

## An Open-economy Real Business Cycle Model with Capital Adjustment cost

Household's period Utility

$$U_t = \frac{[C_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma}$$

Production function

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha}$$

Resource Constraint

$$C_t + X_t = Y_t + B_t - q_t B_{t+1}$$

Law of Motion for Capital

$$K_{t+1} = V_t X_t + (1 - \delta) K_t - \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - \mu_g \right)^2 K_t$$

Price of debt

$$\frac{1}{q_t} = 1 + r^* + \psi \left[ \exp \left( \frac{B_{t+1}}{Z_t} - b \right) - 1 \right]$$

Shock processes

$$\ln V_t = \ln V_{t-1} + g_{v,t}$$

$$g_{v,t} - \mu = \rho(g_{v,t-1} - \mu) + \varepsilon_t; \quad \varepsilon_t \sim N(0, \sigma)$$

Marginal Utility of Consumption:  $U_c = \frac{\gamma(1-\sigma)}{C_t} U_t$

Marginal disutility of Labor:  $U_L = -\frac{(1-\gamma)(1-\sigma)}{1-L_t} U_t$

Set up the Lagrangian

$$\begin{aligned} \mathcal{L} = \sum \beta^t & \left[ \frac{[C_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma} + \lambda_t (A_t K_t^\alpha L_t^{1-\alpha} + B_t - q_t B_{t+1} - C_t - X_t) \right. \\ & \left. + \mu_t \left( V_t X_t + (1 - \delta) K_t - \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - \mu_g \right)^2 K_t - K_{t+1} \right) \right] \end{aligned}$$

First order condition: consumption

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0$$

$$\Rightarrow \frac{(1 - \sigma) [C_t^\gamma (1 - L_t)^{1-\gamma}]^{-\sigma}}{1 - \sigma} \gamma C_t^{\gamma-1} (1 - L_t)^{1-\gamma} - \lambda_t = 0$$

$$\Rightarrow \frac{\gamma}{C_t} [C_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma} = \lambda_t$$

$$\Rightarrow \frac{\gamma(1-\sigma)}{C_t} U_t = \lambda_t$$

FOC: Labor

$$\begin{aligned} \mathcal{L} &= \sum \beta^t \left[ \frac{[C_t^\gamma (1-L_t)^{1-\gamma}]^{1-\sigma}}{1-\sigma} + \lambda_t (A_t K_t^\alpha L_t^{1-\alpha} + B_t - q_t B_{t+1} - C_t - X_t) \right. \\ &\quad \left. + \mu_t \left( V_t X_t + (1-\delta)K_t - \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - \mu_g \right)^2 K_t - K_{t+1} \right) \right] \\ \frac{\partial \mathcal{L}}{\partial L_t} &= 0 \\ \Rightarrow -\frac{(1-\sigma)[C_t^\gamma (1-L_t)^{1-\gamma}]^{-\sigma}}{1-\sigma} (1-\gamma) C_t^\gamma (1-L_t)^{-\gamma} + (1-\alpha) \lambda_t A_t K_t^\alpha L_t^{-\alpha} &= 0 \\ \Rightarrow -\frac{(1-\gamma)}{1-L_t} [C_t^\gamma (1-L_t)^{1-\gamma}]^{1-\sigma} + (1-\alpha) \lambda_t \frac{Y_t}{L_t} &= 0 \\ \Rightarrow -\frac{(1-\gamma)(1-\sigma)}{1-L_t} U_t + (1-\alpha) \lambda_t \frac{Y_t}{L_t} &= 0 \end{aligned}$$

