Technical Appendix for "Financial Crises, Unconventional Monetary Policy Exit Strategies, and Agents' Expectations"

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1 Model Setup

This section describes the basic model. At this point in time, the parameters that will change with regime switching are described simply as time-varying parameters. The next section will describe regime-switching in more detail.

1.1 Households

Households are made up of a fraction $(1-f)$ of workers, and a fraction f of bankers. Bankers become workers with probability $1-\theta$, so a total fraction of $(1-\theta)f$ transition to become workers; the same fraction transition from being workers to being bankers. Upon exit, bankers transfer their net worth to the household, at startup, bankers receive some initial funds from the household. Within the household, there is perfect consumption insurance.

Households consume, supply labor, and save by purchasing bonds/deposits. The HHs maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(C_t - hC_{t-1}\right) - \frac{\varkappa}{1+\varphi} L_t^{1+\varphi} \right\}$$

subject to

$$C_t + B_t + \frac{B_t^n}{P_t} = W_t L_t + \Pi_t + T_t + R_{t-1}B_{t-1} + r_{t-1}\frac{B_{t-1}^n}{P_t}$$

Attach a multiplier ϱ_t on the constraint. The FOCs are

$$C_t : \left(C_t - hC_{t-1}\right)^{-1} - \varrho_t - \beta h \mathbb{E}_t \left(C_{t+1} - hC_t\right)^{-1} = 0$$

$$B_t : -\varrho_t + \beta R_t \mathbb{E}_t \varrho_{t+1} = 0$$

$$B_t^n : -\frac{\varrho_t}{P_t} + \beta r_t \mathbb{E}_t \frac{\varrho_{t+1}}{P_{t+1}} = 0$$

$$L_t : -\varkappa L_t^{\varphi} + \varrho_t W_t = 0$$

The optimality conditions are

$$\left(C_t - hC_{t-1}\right)^{-1} - \beta h \mathbb{E}_t \left(C_{t+1} - hC_t\right)^{-1} = \varrho_t$$

$$\beta R_t \mathbb{E}_t \frac{\varrho_{t+1}}{\varrho_t} = 1$$

$$\beta \frac{r_t}{\varrho_t} \mathbb{E}_t \frac{\varrho_{t+1}}{\Pi_{t+1}} = 1$$

$$\varkappa L_t^{\varphi} = \varrho_t W_t$$

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1.2 Financial Intermediaries

Financial intermediaries, which are indexed by j, accumulate net worth $N_{j,t}$, collect deposits from households $B_{j,t}$, and loan funds to non-financial firms by acquiring claims $S_{j,t}$ at price Q_t. The intermediaries' balance sheet dictates that the overall value of claims on non-financial firms must equal the value of the intermediaries net worth plus deposits.

$$Q_t S_{j,t} = N_{j,t} + B_{j,t}$$

The claims on non-financial firms pay out a stochastic return of $R_{k,t+1}$, and the real interest rate on deposits is the nonstochastic value R_t. Net worth in next period is the difference in interest received from non-financial firms and interest payed out to depositors:

$$\begin{aligned} N_{j,t+1} &= R_{k,t+1} Q_t S_{j,t} - R_t B_{j,t} \\ &= \left(R_{k,t+1} - R_t\right) Q_t S_{j,t} + R_t N_{j,t} \end{aligned}$$

The participation constraint requires that the expected discounted interest rate differential must be in the banker's favor.

$$\mathbb{E}_t \beta^{i+1} \frac{\varrho_{t+i+1}}{\varrho_t} \left(R_{k,t+1+i} - R_{t+i}\right) \geq 0, \text{ for } i \geq 0.$$

Note here the inequality is a key differential with financial frictions. In a standard economy without constrained financial intermediaries this participation constraint exactly binds by no arbitrage.

Each period, bankers exit and become standard workers with probability $(1-\theta)$. The banker's objective function is to maximize the present value of their net worth at exit.

$$\begin{aligned} V_{j,t} &= \mathbb{E}_t \left(1-\theta\right) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} N_{j,t+1+i} \\ &= \mathbb{E}_t \left(1-\theta\right) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} \left(\left(R_{k,t+1+i} - R_{t+i}\right) Q_{t+i} S_{j,t+i} + R_{t+i} N_{j,t+i}\right) \end{aligned}$$

In each period, the banker can divert a fraction λ of the assets back to the household. If he does choose to divert, depositors are able to recover the remaining fraction $(1-\lambda)$ of assets. Consequently, the incentive constraint for the banker requires that the value of not stealing and remaining until exit exceeds the value of stolen funds in each period.

$$V_{j,t} \geq \lambda Q_t S_{j,t}$$

To express $V_{j,t}$ in recursive form, first define the gross growth rate in assets as

$$m_t = \frac{Q_t}{Q_{t-1}} \frac{S_{j,t}}{S_{j,t-1}}$$

and the gross growth rate in net worth by

$$z_t = \frac{N_{j,t}}{N_{j,t-1}}.$$

The above summation for $V_{j,t}$ can be expressed as two separate parts.

$$\begin{aligned} V_{j,t} &= \mathbb{E}_t \left(1-\theta\right) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} \left(\left(R_{k,t+1+i} - R_{t+i}\right) Q_{t+i} S_{j,t+i} + R_{t+i} N_{j,t+i}\right) \\ &= \mathbb{E}_t \left(1-\theta\right) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} \left(R_{k,t+1+i} - R_{t+i}\right) \frac{Q_{t+i} S_{j,t+i}}{Q_t S_{j,t}} Q_t S_{j,t} \\ &\quad + \mathbb{E}_t \left(1-\theta\right) \beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} R_{t+1+i} \frac{N_{j,t+i}}{N_{j,t}} N_{j,t} \\ &= v_t Q_t S_{j,t} + \eta_t N_{j,t} \end{aligned}$$

where

$$\begin{aligned} v_t &= \mathbb{E}_t (1-\theta)\beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} \left(R_{k,t+1+i} - R_{t+i}\right) \frac{Q_{t+i} S_{j,t+i}}{Q_t S_{j,t}} \\ \eta_t &= \mathbb{E}_t (1-\theta)\beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1}}{\varrho_t} R_{t+i} \frac{N_{j,t+i}}{N_{j,t}} \end{aligned}$$

