Optimization Problem:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\Pi_t^f\right)\right]$$

s.t.

$$\Pi_t^f = \Phi_t P_t Y_t - \Phi_t (1 + i_{l.t}) D_t$$

$$D_t = m_{t-1} D_{t-1} + P_t S_t - (1 - \Phi_{t-1}) P_{t-1} Y_{t-1}$$

$$Y_t = S_{t-1}^{\alpha}$$

solving by the first method:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\underbrace{\Phi_t P_t S_{t-1}^{\alpha} - \Phi_t (1 + i_{l.t}) D_t}_{\Pi_t^f} + \mu_t [D_t - m_{t-1} D_{t-1} - P_t S_t + (1 - \Phi_{t-1}) P_{t-1} Y_{t-1}]\right]$$

$$\frac{\partial \mathcal{L}}{\partial s_t (\text{not } s_{t-1})} = \beta^{t+1} \alpha \Phi_{t+1} P_{t+1} S_t^{\alpha-1} - \beta^t \mu_t P_t = 0$$

solving by the second method:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\Phi_t P_t S_{t-1}^{\alpha} - \Phi_t (1 + i_{l.t}) D_t + \mu_t \left[D_t - m_{t-1} D_{t-1} - P_t S_t + (1 - \Phi_{t-1}) P_{t-1} \underbrace{S_{t-2}^{\alpha}}_{Y_{t-1}}\right]\right]$$

$$\frac{\partial \mathcal{L}}{\partial S_t} = \beta^{t+1} \alpha \Phi_{t+1} P_{t+1} S_t^{\alpha-1} - \beta^t \mu_t P_t + \beta^{t+2} \mu_{t+2} \alpha (1 - \Phi_{t+1}) P_{t+1} S_t^{\alpha-1} = 0$$

solving by the third method:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\Phi_t P_t S_{t-1}^{\alpha} - \Phi_t (1 + i_{l.t}) D_t + \Phi_t \mu_t \left[D_t - m_{t-1} D_{t-1} - P_t S_t + (1 - \Phi_{t-1}) P_{t-1} \underbrace{S_{t-2}^{\alpha}}_{Y_{t-1}}\right]\right]$$

$$\frac{\partial \mathcal{L}}{\partial S_t} = \beta^{t+1} \alpha \Phi_{t+1} P_{t+1} S_t^{\alpha-1} - \Phi_t \beta^t \mu_t P_t + \Phi_{t+2} \beta^{t+2} \mu_{t+2} \alpha (1 - \Phi_{t+1}) P_{t+1} S_t^{\alpha-1} = 0$$