FOC: Capital

$$\begin{aligned} \mathcal{L} &= \sum \beta^t \left[ \frac{[C_t^\gamma (1-L_t)^{1-\gamma}]^{1-\sigma}}{1-\sigma} + \lambda_t (A_t K_t^\alpha L_t^{1-\alpha} + B_t - q_t B_{t+1} - C_t - X_t) \right. \\ &\quad \left. + \mu_t \left( V_t X_t + (1-\delta)K_t - \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - \mu_g \right)^2 K_t - K_{t+1} \right) \right] \\ \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= 0 \\ \Rightarrow \mu_t \left[ -\phi \left( \frac{K_{t+1}}{K_t} - \mu_g \right) \frac{K_t}{K_t} - 1 \right] &+ \beta \left[ \alpha \lambda_{t+1} A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} \right. \\ &\quad \left. + \mu_{t+1} \left[ (1-\delta) - \frac{\phi}{2} \left( \frac{K_{t+2}}{K_{t+1}} - \mu_g \right)^2 - \phi \left( \frac{K_{t+2}}{K_{t+1}} - \mu_g \right) \left( -\frac{K_{t+2}}{K_{t+1}^2} \right) K_{t+1} \right] \right] = 0 \\ \Rightarrow \mu_t \left[ \phi \left( \frac{K_{t+1}}{K_t} - \mu_g \right) + 1 \right] &= \beta \left[ \alpha \lambda_{t+1} \frac{Y_{t+1}}{K_{t+1}} \right. \\ &\quad \left. + \mu_{t+1} \left[ (1-\delta) - \frac{\phi}{2} \left( \frac{K_{t+2}}{K_{t+1}} - \mu_g \right)^2 + \phi \left( \frac{K_{t+2}}{K_{t+1}} - \mu_g \right) \left( \frac{K_{t+2}}{K_{t+1}} \right) \right] \right] \end{aligned}$$

FOC: Investment

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial X_t} &= 0 \\ \Rightarrow -\lambda_t + \mu_t V_t &= 0 \\ \Rightarrow \lambda_t &= \mu_t V_t\end{aligned}$$

FOC: Bond

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial B_{t+1}} &= 0 \\ \lambda_t q_t &= \beta \lambda_{t+1}\end{aligned}$$

### Stationarized Model

Scale variable:  $Z_{t-1} = (V_{t-1})^{\frac{\alpha}{1-\alpha}}$

$$\begin{aligned}\hat{c}_t &= \frac{C_t}{Z_{t-1}}; \quad \hat{y}_t = \frac{Y_t}{Z_{t-1}}; \quad \hat{x}_t = \frac{X_t}{Z_{t-1}}; \quad k = \frac{K_t}{V_{t-1} Z_{t-1}}; \\ \hat{u}_t &= \frac{U_t}{(Z_{t-1})^{\gamma(1-\sigma)}}; \quad \hat{u}_{L,t} = \frac{U_L}{(Z_{t-1})^{\gamma(1-\sigma)}} \\ \hat{\lambda}_t = \hat{u}_{c,t} &= \frac{U_c}{(Z_{t-1})^{\gamma(1-\sigma)-1}}; \quad \hat{\mu}_t = \frac{\mu_t}{V_t^{-1} (Z_{t-1})^{\gamma(1-\sigma)-1}}\end{aligned}$$

Production Function

$$\begin{aligned}Y_t &= A_t K_t^\alpha L_t^{1-\alpha} \\ \Rightarrow \frac{Y_t}{Z_{t-1}} &= A_t \frac{K_t^\alpha}{Z_{t-1}} L_t^{1-\alpha} \\ \Rightarrow \hat{y}_t &= A_t \hat{k}_t^\alpha L_t^{1-\alpha}\end{aligned}$$