Putting these into recursive form

$$\begin{aligned} v_t &= \mathbb{E}_t \begin{bmatrix} (1-\theta)\beta \frac{\varrho_{t+1}}{\varrho_t} \left(R_{k,t+1} - R_t\right) \\ +\beta\theta \frac{\varrho_{t+1}}{\varrho_t} \frac{Q_{t+1}S_{j,t+1}}{Q_t S_{j,t}} (1-\theta)\beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1+1}}{\varrho_{t+1}} \left(R_{k,t+1+1+i} - R_{t+1+i}\right) \frac{Q_{t+1+i}S_{j,t+1+i}}{Q_{t+1}S_{j,t+1}} \end{bmatrix} \\ &= \mathbb{E}_t \left[(1-\theta)\beta \frac{\varrho_{t+1}}{\varrho_t} \left(R_{k,t+1} - R_t\right) + \beta\theta \frac{\varrho_{t+1}}{\varrho_t} m_{t+1} v_{t+1} \right] \end{aligned}$$

and

$$\begin{aligned} \eta_t &= \mathbb{E}_t \begin{bmatrix} (1-\theta)\beta \frac{\varrho_{t+1}}{\varrho_t} R_t \\ +\theta\beta \frac{N_{j,t+1}}{N_{j,t}} \frac{\varrho_{t+1}}{\varrho_t} (1-\theta)\beta \sum_{i=0}^{\infty} \beta^i \theta^i \frac{\varrho_{t+i+1+1}}{\varrho_{t+1}} R_{t+1+i} \frac{N_{j,t+1+i}}{N_{j,t+1}} \end{bmatrix} \\ &= \mathbb{E}_t \left[(1-\theta)\beta \frac{\varrho_{t+1}}{\varrho_t} R_t + \theta\beta \frac{\varrho_{t+1}}{\varrho_t} z_{t+1} \eta_{t+1} \right] \end{aligned}$$

so the complete expression is

$$V_{j,t} = \nu_t Q_t S_{j,t} + \eta_t N_{j,t}$$

where

$$\nu_t = \mathbb{E}_t \left[(1-\theta)\beta \frac{\varrho_{t+1}}{\varrho_t} \left(R_{k,t+1} - R_t\right) + \beta\theta \frac{\varrho_{t+1}}{\varrho_t} m_{t+1} v_{t+1} \right]$$

$$\eta_t = \mathbb{E}_t \left[(1-\theta)\beta \frac{\varrho_{t+1}}{\varrho_t} R_t + \theta\beta \frac{\varrho_{t+1}}{\varrho_t} z_{t+1} \eta_{t+1} \right]$$

The term v_t denotes the expected discounted marginal benefit from increasing their asset value $Q_t S_{jt}$. The term η_t is the expected discounted value of increasing their net worth N_{jt}.

Using the new expression, the incentive constraint is

$$\eta_t N_{j,t} + \nu_t Q_t S_{j,t} \geq \lambda Q_t S_{j,t}$$

If the incentive constraint binds, then the claims are expressed as a leverage ratio ϕ_t over net worth

$$Q_t S_{j,t} = \phi_t N_{j,t}$$

where

$$\phi_t = \frac{\eta_t}{\lambda - \nu_t}$$

This incentive constraint binds if and only if $v_t \in (0, \lambda)$. That is, if $v_t > 0$, under which case there is benefit to the banker from increasing his asset holdings, and if $v_t < \lambda$, in which case the marginal benefit from stealing a fraction λ of new assets is greater than the benefit of simply increasing the asset holdings.

Returning to the expression of banker's net worth

$$N_{j,t} = \left[\left(R_{k,t} - R_{t-1}\right) \phi_{t-1} + R_{t-1} \right] N_{j,t-1}$$

this can be used to express the growth in the net worth of the bankers

$$z_t = \frac{N_{j,t}}{N_{j,t-1}} = \left(R_{k,t} - R_{t-1}\right) \phi_{t-1} + R_{t-1}$$

Similarly, the leverage ratio determines an expression for the gross growth rate of assets

$$m_t = \frac{Q_t}{Q_{t-1}} \frac{S_{j,t}}{S_{j,t-1}} = \frac{\phi_t}{\phi_{t-1}} \frac{N_{j,t}}{N_{j,t-1}} = \frac{\phi_t}{\phi_{t-1}} z_t$$

Since the price Q_t and the leverage ratio ϕ_t then are independent of banker-specific characteristics, total intermediary demand is a result of summing over all independent intermediaries j:

$$Q_t S_{p,t} = \phi_t N_t.$$

Now consider the fact that net assets are allocated between new and existing bankers:

$$N_t = N_{e,t} + N_{n,t}$$

Since bankers exit with probability $(1-\theta)$, existing banker net worth makes up a fraction θ of last periods net worth

$$N_{e,t} = \theta \left[(R_{k,t} - R_{t-1}) \phi_{t-1} + R_{t-1} \right] N_{t-1}$$

New bankers receive start-up funds from the household, specifically, they get a fraction $\frac{\omega}{1-\theta}$ of the net worth of exiting bankers $(1-\theta) Q_t S_{p,t-1}$

$$N_{n,t} = \frac{\omega}{1-\theta} (1-\theta) Q_t S_{p,t-1} = \omega Q_t S_{p,t-1}$$

Combining the above terms, the law of motion for net worth is given by

$$N_t = \theta \left[(R_{k,t} - R_{t-1}) \phi_{t-1} + R_{t-1} \right] N_{t-1} + \omega Q_t S_{p,t-1}$$

The credit spread is defined as

$$R_{diff,t} = \mathbb{E}_t R_{k,t+1} - R_t$$

1.3 Credit Policy

Total assets are a combination of private and government assets

$$Q_t S_t = Q_t S_{p,t} + Q_t S_{g,t}$$

The government is willing to supply a fraction ψ_t of intermediated assets

$$Q_t S_{g,t} = \psi_t Q_t S_t$$

Using the private leverage ratio

$$Q_t S_t = \phi_t N_t + \psi_t Q_t S_t$$

so the total leverage ratio satisfies

$$Q_t S_t = \phi_{c,t} N_t$$

where

$$\phi_{c,t} = \frac{\phi_t}{1-\psi_t}.$$

Note that this implies that the net worth of new bankers is given by

$$N_{n,t} = \omega Q_t S_{p,t-1} = \omega \left(1 - \psi_{t-1}\right) Q_t S_{t-1}.$$

1.4 Intermediate Goods Firms

Intermediate goods firms rent capital and labor and produce a product $Y_{m,t}$ and sell it at price $P_{m,t}$. They purchase capital by issuing claims against it to financial intermediaries. So the value of assets is

$$Q_t K_{t-1} = Q_t S_t$$

However, note that Production is Cobb-Douglas and depends upon a utilization rate U_t and the quality of capital ξ_t.

