FOCs and Definitions Baseline Model

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1 Household

The household problem is standard and gives us the expected intertemporal and intratemporal optimality conditions, with one Euler equation per asset type.

Euler bonds:

$$\frac{1}{C_t} = \beta\,\mathbb{E}_t(\frac{R_t}{C_{t+1}}) \tag{1}$$

Labour Supply:

$$L_{j,t}^{\eta_j} C_t = \frac{w_{j,t}}{\varphi_j} \tag{2}$$

Euler Deposits:

$$\frac{1}{C_t} = \beta\,\mathbb{E}_t\,\big[\frac{R_{d,t} - (1-\kappa)\Omega_{t+1}}{C_{t+1}}\big] \tag{3}$$

Household budget constraint (note bonds are in zero net supply):

$$C_t + D_t = w_{f,t}L_{f,t} + \tilde{R}_{d,t}D_{t-1} - T_t + \Pi_t + \Xi_t \tag{4}$$

where profits of entrepreneurs and bankers are given by:

$$\Pi_t = (1-\chi_i)(1-\theta_i)\rho_{i,t}n_{i,t-1} \tag{5}$$

$$\Xi_t = (1-\chi_e)(1-\theta_e)\rho_{f,t}n_{e,t-1} \tag{6}$$

and taxes are defined at the end of this document. Note that Ω is the average unit loss on deposits, given by:

$$\Omega_t = \frac{[\bar{\omega}_{i,t} - \Gamma_i(\bar{\omega}_{i,t}) + \mu_i H_i(\bar{\omega}_{i,t})]\tilde{R}_{f,t}}{(1-\phi_f)} \tag{7}$$

2 Entrepreneurs

Guessing that the entrepreneur's value function is linear in net worth, optimality implies that we get the following condition for entrepreneurial equity:

$$\mathbb{E}_t[\lambda_{e,t+1}\rho_{f,t+1}] = v_{e,t} \tag{8}$$

Eventually, we also get the law of motion of aggregate entrepreneurial net worth:

$$n_{e,t+1} = \big[\theta_e + \chi_e(1-\theta_e)\big](\rho_{f,t+1}A_{f,t}) \tag{9}$$

3 Entrepreneurial Firms

To write the problem more compactly, define some auxiliary variables. First, let the return on capital be given by:

$$R^k_{t+1} = \frac{r_{f,t+1} + (1-\delta)Q_{f,t+1}}{Q_{f,t}} \tag{10}$$

and define the leverage of the firm as:

$$lev_{f,t} = \frac{R_{f,t}B_{f,t}}{Q_{f,t}K_{f,t}} \tag{11}$$

which then implies that $\bar{\omega}_{f,t+1} = \frac{lev_{f,t}}{R^k_{t+1}}$. Use the budget constraint to replace for $A_{f,t} = Q_{f,t}K_{f,t} - B_{f,t}$ and replace for loans using the leverage formula, to get a problem in three variables: leverage, capital and the loan rate for entrepreneurial firms of type f. Finally, we need to add the constraint from the bankers, which holds with equality and to which we attach a multiplier called o.

The problem is then given by:

$$\begin{aligned}\max_{R_{f,t}, lev_{f,t}, K_{f,t}} \mathbb{E}_t \left[\lambda_{e,t+1}\big(1 - \Gamma_f(\frac{lev_{f,t}}{R^K_{t+1}})\big)R^k_{t+1}\right]Q_{f,t}K_{f,t} - v_{e,t}[Q_{f,t}K_{f,t} - \frac{lev_{f,t}Q_{f,t}K_{f,t}}{R_{f,t}}] \\ +o_t\left\{E_t\left[\lambda_{i,t+1}\big(1-\Gamma_i(\bar{\omega}_{i,t+1})\big)\big(\Gamma_f(\frac{lev_{f,t}}{R^K_{t+1}}) - \mu_f H_f(\frac{lev_{f,t}}{R^K_{t+1}})\big)R^k_{t+1}\right]Q_{f,t}K_{f,t} - v_{it}\phi_{f,t}\frac{lev_{f,t}Q_{f,t}K_{f,t}}{R_{f,t}}]\right\}\end{aligned} \tag{12}$$

Solving this yields three first-order conditions. For leverage, we obtain:

$$\mathbb{E}_t\left[\lambda_{e,t+1}\Gamma'_f(\bar{\omega}_{t+1})\right] = \frac{v_{e,t}}{R_{f,t}} + o_t\left\{E_t\left[\lambda_{i,t+1}\big(1-\Gamma_i(\bar{\omega}_{i,t+1})\big)\big(\Gamma'_f(\bar{\omega}_{f,t+1}) - \mu_f H'_f(\bar{\omega}_{f,t+1})\big)\right] - \frac{v_{i,t}\phi_{f,t}}{R_{f,t}}\right\} \tag{13}$$

For the interest rate on loans, the FOC can be written as:

$$v_{e,t} = o_t[v_{i,t}\phi_{f,t}] \tag{14}$$

and for capital, we get an FOC that can be written as:

$$\begin{aligned}\mathbb{E}_t\left[\lambda_{e,t+1}\big(1-\Gamma_f(\bar{\omega}_{f,t+1})\big)R^k_{t+1}\right] - v_{e,t}[1 - \frac{lev_{f,t}}{R_{f,t}}] = o_t\Big\{\left[v_{it}\phi_{f,t}\frac{lev_{f,t}}{R_{f,t}}\right] \\ -E_t\left[\lambda_{i,t+1}\big(1-\Gamma_i(\bar{\omega}_{i,t+1})\big)\big(\Gamma_f(\bar{\omega}_{f,t+1}) - \mu_f H_f(\bar{\omega}_{f,t+1})\big)R^k_{f,t+1}\right]\Big\}\end{aligned} \tag{15}$$