Resource Constraint

$$\begin{aligned}C_t + X_t &= Y_t + B_t - B_{t+1} \\ \Rightarrow \frac{C_t}{Z_{t-1}} + \frac{X_t}{Z_{t-1}} &= \frac{Y_t}{Z_{t-1}} + \frac{B_t}{Z_{t-1}} - \frac{B_{t+1}}{Z_t} \frac{Z_t}{Z_{t-1}}\end{aligned}$$

$$\Rightarrow \hat{c}_t + \hat{x}_t = \hat{y}_t + \hat{b}_t - \hat{b}_{t+1} \left( \frac{V_t}{V_{t-1}} \right)^{\frac{\alpha}{1-\alpha}}$$

$$\Rightarrow \hat{c}_t + \hat{x}_t = \hat{y}_t + \hat{b}_t - \hat{b}_{t+1} (e^{gt})^{\frac{\alpha}{1-\alpha}}$$

Law of Motion for Capital

$$K_{t+1} = V_t X_t + (1 - \delta) K_t - \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - \mu_g \right)^2 K_t$$

Dividing by  $V_t Z_{t-1}$

$$\begin{aligned} \Rightarrow \frac{K_{t+1}}{V_t Z_t} \frac{Z_t}{Z_{t-1}} &= \frac{X_t}{Z_{t-1}} + (1 - \delta) \frac{K_t}{V_{t-1} Z_{t-1}} \frac{V_{t-1}}{V_t} \\ &\quad - \frac{\phi}{2} \left( \frac{\frac{K_{t+1}}{V_t Z_t} V_t Z_t}{\frac{K_t}{V_{t-1} Z_{t-1}} V_{t-1} Z_{t-1}} - \mu_g \right)^2 \frac{K_t}{V_{t-1} Z_{t-1}} \frac{V_{t-1}}{V_t} \\ \Rightarrow \hat{k}_{t+1} \left( \frac{V_t}{V_{t-1}} \right)^{\frac{\alpha}{1-\alpha}} &= \hat{x}_t + (1 - \delta) \hat{k}_t \frac{V_{t-1}}{V_t} - \frac{\phi}{2} \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} \left( \frac{V_t}{V_{t-1}} \right)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \hat{k}_t \frac{V_{t-1}}{V_t} \\ \Rightarrow \hat{k}_{t+1} (e^{gt})^{\frac{\alpha}{1-\alpha}} &= \hat{x}_t + (1 - \delta) \hat{k}_t e^{-gt} - \frac{\phi}{2} \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} (e^{gt})^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \hat{k}_t e^{-gt} \end{aligned}$$

Utility

$$\begin{aligned} U_t &= \frac{[C_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma} \\ \Rightarrow \frac{U_t}{(Z_{t-1})^{\gamma(1-\sigma)}} &= \frac{\left( \frac{C_t}{Z_{t-1}} \right)^{\gamma(1-\sigma)} [(1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma} \\ \Rightarrow \hat{u}_t &= \frac{[\hat{c}_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma} \end{aligned}$$

Marginal Utility of consumption

$$\begin{aligned} U_c &= \frac{\gamma(1 - \sigma)}{C_t} U_t \\ \Rightarrow \frac{U_c}{(Z_{t-1})^{\gamma(1-\sigma)-1}} &= \frac{\gamma(1 - \sigma)}{\frac{C_t}{Z_{t-1}}} \frac{U_t}{(Z_{t-1})^{\gamma(1-\sigma)}} \\ \Rightarrow \hat{u}_c &= \frac{\gamma(1 - \sigma)}{\hat{c}_t} \hat{u}_t \end{aligned}$$

Marginal disutility of Labor

$$\begin{aligned}
 U_L &= -\frac{(1-\gamma)(1-\sigma)}{1-L_t} U_t \\
 \Rightarrow \frac{U_L}{(Z_{t-1})^{\gamma(1-\sigma)}} &= -\frac{(1-\gamma)(1-\sigma)}{1-L_t} \frac{U_t}{(Z_{t-1})^{\gamma(1-\sigma)}} \\
 \Rightarrow \hat{u}_L &= -\frac{(1-\gamma)(1-\sigma)}{1-L_t} \hat{u}_t
 \end{aligned}$$