$$Y_{m,t} = \left(U_t \xi_t K_{t-1}\right)^{\alpha} L_t^{1-\alpha}$$

Given the level of capital stock K_{t-1}, the firm chooses a capital utilization rate U_t and a labor rate L_t, produces $Y_{m,t}$, sells its product at price $P_{m,t}$, makes wage payments at wage rate W_t, pays to repair depreciated capital at a price of unity, and sells it's capital on the market at a price of Q_t. So it's profits at time t are given by

$$\begin{aligned} profits_t &= P_{m,t} Y_{m,t} - W_t L_t + Q_t \xi_t K_{t-1} - \delta\left(U_t\right) \xi_t K_{t-1} \\ &= P_{m,t} Y_{m,t} - W_t L_t + \left(Q_t - \delta\left(U_t\right)\right) \xi_t K_{t-1} \end{aligned}$$

The FOCs for maximizing profits are given by

$$\begin{aligned} L_t &: P_{m,t}\left(1-\alpha\right)\left(U_t \xi_t K_{t-1}\right)^{\alpha} L_t^{-\alpha} - W_t = 0 \\ U_t &: P_{m,t}\alpha\left(U_t \xi_t K_{t-1}\right)^{\alpha-1} L_t^{1-\alpha} \xi_t K_{t-1} - \delta'\left(U_t\right) \xi_t K_{t-1} = 0 \end{aligned}$$

And so the optimality conditions are

$$P_{m,t}\left(1-\alpha\right)\frac{Y_{m,t}}{L_t} = W_t$$

$$P_{m,t}\alpha\frac{Y_{m,t}}{U_t} = \delta'\left(U_t\right) \xi_t K_{t-1}$$

The firm earns zero profits state-by-state, it pays the realized return on capital to the intermediary, which is given by

$$R_{k,t} = \frac{\left[P_{m,t}\alpha\frac{Y_{m,t}}{\xi_t K_{t-1}} + Q_t - \delta\left(U_t\right)\right]\xi_t}{Q_{t-1}}$$

1.5 Capital Producing Firms

Capital producers are competitive firms that buy used capital from intermediate goods firms, repair depreciated capital, build new capital, and sell it to the intermediate goods firms. They face adjustment costs on construction of new capital, but not on refurbishing capital. The cost of repairing depreciated capital is unity. So profits are given by

$$\begin{aligned} profits_t &= \delta\left(U_t\right)\xi_t K_{t-1} - Q_t \xi_t K_{t-1} + Q_t K_t - \left(K_t - \left(1-\delta\left(U_t\right)\right)\xi_t K_{t-1}\right) \\ &\quad - f\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right)\left(I_{n,t}+I_{ss}\right) \\ &= \left(Q_t - 1\right) I_{n,t} - f\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right)\left(I_{n,t}+I_{ss}\right) \end{aligned}$$

where $f\left(1\right) = f'\left(1\right) = 0$, $f''\left(1\right) > 0$.

The present value of profits for a capital producer are

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{\varrho_t}{\varrho_0}\left\{\left(Q_t - 1\right) I_{n,t} - f\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right)\left(I_{n,t}+I_{ss}\right)\right\}$$

where

$$I_t = K_t - \left(1-\delta\left(U_t\right)\right)\xi_t K_{t-1}$$

and

$$I_{n,t} = I_t - \delta(U_t)\,\xi_t K_{t-1}$$

The FOC is

$$I_{n,t} : \left\{ \begin{array}{c} (Q_t - 1) - f\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right) - f'\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right)\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right) \\ +\mathbb{E}_t\beta\frac{\varrho_{t+1}}{\varrho_t} f'\left(\frac{I_{n,t+1}+I_{ss}}{I_{n,t}+I_{ss}}\right)\left(\frac{I_{n,t+1}+I_{ss}}{I_{n,t}+I_{ss}}\right)^2 \end{array} \right\} = 0$$

So the net investment Q relation is given by

$$Q_t = 1 + f\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right) + f'\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right)\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right) - \mathbb{E}_t\beta\frac{\varrho_{t+1}}{\varrho_t} f'\left(\frac{I_{n,t+1}+I_{ss}}{I_{n,t}+I_{ss}}\right)\left(\frac{I_{n,t+1}+I_{ss}}{I_{n,t}+I_{ss}}\right)^2$$

The depreciation rate is assumed to follow the functional form

$$\delta(U_t) = \bar{\delta} - \frac{\tilde{\delta}}{1+\zeta} + \frac{\tilde{\delta}}{1+\zeta}U_t^{1+\zeta}$$

where $\tilde{\delta}$ is determined by the steady state.

In addition the functional form for f is given by

$$f\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}}\right) = \frac{\eta_i}{2}\left(\frac{I_{n,t}+I_{ss}}{I_{n,t-1}+I_{ss}} - 1\right)^2$$

1.6 Retail Firms

Final output is a CES aggregate of retail firm goods, so

$$Y_t = \left(\int_0^1 Y_{f,t}^{\frac{\varepsilon-1}{\varepsilon}} df\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Consumers of the final good use cost minimization, so standard optimality conditions imply

$$Y_{f,t} = \left(\frac{P_{f,t}}{P_t}\right)^{-\varepsilon} Y_t$$

and

$$P_t^{1-\varepsilon} = \int_0^1 P_{f,t}^{1-\varepsilon} df$$

Retail firms repackage output from intermediate firms and add a markup, with Calvo pricing where they can re-optimize their prices with probability $(1-\gamma)$. Those firms that do not re-optimize prices re-index prices with respect to inflation and the parameter μ. A firm that can choose its price at time t maximizes the present value of profits according to

$$\max_{P_{f,t}} \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t}\left(\prod_{k=1}^{i} \Pi_{t+k-1}^{\mu}\frac{P_{f,t}}{P_{t+i}} - P_{m,t+i}\right) Y_{f,t+i}$$

subject to

$$Y_{f,t+i} = \left(\prod_{k=1}^{i} \Pi_{t+k-1}^{\mu}\frac{P_{f,t}}{P_{t+i}}\right)^{-\varepsilon} Y_{t+i}$$

Substituting the constraint in gives

$$\max_{P_{f,t}} \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t}\left(\prod_{k=1}^{i} \Pi_{t+k-1}^{\mu}\frac{P_{f,t}}{P_{t+i}} - P_{m,t+i}\right)\left(\prod_{k=1}^{i} \Pi_{t+k-1}^{\mu}\frac{P_{f,t}}{P_{t+i}}\right)^{-\varepsilon} Y_{t+i}$$

and simplifying produces

$$\max_{P_{f,t}} \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t} \left(\left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \frac{P_{f,t}}{P_t} \right)^{1-\varepsilon} - \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \frac{P_{f,t}}{P_t} \right)^{-\varepsilon} P^m_{t+i} \right) Y_{t+i}$$

