \quad \text{.... (7)}$$

$$\frac{\partial H}{\partial k_B} = -\dot{\mu}_B \quad \rightarrow \quad \lambda\left[\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_B^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_B^{\sigma-1}\right] - \mu_B\delta = -\dot{\mu}_B$$

We define net MPL as $\widetilde{f_A} = \left[\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_A^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_A^{\sigma-1}\right]$ for city A

and $\widetilde{f_B} = \left[\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_B^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_B^{\sigma-1}\right]$ for city B

Taking log and time derivative of equation (4) we get,

$$-\rho - \frac{\dot{C}}{C} = \frac{\dot{\lambda}}{\lambda} \quad \text{...(8)}$$

Using the fact that $i_A > 0$ in time period $\left(0, t_A^p\right]$ which means that due to complementary slackness (?) $\frac{\partial H}{\partial i_A} = 0$

Therefore, from equation (5) we get

$$\lambda = \mu_A$$

Substituting this into equation (7)

$$\lambda\left[\widetilde{f_A}\right] - \lambda\delta = -\dot{\lambda}$$

$$\widetilde{f_A} - \delta = -\frac{\dot{\lambda}}{\lambda} \quad \text{...(9)}$$

Combining (8) and (9) gives us growth rate of consumption (γ_c) in time period $\left(0, t_A^p\right]$

$$\gamma_c = \widetilde{f_A} - \delta - \rho$$

For city A

$$\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_A^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_A^{\sigma-1} - \delta - \rho = \gamma_{ca} \quad \text{...................................(a)}$$

For city B

$$\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_B^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_B^{\sigma-1} - \delta - \rho = \gamma_{cb} \quad \text{.......................................(b)}$$

For Dynare

To write the social planner Ramsey model model in Dynare, we need to write first order conditions of the model along with capital accumulation equation, resource constraint, and equation of production function. In our case, we need separate equations and time paths of evolution of capital in both cities, and thus we need to define separate equations for both cities, but connected with each other through the overall resource constraint.

The equations we used for the model are:

$$y = y_A + y_B$$ --(eq1)

$$y_A = N^{\alpha+\beta-1}\left(\frac{k_B^{\frac{\beta}{(\alpha-1)}}}{\left(k_A^{\frac{\beta}{(\alpha-1)}}+k_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} k_A^{\beta}$$ -----------------------------------(eq2)

$$y_B = N^{\alpha+\beta-1}\left(\frac{k_A^{\frac{\beta}{(\alpha-1)}}}{\left(k_A^{\frac{\beta}{(\alpha-1)}}+k_B^{\frac{\beta}{(\alpha-1)}}\right)}\right)^{\alpha} k_B^{\beta}$$ ----------------------------(eq3)

$$\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_A^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_A^{\sigma-1} - \delta - \rho = \gamma_{ca}$$ ------------(eq4)

$$\beta\, N^{\alpha+\beta-1}\left(k_B^{\frac{\beta}{(1-\alpha)}} + k_A^{\frac{\beta}{(1-\alpha)}}\right)^{-\alpha} k_B^{\frac{\beta}{(1-\alpha)}-1} - \sigma k_B^{\sigma-1} - \delta - \rho = \gamma_{cb}$$ --------------(eq5)

$$y_A = i_A + c_A - k_A^{\sigma}$$ --(eq6)

$$y - y_A = i_B + c_B - k_B^{\sigma}$$ -- (eq7)

$$k_A(+1) = (1-\delta)k_A + i_A$$ ---(eq8)

$$k_B(+1) = (1-\delta)k_B + i_B$$ ---(eq9)