FOC: Labor

$$\begin{aligned}
 -\frac{(1-\gamma)(1-\sigma)}{1-L_t} U_t + (1-\alpha)\lambda_t \frac{Y_t}{L_t} &= 0 \\
 \Rightarrow U_L + (1-\alpha)U_c \frac{Y_t}{L_t} &= 0 \\
 \Rightarrow \frac{U_L}{(Z_{t-1})^{\gamma(1-\sigma)}} + (1-\alpha) \frac{U_c}{(Z_{t-1})^{\gamma(1-\sigma)-1}} \frac{Y_t}{(Z_{t-1})L_t} &= 0 \\
 \Rightarrow \hat{u}_L + (1-\alpha)\hat{u}_c \frac{\hat{y}_t}{L_t} &= 0
 \end{aligned}$$

FOC: Capital  $\mu_t$  and  $\lambda_t$  should have different trends as  $\lambda_t = \mu_t V_t$

$$\begin{aligned}
 &\mu_t \left[ \phi \left( \frac{K_{t+1}}{K_t} - \mu_g \right) + 1 \right] \\
 &= \beta \left[ \alpha \lambda_{t+1} \frac{Y_{t+1}}{K_{t+1}} \right. \\
 &\quad \left. + \mu_{t+1} \left[ (1-\delta) - \frac{\phi}{2} \left( \frac{K_{t+2}}{K_{t+1}} - \mu_g \right)^2 + \phi \left( \frac{K_{t+2}}{K_{t+1}} - \mu_g \right) \left( \frac{K_{t+2}}{K_{t+1}} \right) \right] \right]
 \end{aligned}$$

Multiplying each term by  $V_t$  and detrending each variable by its respective trend

$$\begin{aligned}
& \Rightarrow \frac{\mu_t V_t}{(Z_{t-1})^{\gamma(1-\sigma)-1}} \frac{(Z_{t-1})^{\gamma(1-\sigma)-1}}{(Z_t)^{\gamma(1-\sigma)-1}} \left[ \phi \left( \frac{\frac{K_{t+1}}{V_t Z_t} V_t Z_t}{\frac{K_t}{V_{t-1} Z_{t-1}} V_{t-1} Z_{t-1}} - \mu_g \right) + 1 \right] \\
& = \beta \left[ \alpha \frac{U_{c,t+1}}{(Z_t)^{\gamma(1-\sigma)-1}} \frac{Y_{t+1}/Z_t}{K_{t+1}/V_t Z_t} \right. \\
& \quad + \frac{\mu_{t+1} V_{t+1}}{(Z_t)^{\gamma(1-\sigma)-1}} \frac{V_t}{V_{t+1}} \left\{ (1-\delta) - \frac{\phi}{2} \left( \frac{\frac{K_{t+2}}{V_{t+1} Z_{t+1}} V_{t+1} Z_{t+1}}{\frac{K_{t+1}}{V_t Z_t} V_t Z_t} - \mu_g \right)^2 \right. \\
& \quad \left. \left. + \phi \left( \frac{\frac{K_{t+2}}{V_{t+1} Z_{t+1}} V_{t+1} Z_{t+1}}{\frac{K_{t+1}}{V_t Z_t} V_t Z_t} - \mu_g \right) \left( \frac{\frac{K_{t+2}}{V_{t+1} Z_{t+1}} V_{t+1} Z_{t+1}}{\frac{K_{t+1}}{V_t Z_t} V_t Z_t} \right) \right\} \right] \\
& \Rightarrow \hat{\mu}_t \left( \left( \frac{V_{t-1}}{V_t} \right)^{\alpha/1-\alpha} \right)^{\gamma(1-\sigma)-1} \left[ \phi \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} \left( \frac{V_t}{V_{t-1}} \right)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \right] \\
& = \beta \left[ \alpha \hat{u}_{c,t+1} \frac{\hat{y}_{t+1}}{\hat{k}_{t+1}} + \hat{\mu}_{t+1} e^{-g_{t+1}} \left\{ (1-\delta) - \frac{\phi}{2} \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} \left( \frac{V_{t+1}}{V_t} \right)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \right. \right. \\
& \quad \left. \left. + \phi \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} \left( \frac{V_{t+1}}{V_t} \right)^{\frac{1}{1-\alpha}} - \mu_g \right) \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} \left( \frac{V_{t+1}}{V_t} \right)^{\frac{1}{1-\alpha}} \right\} \right] \\
& \Rightarrow \hat{\mu}_t ((e^{-g_t})^{\alpha/1-\alpha})^{\gamma(1-\sigma)-1} \left[ \phi \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} (e^{g_t})^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \right] \\
& = \beta \left[ \alpha \hat{u}_{c,t+1} \frac{\hat{y}_{t+1}}{\hat{k}_{t+1}} + \hat{\mu}_{t+1} e^{-g_{t+1}} \left\{ (1-\delta) - \frac{\phi}{2} \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \right. \right. \\
& \quad \left. \left. + \phi \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} - \mu_g \right) \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} \right\} \right]
\end{aligned}$$