The FOC produces the optimal price setting level $P^*_{f,t}$

$$P_{f,t} : \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t} \left((1-\varepsilon) \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \frac{P^*_{f,t}}{P_t} \right)^{1-\varepsilon} \frac{1}{P^*_{f,t}} + \varepsilon \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \frac{P^*_{f,t}}{P_t} \right)^{-\varepsilon} \frac{1}{P^*_{f,t}} P_{m,t+i} \right) Y_{t+i} = 0$$

and simplifying

$$\sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t} \left((1-\varepsilon) \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \right)^{1-\varepsilon} \left(\frac{P^*_{f,t}}{P_t} \right) + \varepsilon \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \right)^{-\varepsilon} P_{m,t+i} \right) Y_{t+i} = 0$$

and now using the symmetric equilibrium $P^*_{f,t} = P^*_t$

$$\sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t} \left((\varepsilon - 1) \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \right)^{1-\varepsilon} \frac{P^*_t}{P_t} - \varepsilon \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \right)^{-\varepsilon} P_{m,t+i} \right) Y_{t+i} = 0$$

Splitting this term produces

$$(\varepsilon - 1)\, a_{1,t} = \varepsilon a_{2,t}$$

$$a_{1,t} = \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t} \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \right)^{1-\varepsilon} \frac{P^*_t}{P_t} Y_{t+i}$$

$$a_{2,t} = \sum_{i=0}^{\infty} \gamma^i \beta^i \frac{\varrho_{t+i}}{\varrho_t} \left(\prod_{k=1}^{i} \frac{\Pi^{\mu}_{t+k-1}}{\Pi_{t+k}} \right)^{-\varepsilon} P_{m,t+i} Y_{t+i}$$

Writing these in recursive form

$$(\varepsilon - 1)\, a_{1,t} = \varepsilon a_{2,t}$$

$$a_{1,t} = \tilde{P}^*_t Y_t + \gamma\beta \frac{\tilde{P}^*_t}{\tilde{P}^*_{t+1}} \left(\frac{\Pi^{\mu}_t}{\Pi_{t+1}} \right)^{1-\varepsilon} \frac{\varrho_{t+1}}{\varrho_t} a_{1,t+1}$$

$$a_{2,t} = P_{m,t} Y_t + \gamma\beta \left(\frac{\Pi^{\mu}_t}{\Pi_{t+1}} \right)^{-\varepsilon} \frac{\varrho_{t+1}}{\varrho_t} a_{2,t+1}$$

Given indexation, the evolution of the price level satisfies

$$P_t^{1-\varepsilon} = (1-\gamma) \left(P^*_t \right)^{1-\varepsilon} + \gamma \left(\Pi^{\mu}_{t-1} P_{t-1} \right)^{1-\varepsilon}$$

and putting the price in relative terms

$$1 = (1-\gamma)\, \tilde{P}^{*1-\varepsilon}_t + \gamma \left(\frac{\Pi^{\mu}_{t-1}}{\Pi_t} \right)^{1-\varepsilon}$$

Integrating the above over f, and using the fact that $(1-\gamma)$ firms reoptimize

$$\int_0^1 Y_{f,t} df = \int_0^1 \left(\frac{P_{f,t}}{P_t} \right)^{-\varepsilon} df Y_t = \varsigma_t Y_t$$

so

$$
\begin{aligned}
\varsigma_t &= \int_0^1 \left(\frac{P_{f,t}}{P_t}\right)^{-\varepsilon} df \\
&= (1-\gamma)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} + \int_\gamma \left(\frac{\Pi_{t-1}^{\mu} P_{f,t-1}}{P_t}\right)^{-\varepsilon} df \\
&= (1-\gamma)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} + \int_\gamma \left(\Pi_{t-1}^{\mu} \frac{P_{f,t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t}\right)^{-\varepsilon} df \\
&= (1-\gamma)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} + \gamma \left(\Pi_{t-1}^{\mu} \frac{P_{t-1}}{P_t}\right)^{-\varepsilon} \varsigma_{t-1}
\end{aligned}
$$

and hence

$$
\varsigma_t = (1-\gamma)\left(\tilde{P}_t^*\right)^{-\varepsilon} + \gamma \left(\frac{\Pi_{t-1}^{\mu}}{\Pi_t}\right)^{-\varepsilon} \varsigma_{t-1}
$$

Since $Y_{f,t} = Y_{m,t}$, then

$$
Y_{m,t} = \varsigma_t Y_t
$$

1.7 Resource Constraint

The economy wide resource constraint is

$$
Y_t = C_t + I_t + f\left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}}\right)(I_{n,t} + I_{ss}) + G + \tau \psi_t Q_t K_t
$$

Assume that the government spending is constant and a fraction $\bar{g}$ of steady state output, $G = \bar{g}\bar{Y}$, so

$$
Y_t = C_t + I_t + f\left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}}\right)(I_{n,t} + I_{ss}) + \bar{g}\bar{Y} + \tau \psi_t Q_t K_t
$$

The evolution of capital follows

$$
K_t = \xi_t K_{t-1} + I_{n,t}
$$

1.8 Government Policy

The government uses the return on its bonds and lump-sum taxes to finance government spending and any credit market interventions, so the government budget constraint is

$$
G + \tau \psi_t Q_t K_t = T_t + \left(R_{k,t} - R_{t-1}\right) B_{g,t-1}
$$

Monetary policy follows a a Taylor rule where the smoothing parameter $\rho_{r,t}$ is time-varying, target interest rate $\bar{r}$, target inflation rate $\bar{\Pi}$

$$
\left(\frac{r_t}{r_{ss}}\right) = \left(\frac{\Pi_t}{\Pi_{ss}}\right)^{\kappa_\pi} \left(\frac{Y_t}{Y_t^*}\right)^{\kappa_y} \exp\left(\sigma_r \varepsilon_{r,t}\right)
$$

The government also follows the following rule for credit policy where the steady state, persistance parameter, and reaction parameter are time varying

$$
\psi_t = \rho_\psi(s_t)\,\psi_{t-1} + \nu(s_t)\left(R_{diff,t} - R_{diff,ss}\right)
$$

1.9 Exogenous Proccesses

The process for capital quality is time-varying in the mean and persistance

$$
\log \xi_t = \left(1 - \rho_\xi(s_t)\right) \log \xi_m(s_t) + \rho_\xi(s_t) \log \xi_{t-1} + \sigma_\xi \varepsilon_{\xi,t}
$$

and total factor productivity satisfies

$$
\log A_t = \rho_a \log A_{t-1} + \sigma_a \varepsilon_{a,t}
$$

1.10 Welfare

For welfare calculations the value function is

$$V_t = \log\left(C_t - hC_{t-1}\right) - \frac{\varkappa}{1+\varphi} L_t^{1+\varphi} + \beta \mathbb{E}_t V_{t+1}$$

1.11 Auxiliary Variables

Since the variables for C_t, Π_t, and $I_{n,t}$ appear at times $t-1$ and $t+1$, they will be both predetermined and nonpredetermined, so define the following auxiliary variables

$$C_{1,t} = C_t$$

$$\Pi_{1,t} = \Pi_t$$

$$I_{n,1,t} = I_{n,t}$$

2 Model with Regime Switching

2.1 Regime Switching

The economy switches between regimes denoted $s_t \in S = \{1, ..., n_s\}$. There are $n_s = 4$ regimes, with regime $s_t = 1$ called "normal times," $s_t = 2$ "financial crisis with no intervention," $s_t = 3$ "financial crisis with intervention," and $s_t = 4$ "post crisis with intervention."