4 Bankers

Banker's problem is similar to that of entrepreneurs but even simpler, so we just get one condition for bank equity:

$$v_{i,t} = \mathbb{E}_t[\lambda_{i,t+1}\rho_{i,t+1}] \tag{16}$$

and finally, we can also write the law of motion for banker's net worth:

$$n_{i,t+1} = [\theta_i + \chi_i(1-\theta_i)](\rho_{i,t+1}E_t) \tag{17}$$

5 Banks

The fact that deposit is cheaper than equity finance, together with the capital requirements constraint and the budget constraint, directly yield us equity and deposits as:

$$E_t = \phi_{f,t} B_{f,t} \tag{18}$$
$$D_t = (1 - \phi_{f,t}) B_{f,t} \tag{19}$$

6 Final Goods Production

For this simpler model, this is a standard Cobb-Douglas problem under perfect competition. The only difference is the timing of capital, and the two FOCs are:

$$r_{f,t} = \frac{\alpha_f y_t}{K_{f,t-1}} \tag{20}$$

$$w_{f,t} = \frac{(1 - \alpha_f) y_t}{L_{f,t}} \tag{21}$$

Finally, note that the law of motion for technology is given by:

$$ln\Upsilon_{j,t+1} = ln\bar{\Upsilon} + \varrho_j (ln\Upsilon_{j,t} - ln\bar{\Upsilon}) + u_{j,t+1} \tag{22}$$

7 Capital Producers

The problem of capital producers is a standard unconstrained optimisation under adjustment costs. Taking the FOC with respect to investment, we get an optimality condition for the price of capital:

$$Q_{f,t} = \frac{1}{S'(\frac{I_{f,t}}{K_{f,t-1}})} \tag{23}$$

noting that $S'_{j,t}(\frac{I_{j,t}}{K_{j,t-1}}) = a_{f,1}[\frac{I_{j,t}}{K_{j,t-1}}]^{\frac{-1}{\psi_j}}$. The law of motion for capital is given by:

$$K_{j,t+1} = (1 - \delta_j) K_{j,t} + S_{j,t+1}(\frac{I_{j,t+1}}{K_{j,t}}) K_{j,t} \tag{24}$$

8 Market Clearing

Market clearing requires that markets for labour, capital, the final good, entrepreneurial and bank equity, loans and deposits clear. The first and latter two have already been imposed on the model. The consumption goods market condition is implied by using all other market clearing conditions (Walras Law). The other two conditions are added and finally, I define the lump sum taxes needed to finance the deposit insurance.

8.1 Consumption good market

$$Y_t = C_t + I_{f,t} + \mu_f H_f(\bar{\omega}_{f,t}) r_{f,t} Q_{f,t} K_{f,t-1} + \mu_i H_i(\bar{\omega}_{i,t})(\tilde{R}_{f,t} B_{f,t-1}) \tag{25}$$

8.2 Entrepreneurial equity market

$$n_{e,t} = A_{f,t} \tag{26}$$

8.3 Bank equity market

$$n_{i,t} = E_t \tag{27}$$

8.4 DIS and lump-sum taxes

$$T_t = \kappa \Omega_t D_{t-1} \tag{28}$$

9 Definition of Variables and Parameters

- A_j: Entrepreneurial Equity
- B_j: Loans to Entrepreneurial Firms
- C: Consumption
- Γ_f, Γ_i: Gross Asset Share going to a lender
- D: Deposits
- E: Bank Equity
- F_f, F_i: CDF Idiosyncratic Shocks
- H_f, H_i: Share of assets owned by borrower that ends up in default
- I_f: Investment
- K_f: Capital Stock
- L_f: Labour
- n_e: Entrepreneurial Net Worth
- Ω: Average per unit loss to the fraction of uninsured deposits
- Q_f: Capital Price
- R: Bond Return
- R_d: Deposit Interest Rate
- r_f: Rental Rate Capital
- ρ_f Return on entrepreneurial equity
- ρ_i Return on bank equity
- $\tilde{R}_f$: Gross Realised Return on Loans
- S_f: Adjustment Costs
- T: Lump-sum Taxes
- u_f: Technology shock
- v_x: Shadow Value of one unit of net worth of entrepreneurs (e) or bankers (i)
- w_f: Wage
- Y: Output
- Υ_f: Technology
- Ξ: Aggregate transfers from entrepreneurs to the household dynasty
- Π: Aggregate transfers from bankers to the household dynasty

- $a_{f,1}, a_{f,2}, \psi_f$: Adjustment Cost Parameters
- α_f: Capital Share

- β: HH Discount Factor
- δ_f: Depreciation Rate
- η_f: Inverse Frisch Elasticity
- θ_x: Share of continuing Entrepreneurs (e) and Bankers (i) each period
- κ: Share of insured deposits
- λ_{t+1}: SDF Household
- $\lambda_{x,t+1}$: SDF of Entrepreneurs (e) and Bankers (i)
- μ_f: Asset repossession cost parameter
- ϱ_f: AR coefficient technology
- σ_m Standard deviation of the idiosyncratic return shock to banks (i) or firms (j)
- $\sigma_{a,f}$: standard deviation of the technology shock to a firm of type j
- ϕ_f: Capital Requirement
- φ_f: Labour disutility
- χ_x: Share of net worth of retiring entrepreneurs (e) or bankers (i) given to new agents
- ω_f, ω_i: Idiosyncratic shocks to entrepreneurial firms or banks (i)