Euler equation for bond

$$\begin{aligned}
\lambda_t q_t &= \beta \lambda_{t+1} \\
\Rightarrow U_{c,t} q_t &= \beta U_{c,t+1} \\
\frac{U_{c,t}}{(Z_{t-1})^{\gamma(1-\sigma)-1}} q_t &= \beta \frac{U_{c,t+1}}{(Z_t)^{\gamma(1-\sigma)-1}} \frac{(Z_t)^{\gamma(1-\sigma)-1}}{(Z_{t-1})^{\gamma(1-\sigma)-1}} \\
\hat{u}_{c,t} q_t &= \beta \hat{u}_{c,t+1} \left( \frac{V_t}{V_{t-1}} \right)^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}
\end{aligned}$$

$$\hat{u}_{c,t} q_t = \beta \hat{u}_{c,t+1} (e^{g_t})^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}$$

FOC: Investment

$$\frac{\mu_t V_t}{(Z_{t-1})^{\gamma(1-\sigma)-1}} = \frac{\lambda_t}{(Z_{t-1})^{\gamma(1-\sigma)-1}}$$

$$\hat{\mu}_t = \hat{\lambda}_t$$

### Summary of Equilibrium

Production Function:

$$\Rightarrow \hat{y}_t = A_t \hat{k}_t^\alpha L_t^{1-\alpha}$$

Utility:

$$\Rightarrow \hat{u}_t = \frac{[\hat{c}_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma}$$

Marginal utility of consumption:

$$\Rightarrow \hat{u}_c = \frac{\gamma(1-\sigma)}{\hat{c}_t} \hat{u}_t$$

Marginal disutility of labor:

$$\hat{u}_L = -\frac{(1-\gamma)(1-\sigma)}{1-L_t} \hat{u}_t$$

FOC for Labor:

$$\hat{u}_L + (1-\alpha) \hat{u}_c \frac{\hat{y}_t}{L_t} = 0$$

FOC for Capital:

$$\begin{aligned} \Rightarrow \hat{\mu}_t ((e^{-g_t})^{\alpha/(1-\alpha)})^{\gamma(1-\sigma)-1} & \left[ \phi \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} (e^{g_t})^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \right] \\ & = \beta \left[ \alpha \hat{u}_{c,t+1} \frac{\hat{y}_{t+1}}{\hat{k}_{t+1}} + \hat{\mu}_{t+1} e^{-g_{t+1}} \left\{ (1-\delta) - \frac{\phi}{2} \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \right. \right. \\ & \quad \left. \left. + \phi \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} - \mu_g \right) \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} \right\} \right] \end{aligned}$$

Euler equation for bond:

$$\hat{u}_{c,t} q_t = \beta \hat{u}_{c,t+1} (e^{g_t})^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}$$