The economy experiences a financial crisis with probability p_c, and once in a financial crisis exits with probability p_e. Conditional on a financial crisis, the government decides to intervene or not with probability p_b. If there was intervention and the crisis ends, policy continuation occurs and ends with probability p_s. If a crisis occurs during policy continuation, it is assumed the govenrment makes the same intervention decision. So the transition matrix between the regimes is given by $\mathbb{P}$, with element $p_{i,j} = \Pr\left(s_t = j | s_{t-1} = i\right)$, and

$$\begin{aligned}
\mathbb{P} &= \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \\ p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} \end{bmatrix} \\
&= \begin{bmatrix} 1-p_c & p_c\left(1-p_b\right) & p_c p_b & 0 \\ p_e & 1-p_e & 0 & 0 \\ p_e p_s & 0 & 1-p_e & p_e\left(1-p_s\right) \\ \left(1-p_c\right) p_e & 0 & p_c & \left(1-p_c\right)\left(1-p_s\right) \end{bmatrix}
\end{aligned}$$

The equilibrium conditions for the economy with regime switching are identical to those in the previous section, with the exception of the monetary policy and credit policy rules, and the law of motion for the quality of capital ξ_t. For all variables, the subindex i indicates the regime, so for example $x_{i,t}$ is the variable x_t assuming $s_t = i$. In addition, the parameters that are affected by regime switching control the steady state of the model, so each regime will have it's own "steady state" which is the steady state that would occur if there was no regime switching and only that regime occured. So the steady state for the variable x_t in regime $s_t = i$ will be denoted $x_{i,ss}$.

2.2 Regime Switching Equations

The second equation is the policy rule for credit market intervention, which is

$$\psi_t = \rho_\psi\left(s_t\right)\psi_{t-1} + \nu\left(s_t\right)\left(R_{diff,t} - R_{diff,ss}\right)$$

The capital quality equation has switching, and is

$$\log \xi_t = \left(1 - \rho_\xi\left(s_t\right)\right)\log \xi_m\left(s_t\right) + \rho_\xi\left(s_t\right)\log \xi_{t-1}$$

2.3 Regime Switching: Parameters

To summarize, there are $n_s = 4$ regimes, and the switching parameters are

$$\left\{\xi_m\left(s_t\right), \rho_\xi\left(s_t\right), \rho_\psi\left(s_t\right), \nu\left(s_t\right)\right\}$$

The following table summarizes the parameterizations across regimes.

s_t		$\xi_m\left(s_t\right)$	$\rho_\xi\left(s_t\right)$	$\nu\left(s_t\right)$	$\rho_\psi\left(s_t\right)$
1)	"Normal"	1	ρ_ξ^n	0	ρ_ψ^n
2)	"Crisis without Intervention"	ξ_m^c	0	0	ρ_ψ^n
3)	"Crisis with Intervention"	ξ_m^c	0	ν^i	0
4)	"Post-Crisis with Intervention"	1	ρ_ξ^n	0	1

3 Perturbation Setup and Equilibrium

3.1 Perturbation with Regime Switching

Given the regime switching in the previous section, the regime switching affects the steady state of the model through the steady state values $\left\{\xi_{i,ss}\right\}$. For perturbation, these values will be substituted by deviations from some perturbation point. So, rewrite the steady-state switching values as

$$\xi_m\left(s_t\right) = \bar{\xi} + \chi\hat{\xi}_m\left(s_t\right) = \bar{\xi} + \chi\left(\xi_m\left(s_t\right) - \bar{\xi}\right)$$

where χ is the perturbation parameter. So when $\chi = 0$, all the steady state switches are reduced to the perturbation point, and when $\chi = 1$, the economy has steady state switches.

3.2 Equilibrium Conditions

The equilibrium conditions are optimality conditions for the household (5 equations)

$$\left(C_t - hC_{t-1}\right)^{-1} - \beta h\mathbb{E}_t\left(C_{t+1} - hC_t\right)^{-1} = \varrho_t$$

$$\beta R_t \mathbb{E}_t \Lambda_{t+1} = 1$$

$$\varkappa L_t^\varphi = \varrho_t W_t$$

$$\Lambda_t = \frac{\varrho_t}{\varrho_{t-1}}$$

$$\beta r_t \mathbb{E}_t \frac{\Lambda_{t+1}}{\Pi_{t+1}} = 1$$

optimality conditions for the financial intermediaries (7 equations)

$$\nu_t = \mathbb{E}_t\left[\left(1-\theta\right)\beta\Lambda_{t+1}\left(R_{k,t+1} - R_t\right) + \beta\theta\Lambda_{t+1}m_{t+1}v_{t+1}\right]$$

$$\eta_t = \mathbb{E}_t\left[\left(1-\theta\right)\beta\Lambda_{t+1}R_t + \theta\beta\Lambda_{t+1}z_{t+1}\eta_{t+1}\right]$$

$$\phi_t = \frac{\eta_t}{\lambda - \nu_t}$$

$$z_t = \left(R_{k,t} - R_{t-1}\right)\phi_{t-1} + R_{t-1}$$

$$m_t = \frac{\phi_t}{\phi_{t-1}} z_t$$

$$N_t = \theta z_t N_{t-1} + \omega \left(1 - \psi_{t-1}\right) Q_t \xi_t K_{t-1}$$

$$R_{diff,t} = \mathbb{E}_t R_{k,t+1} - R_t$$

conditions for credit policy (2 equations)

$$Q_t K_t = \phi_{c,t} N_t$$

$$\phi_{c,t} = \frac{\phi_t}{1 - \psi_t}$$

optimality conditons for the intermediate goods firm (4 equations)