FOC for Investment:

$$\hat{\mu}_t = \hat{\lambda}_t$$

Resource constraint:

$$\Rightarrow \hat{c}_t + \hat{x}_t = \hat{y}_t + \hat{b}_t - \hat{b}_{t+1}(e^{g_t})^{\frac{\alpha}{1-\alpha}}$$

Law of motion for capital:

$$\hat{k}_{t+1}(e^{g_t})^{\frac{\alpha}{1-\alpha}} = \hat{x}_t + (1 - \delta)\hat{k}_t e^{-g_t} - \frac{\phi}{2} \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} (e^{g_t})^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \hat{k}_t e^{-g_t}$$

Price of Debt:

$$\frac{1}{q_t} = 1 + r^* + \psi \left[ \exp \left( \frac{B_{t+1}}{Z_t} - b \right) - 1 \right]$$

Shock Processes:

$$\ln V_t = \ln V_{t-1} + g_{v,t}$$

$$g_{v,t} - \mu = \rho(g_{v,t-1} - \mu) + \varepsilon_{v,t}; \quad \varepsilon_t \sim N(0, \sigma)$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_{a,t}$$

### Steady-State Model

V = 1;

b = bstar

g = mu\_g

Euler equation for bond:

$$\hat{u}_{c,t} q_t = \beta \hat{u}_{c,t+1} (e^{g_t})^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}$$

$$\Rightarrow \hat{u}_c q = \beta \hat{u}_c (e^g)^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}$$

$$\Rightarrow q = \beta (e^g)^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}$$

FOC for Capital:

$$\begin{aligned} \Rightarrow \hat{\mu}_t ((e^{-g_t})^{\alpha/(1-\alpha)})^{\gamma(1-\sigma)-1} & \left[ \phi \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} (e^{g_t})^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \right] \\ & = \beta \left[ \alpha \hat{u}_{c,t+1} \frac{\hat{y}_{t+1}}{\hat{k}_{t+1}} + \hat{\mu}_{t+1} e^{-g_{t+1}} \left\{ (1 - \delta) - \frac{\phi}{2} \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \right. \right. \\ & \quad \left. \left. + \phi \left( \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} - \mu_g \right) \frac{\hat{k}_{t+2}}{\hat{k}_{t+1}} (e^{g_{t+1}})^{\frac{1}{1-\alpha}} \right\} \right] \end{aligned}$$



$$\begin{aligned}
&\Rightarrow \hat{\mu}((e^{-g})^{\alpha/1-\alpha})^{\gamma(1-\sigma)-1} \left[ \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \right] \\
&= \beta \left[ \alpha \hat{u}_c \frac{\hat{y}}{\hat{k}} \right. \\
&\quad \left. + \hat{\mu} e^{-g} \left\{ (1-\delta) - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 + \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) (e^g)^{\frac{1}{1-\alpha}} \right\} \right]
\end{aligned}$$

Since  $\hat{\lambda} = \hat{u}_c = \hat{\mu}$

$$\begin{aligned}
&\Rightarrow ((e^{-g})^{\alpha/1-\alpha})^{\gamma(1-\sigma)-1} \left[ \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \right] \\
&= \beta \left[ \alpha \frac{\hat{y}}{\hat{k}} \right. \\
&\quad \left. + e^{-g} \left\{ (1-\delta) - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 + \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) (e^g)^{\frac{1}{1-\alpha}} \right\} \right] \\
&\Rightarrow \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1 \\
&= \beta ((e^g)^{\alpha/1-\alpha})^{\gamma(1-\sigma)-1} \left[ \alpha \frac{\hat{y}}{\hat{k}} \right. \\
&\quad \left. + e^{-g} \left\{ (1-\delta) - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 + \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) (e^g)^{\frac{1}{1-\alpha}} \right\} \right] \\
&\Rightarrow \frac{\phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1}{q} \\
&= \alpha \frac{\hat{y}}{\hat{k}} + e^{-g} \left\{ (1-\delta) - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 + \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) (e^g)^{\frac{1}{1-\alpha}} \right\} \\
&\Rightarrow \frac{\hat{y}}{\hat{k}} \\
&= \frac{\frac{\phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1}{q} - e^{-g} \left\{ (1-\delta) - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 + \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) (e^g)^{\frac{1}{1-\alpha}} \right\}}{\alpha}
\end{aligned}$$