$$Y_{m,t} = A_t \left(U_t \xi_t K_{t-1}\right)^{\alpha} L_t^{1-\alpha}$$

$$P_{m,t} \left(1 - \alpha\right) \frac{Y_{m,t}}{L_t} = W_t$$

$$P_{m,t} \alpha \frac{Y_{m,t}}{U_t} = \delta' \left(U_t\right) \xi_t K_{t-1}$$

$$R_{k,t} = \frac{\left[P_{m,t} \alpha \frac{Y_{m,t}}{\xi_t K_{t-1}} + Q_t - \delta\left(U_t\right)\right] \xi_t}{Q_{t-1}}$$

optimality conditions for the capital producing firms (3 equations)

$$I_{n,t} = I_t - \delta\left(U_t\right) \xi_t K_{t-1}$$

$$K_t = I_{n,t} + \xi_t K_{t-1}$$

$$\begin{aligned} Q_t \quad = \quad & 1 + f\left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}}\right) + f'\left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}}\right)\left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}}\right) \\ & - \mathbb{E}_t \beta \Lambda_{t+1} f'\left(\frac{I_{n,t+1} + I_{ss}}{I_{n,t} + I_{ss}}\right)\left(\frac{I_{n,t+1} + I_{ss}}{I_{n,t} + I_{ss}}\right)^2 \end{aligned}$$

optimality conditions for the retail firms (6 equations)

$$\left(\varepsilon - 1\right) a_{1,t} = \varepsilon a_{2,t}$$

$$a_{1,t} = \tilde{P}_t^* Y_t + \gamma \beta \frac{\tilde{P}_t^*}{\tilde{P}_{t+1}^*} \left(\frac{\Pi_t^{\mu}}{\Pi_{t+1}}\right)^{1-\varepsilon} \Lambda_{t+1} a_{1,t+1}$$

$$a_{2,t} = P_{m,t} Y_t + \gamma \beta \left(\frac{\Pi_t^{\mu}}{\Pi_{t+1}}\right)^{-\varepsilon} \Lambda_{t+1} a_{2,t+1}$$

$$1 = \left(1 - \gamma\right) \tilde{P}_t^{*1-\varepsilon} + \gamma \left(\frac{\Pi_{t-1}^{\mu}}{\Pi_t}\right)^{1-\varepsilon}$$

$$\varsigma_t = \left(1 - \gamma\right) \left(\tilde{P}_t^*\right)^{-\varepsilon} + \gamma \left(\frac{\Pi_{t-1}^{\mu}}{\Pi_t}\right)^{-\varepsilon} \varsigma_{t-1}$$

$$Y_{m,t} = \varsigma_t Y_t$$

a resource constraint (1 equation)

$$Y_t = C_t + I_t + f\left(\frac{I_{n,t} + I_{ss}}{I_{n,t-1} + I_{ss}}\right)(I_{n,t} + I_{ss}) + \bar{g}Y_{ss} + \tau\psi_t Q_t K_t$$

policy rules (2 equations)

$$\left(\frac{r_t}{r_{ss}}\right) = \left(\frac{\Pi_t}{\Pi_{ss}}\right)^{\kappa_\pi}\left(\frac{Y_t}{Y_t^*}\right)^{\kappa_y}\exp\left(\sigma_r \varepsilon_{r,t}\right)$$

$$\psi_t = \rho_\psi\left(s_t\right)\psi_{t-1} + \nu\left(s_t\right)\left(R_{diff,t} - R_{diff,ss}\right)$$

exogenous proceses (2 equations)

$$\log\xi_t = \left(1 - \rho_\xi\left(s_t\right)\right)\log\xi_m\left(s_t\right) + \rho_\xi\left(s_t\right)\log\xi_{t-1} + \sigma_\xi\varepsilon_{\xi,t}$$

$$\log A_t = \rho_a \log A_{t-1} + \sigma_a \varepsilon_{a,t}$$

a value function equation (1 equation)

$$V_t = \log\left(C_t - hC_{t-1}\right) - \frac{\varkappa}{1+\varphi}L_t^{1+\varphi} + \beta\mathbb{E}_t V_{t+1}$$

and auxiliary definitions (3 equations)

$$C_{1,t} = C_t$$

$$\Pi_{1,t} = \Pi_t$$

$$I_{n,1,t} = I_{n,t}$$

These equilibrium conditions are written as

$$\mathbb{E}_t f\left(y_{t+1}, y_t, x_t, x_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta_{t+1}, \theta_t\right) = 0$$

where the variables are separated into the predetermined variables x_{t-1} and the nonpredermined variables y_t. For a variable var_t, define $\widetilde{var}_t = \log var_t$, so the variables are

$$x_{t-1} = \left[\tilde{C}_{t-1}, \tilde{\Pi}_{t-1}, \tilde{\varsigma}_{t-1}, \tilde{k}_{t-1}, I_{n,t-1}, \tilde{Q}_{t-1}, \tilde{\xi}_{t-1}, \tilde{\phi}_{t-1}, \tilde{N}_{t-1}, \tilde{R}_{t-1}, \psi_{t-1}, \tilde{\varrho}_{t-1}, \tilde{A}_{t-1}\right]$$

$$y_t = \left[\tilde{R}_{k,t}, \tilde{Y}_t, \tilde{C}_{1,t}, \tilde{\Pi}_{1,t}, \tilde{a}_{1,t}, \tilde{a}_{2,t}, \tilde{u}_t, \tilde{\nu}_t, \tilde{\eta}_t, I_{n,1,t}, R_{diff,t}, \tilde{\Lambda}_t, \tilde{L}_t, \tilde{P}_{m,t}, \tilde{P}_t^*, \tilde{I}_t, \tilde{Y}_{m,t}, \tilde{z}_t, \tilde{m}_t, \tilde{r}_t, \tilde{W}_t, \tilde{\phi}_{c,t}, V_t\right]$$

the shocks are

$$\epsilon_t = \left[\varepsilon_{\xi,t}, \varepsilon_{r,t}, \varepsilon_{a,t}\right]$$

and the switching variables are

$$\theta_t = \begin{bmatrix} \theta_{1,t} \\ \theta_{2,t} \end{bmatrix} = \begin{bmatrix} \xi_m\left(s_t\right) \\ \rho_\xi\left(s_t\right), \rho_\psi\left(s_t\right), \nu\left(s_t\right) \end{bmatrix}.$$