Law of motion for capital:

$$\begin{aligned}
\hat{k}_{t+1} (e^{gt})^{\frac{\alpha}{1-\alpha}} &= \hat{x}_t + (1-\delta) \hat{k}_t e^{-gt} - \frac{\phi}{2} \left( \frac{\hat{k}_{t+1}}{\hat{k}_t} (e^{gt})^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \hat{k}_t e^{-gt} \\
&\Rightarrow \hat{k} (e^g)^{\frac{\alpha}{1-\alpha}} = \hat{x} + (1-\delta) \hat{k} e^{-g} - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \hat{k} e^{-g} \\
&\Rightarrow \frac{\hat{x}}{\hat{y}} = \frac{\hat{k}}{\hat{y}} (e^g)^{\frac{\alpha}{1-\alpha}} - (1-\delta) \frac{\hat{k}}{\hat{y}} e^{-g} + \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 \frac{\hat{k}}{\hat{y}} e^{-g}
\end{aligned}$$

$$\Rightarrow \frac{\hat{x}}{\hat{y}} = \frac{(e^g)^{\frac{\alpha}{1-\alpha}} - (1-\delta)e^{-g} + \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 e^{-g}}{\frac{\hat{y}}{\hat{k}}}$$

Resource constraint:

$$\begin{aligned}\hat{c}_t + \hat{x}_t &= \hat{y}_t + \hat{b}_t - \hat{b}_{t+1}(e^{gt})^{\frac{\alpha}{1-\alpha}} \\ \Rightarrow \frac{\hat{c}}{\hat{y}} + \frac{\hat{x}}{\hat{y}} &= 1 + \frac{\hat{b}}{\hat{y}}(1 - (e^g)^{\frac{\alpha}{1-\alpha}}) \\ \Rightarrow \frac{\hat{c}}{\hat{y}} &= 1 + b\_share(1 - (e^g)^{\frac{\alpha}{1-\alpha}}) - \frac{\hat{x}}{\hat{y}}\end{aligned}$$

FOC for Labor:

$$\begin{aligned}\hat{u}_L + (1-\alpha)\hat{u}_c \frac{\hat{y}_t}{L_t} &= 0 \\ \Rightarrow -\frac{(1-\gamma)(1-\sigma)}{1-L_t} \hat{u}_t + (1-\alpha) \frac{\gamma(1-\sigma)}{\hat{c}_t} \hat{u}_t \frac{\hat{y}_t}{L_t} &= 0 \\ \Rightarrow \frac{(1-\gamma)(1-\sigma)}{1-L} \hat{u} &= (1-\alpha) \frac{\gamma(1-\sigma)}{\hat{c}} \hat{u} \frac{\hat{y}}{L} \\ \Rightarrow \frac{(1-\gamma)}{1-L} &= (1-\alpha) \frac{\gamma \hat{y}}{\hat{c} L} \\ \Rightarrow \frac{L}{1-L} &= \frac{\gamma(1-\alpha) \hat{y}}{(1-\gamma) \hat{c}} \\ \Rightarrow \frac{1-L}{L} &= \frac{(1-\gamma) \hat{c}}{\gamma(1-\alpha) \hat{y}} \\ \Rightarrow \frac{1}{L} - 1 &= \frac{(1-\gamma) \hat{c}}{\gamma(1-\alpha) \hat{y}} \\ \Rightarrow \frac{1}{L} &= 1 + \frac{(1-\gamma) \hat{c}}{\gamma(1-\alpha) \hat{y}} \\ \Rightarrow \frac{1}{L} &= \frac{\gamma(1-\alpha) + (1-\gamma) \frac{\hat{c}}{\hat{y}}}{\gamma(1-\alpha)} \\ \Rightarrow L &= \frac{\gamma(1-\alpha)}{\gamma(1-\alpha) + (1-\gamma) \frac{\hat{c}}{\hat{y}}}\end{aligned}$$

Production Function:

$$\hat{y}_t = A_t \hat{k}_t^\alpha L_t^{1-\alpha}$$

$$\Rightarrow \frac{\hat{y}}{\hat{k}} = \hat{k}^{\alpha-1} L^{1-\alpha}$$

$$\Rightarrow \hat{k} = \left( \frac{\hat{y}}{\hat{k}} \right)^{1/\alpha-1}$$

Output:  $\hat{y} = \hat{k}^\alpha L^{1-\alpha}$

Consumption:  $\hat{c} = \frac{\hat{c}}{\hat{y}} \hat{y}$

Investment:  $\hat{x} = \frac{\hat{x}}{\hat{y}} \hat{y}$

Utility:

$$\Rightarrow \hat{u}_t = \frac{[\hat{c}_t^\gamma (1 - L_t)^{1-\gamma}]^{1-\sigma}}{1 - \sigma}$$

Marginal utility of consumption:

$$\Rightarrow \hat{u}_c = \frac{\gamma(1 - \sigma)}{\hat{c}} \hat{u}$$

Marginal disutility of labor:

$$\Rightarrow \hat{u}_L = - \frac{(1 - \gamma)(1 - \sigma)}{1 - L} \hat{u}$$

FOC for Investment:

$$\hat{\mu} = \hat{u}_c$$

### Steady-state model summary

$b = b_{\text{star}}$

$g = \mu_g$

$$q = \beta(e^g)^{\frac{\alpha(\gamma(1-\sigma)-1)}{1-\alpha}}$$

$$\frac{\hat{y}}{\hat{k}}$$

$$= \frac{\frac{\phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) + 1}{q} - e^{-g} \left\{ (1 - \delta) - \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 + \phi \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right) (e^g)^{\frac{1}{1-\alpha}} \right\}}{\alpha}$$

$$\frac{\hat{x}}{\hat{y}} = \frac{(e^g)^{\frac{\alpha}{1-\alpha}} - (1-\delta)e^{-g} + \frac{\phi}{2} \left( (e^g)^{\frac{1}{1-\alpha}} - \mu_g \right)^2 e^{-g}}{\frac{\hat{y}}{\hat{k}}}$$

$$\frac{\hat{c}}{\hat{y}} = 1 + b\_share(1 - (e^g)^{\frac{\alpha}{1-\alpha}}) - \frac{\hat{x}}{\hat{y}}$$

$$L = \frac{\gamma(1-\alpha)}{\gamma(1-\alpha) + (1-\gamma)\frac{\hat{c}}{\hat{y}}}$$

$$\Rightarrow \hat{k} = \left( \frac{\frac{\hat{y}}{\hat{k}}}{L^{1-\alpha}} \right)^{1/\alpha-1}$$

$$\hat{y} = \hat{k}^\alpha L^{1-\alpha}$$

$$\hat{c} = \frac{\hat{c}}{\hat{y}} \hat{y}$$

$$\hat{x} = \frac{\hat{x}}{\hat{y}} \hat{y}$$

$$\hat{u}_t = \frac{[\hat{c}_t^\gamma (1-L_t)^{1-\gamma}]^{1-\sigma}}{1-\sigma}$$

$$\hat{u}_c = \frac{\gamma(1-\sigma)}{\hat{c}} \hat{u}$$

$$\hat{u}_L = -\frac{(1-\gamma)(1-\sigma)}{1-L} \hat{u}$$

$$\mu = \hat{u}_c$$