3.3 Steady State

The steady state results when $\chi = 0$, and so $\xi_m(s_t) = \bar{\xi}$, and the exogenous shocks $\epsilon_t = 0$. Solving for this point analytically using the full set of equilibrium conditions produces the following. The perturbation point interest rate differential $R_{diff,ss} = R_{k,ss} - R_{ss}$ is the positive solution to the quadratic equation $\bar{A}(R_{diff,ss})^2 + \bar{B}R_{diff,ss} + \bar{C} = 0$, where

$$\begin{aligned}
\bar{A} &= -\theta\beta \\
\bar{B} &= \frac{\lambda\theta(1-\beta)\left(1-\frac{\theta}{\beta}\right)}{(1-\theta)} - (\beta+\theta)\frac{\omega\bar{\xi}}{1-\bar{\psi}} \\
\bar{C} &= \left(\lambda\left(1-\frac{\theta}{\beta}\right) - \frac{\omega\bar{\xi}}{1-\bar{\psi}}\right)\frac{\omega\bar{\xi}}{1-\bar{\psi}}
\end{aligned}$$

and then the remaining conditions are

$$\xi_{ss} = \bar{\xi}$$

$$\Pi_{ss} = 1$$

$$R_{ss} = \frac{1}{\beta}$$

$$Q_{ss} = 1$$

$$r_{ss} = R_{ss}$$

$$R_{k,ss} = R_{diff,ss} + R_{ss}$$

$$\psi_{ss} = 0$$

$$P^*_{ss} = 1$$

$$\varsigma_{ss} = 1$$

$$P_{m,ss} = \frac{(\varepsilon-1)}{\varepsilon}$$

$$\Omega = \left(\frac{\frac{R_{k,ss}}{\xi_{ss}} - 1 + \bar{\delta}}{\alpha P_{m,ss}}\right)^{\frac{1}{\alpha-1}}$$

$$W_{ss} = (1-\alpha)P_{m,ss}\Omega^{\alpha}$$

$$\tilde{\delta} = \frac{R_{k,ss}}{\bar{\xi}} - 1 + \bar{\delta}$$

$$L_{ss} = \left\{\frac{(1-\beta h)W_{ss}}{(1-h)\varkappa\left((1-\bar{g})\Omega^{\alpha} - \left[1-\bar{\xi}+\tilde{\delta}\bar{\xi}\right]\frac{\Omega}{\bar{\xi}}\right)}\right\}^{\frac{1}{\varphi+1}}$$

$$\varrho_{ss} = \frac{\varkappa L_{ss}^{\varphi}}{W_{ss}}$$

$$C_{ss} = \frac{(1-\beta h)}{(1-h)\varrho_{ss}}$$

$$Y_{m,ss} = \Omega^{\alpha}L_{ss}$$

$$Y_{ss} = Y_{m,ss}$$

$$K_{ss} = \frac{\Omega L_{ss}}{\bar{\xi}}$$

$$I_{n,ss} = \left(1-\bar{\xi}\right)K_{ss}$$

$$I_{ss} = I_{n,ss} + \bar{\delta}\bar{\xi} K_{ss}$$

$$a_{1,ss} = \frac{P_{ss}^{*} Y_{ss}}{1 - \gamma\beta}$$

$$a_{2,ss} = \frac{P_{m,ss} Y_{ss}}{1 - \gamma\beta}$$

$$\phi_{ss} = \frac{1 - \theta R_{ss}}{\theta \left(R_{k,ss} - R_{ss} \right) + \omega\bar{\xi}}$$

$$\phi_{c,ss} = \phi_{ss}$$

$$N_{ss} = \frac{K_{ss}}{\phi_{c,ss}}$$

$$z_{ss} = \left(R_{k,ss} - R_{ss} \right) \phi_{ss} + R_{ss}$$

$$x_{ss} = z_{ss}$$

$$\nu_{ss} = \frac{(1-\theta)\,\beta \left(R_{k,ss} - R_{ss} \right)}{(1 - \beta\theta x_{ss})}$$

$$\eta_{ss} = \frac{(1-\theta)}{(1 - \theta\beta z_{ss})}$$

$$V_{ss} = \frac{\log (C_{ss}) + \log (1-h) - \frac{\varkappa}{1+\varphi} L_{ss}^{1+\varphi}}{1-\beta}$$

4 Model Solution

The solution to the model is a set of transition equations for the predetermined variables that make up the state, and observation equations for the nonpredetermined variables. The decision rules are found by writing

$$\mathbb{E}_t f \left(y_{t+1}, y_t, x_t, x_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta_{t+1}, \theta_t \right) = 0$$

with the functional forms

$$y_t = g \left(x_{t-1}, \varepsilon_t, \chi; s_t \right)$$

$$x_t = h \left(x_{t-1}, \varepsilon_t, \chi; s_t \right)$$

$$y_{t+1} = g \left(x_t, \chi\varepsilon_{t+1}, \chi; s_{t+1} \right) = g \left(h \left(x_{t-1}, \varepsilon_t, \chi; s_t \right), \chi\varepsilon_{t+1}, \chi; s_{t+1} \right)$$

and

$$\theta_t = \theta \left(\chi; s_t \right)$$

$$\theta_{t+1} = \theta \left(\chi; s_{t+1} \right).$$

The state at time t is given by

$$\mathcal{S}_{t-1} = [x_{t-1}, \epsilon_t, \chi]$$

and the first order transition equation is

$$x_t = x_{ss} + H \{ s_t \} \widehat{\mathcal{S}}_{t-1}$$

and the nonpredetermined variables follow

$$y_t = y_{ss} + G \{ s_t \} \widehat{\mathcal{S}}_{t-1}$$

The second order solution is given by

$$x_t = x_{ss} + H \{ s_t \} \widehat{\mathcal{S}}_{t-1} + \frac{1}{2} H_2 \{ s_t \} \left(\widehat{\mathcal{S}}_{t-1} \otimes \widehat{\mathcal{S}}_{t-1} \right)$$

and

$$y_t = y_{ss} + G \{ s_t \} \widehat{\mathcal{S}}_{t-1} + \frac{1}{2} G_2 \{ s_t \} \left(\widehat{\mathcal{S}}_{t-1} \otimes \widehat{\mathcal{S}}_{t-1} \right)$$

5 Results

5.1 Differences in Stochastic Steady States

Consider economies with two slightly different transition matrices, so all paramters are identical except the probability associated with intervention. These economies will in general have different stochastic steady states asscoiated with the normal regime $s_t = 1$. Using pruning, the first order effect for the predetermined variables is

$$\bar{x}^f = H\{1\} * \begin{bmatrix} \bar{x}^{f\prime} & 0' & 1 \end{bmatrix}'$$

the second order effect is

$$\bar{x}^s = H\{1\} * \begin{bmatrix} \bar{x}^{s\prime} & 0' & 0 \end{bmatrix}' + \frac{1}{2} H_2\{1\} * \left(\begin{bmatrix} \bar{x}^{f\prime} & 0' & 1 \end{bmatrix}' \otimes \begin{bmatrix} \bar{x}^{f\prime} & 0' & 1 \end{bmatrix}' \right)$$

and the total effect is

$$\bar{x} = x_{ss} + \bar{x}^f + \bar{x}^s$$

Similarly, for the nonpredetermined variables, the first order effect is

$$\bar{y}^f = G\{1\} * \begin{bmatrix} \bar{x}^{f\prime} & 0' & 1 \end{bmatrix}'$$

the second order effect is

$$\bar{y}^s = G\{1\} * \begin{bmatrix} \bar{x}^{s\prime} & 0' & 0 \end{bmatrix}' + \frac{1}{2} G_2\{1\} * \left(\begin{bmatrix} \bar{x}^{f\prime} & 0' & 1 \end{bmatrix}' \otimes \begin{bmatrix} \bar{x}^{f\prime} & 0' & 1 \end{bmatrix}' \right)$$

and the total effect is

$$\bar{y} = y_{ss} + \bar{y}^f + \bar{y}^s.$$

5.2 Welfare Calculations

5.2.1 Welfare Measure

Consider economies with two sets of policy parameters, one determining the intervention probability p_b, and one for the resource cost τ. The first economy, denoted economy A, has $p_b = 0$ (and so τ is irrelevant). The second economy, denoted B, has $p_b \in [0, 1]$ and $\tau \geq 0$.

The welfare associated with economy A is

$$V_0^A = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(C_t^A - hC_{t-1}^A\right) - \frac{\varkappa}{1+\varphi} \left(L_t^A\right)^{1+\varphi} \right\}$$

and the welfare for economy B is

$$V_0^B = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(C_t^B - hC_{t-1}^B\right) - \frac{\varkappa}{1+\varphi} \left(L_t^B\right)^{1+\varphi} \right\}$$

Considering the welfare between the two, the household will be indifferent between economy B and a scaled version of economy A, where consumption is $(1 - \Upsilon) C_t^A$ in every period. So Υ is the percentage increase in consumption under A that would make the household indifferent between economy A and B. That is

$$\begin{aligned} V_t^B &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(C_t^A (1+\Upsilon) - hC_{t-1}^A (1+\Upsilon)\right) - \frac{\varkappa}{1+\varphi} \left(L_t^A\right)^{1+\varphi} \right\} \\ &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(C_t^A - hC_{t-1}^A\right) + \log(1+\Upsilon) - \frac{\varkappa}{1+\varphi} \left(L_t^A\right)^{1+\varphi} \right\} \\ &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log\left(C_t^A - hC_{t-1}^0\right) - \frac{\varkappa}{1+\varphi} \left(L_t^A\right)^{1+\varphi} \right\} + \beta^t \log(1+\Upsilon) \\ &= V_t^A + \frac{\log(1+\Upsilon)}{1-\beta} \end{aligned}$$

Note that the welfare function is written as

$$V_t = \left\{ \log\left(C_t - hC_{t-1}\right) - \frac{\varkappa}{1+\varphi}\left(L_t\right)^{1+\varphi} \right\} + \beta \mathbb{E}_t V_{t+1}$$

and so a second order expansion is of the form

$$V_t = V_{ss} + H\left\{s_t\right\} \widehat{\mathcal{S}}_{t-1} + \frac{1}{2} H_2\left\{s_t\right\}\left(\widehat{\mathcal{S}}_{t-1} \otimes \widehat{\mathcal{S}}_{t-1}\right).$$

Similar to above, the two expansions are, for economy A:

$$V_t^A = V_{ss}^A + H^A\left\{s_t\right\} \widehat{\mathcal{S}}_{t-1}^A + \frac{1}{2} H_2^A\left\{s_t\right\}\left(\widehat{\mathcal{S}}_{t-1}^A \otimes \widehat{\mathcal{S}}_{t-1}^A\right)$$

and for B:

$$V_t^B = V_{ss}^B + H^B\left\{s_t\right\} \widehat{\mathcal{S}}_{t-1}^B + \frac{1}{2} H_2^B\left\{s_t\right\}\left(\widehat{\mathcal{S}}_{t-1}^B \otimes \widehat{\mathcal{S}}_{t-1}^B\right).$$

5.2.2 Ex-ante Welfare

Now suppose that the economy A is at it's stochastic steady state associated with regime 1. This level is given by

$$\bar{x}^A = x_{ss}^A + H^A\{1\}\bar{\mathcal{S}}^A + \frac{1}{2} H_2^A\{1\}\left(\bar{\mathcal{S}}^A \otimes \bar{\mathcal{S}}^A\right)$$

where

$$\bar{\mathcal{S}}^A = \left\{\bar{x}^A - x_{ss}^A, 0, 1\right\}.$$

At this point $\bar{\mathcal{S}}^A$, the value of being in economy A is

$$V^A = V_{ss}^A + H^A\{1\}\bar{\mathcal{S}}^A + \frac{1}{2} H_2^A\{1\}\left(\bar{\mathcal{S}}^A \otimes \bar{\mathcal{S}}^A\right)$$

and the value of being in economy B is

$$V^B = V_{ss}^B + H^B\{1\}\bar{\mathcal{S}}^B + \frac{1}{2} H_2^B\{1\}\left(\bar{\mathcal{S}}^B \otimes \bar{\mathcal{S}}^B\right)$$

where

$$\bar{\mathcal{S}}^B = \left\{\bar{x}^A - x_{ss}^B, 0, 1\right\}$$

and so the *ex-ante* welfare measure for B is

$$\begin{aligned} \Upsilon &= \exp\left\{(1-\beta)\left(V_t^B - V_t^A\right)\right\} - 1 \\ &\simeq \exp\left\{(1-\beta)\left[\begin{array}{c} \left(V_{ss}^B + H^B\{1\}\bar{\mathcal{S}}^B + \frac{1}{2}H_2^B\{1\}\left(\bar{\mathcal{S}}^B \otimes \bar{\mathcal{S}}^B\right)\right) \\ -\left(V_{ss}^A + H^A\{1\}\bar{\mathcal{S}}^A + \frac{1}{2}H_2^A\{1\}\left(\bar{\mathcal{S}}^A \otimes \bar{\mathcal{S}}^A\right)\right) \end{array}\right]\right\} - 1 \end{aligned}$$

5.2.3 Ex-post Welfare

Now suppose that the economy A is at it's stochastic steady state associated with regime 1. Again, this level is given by

$$\bar{x}^A = x_{ss}^A + H^A\{1\}\bar{\mathcal{S}}^A + \frac{1}{2} H_2^A\{1\}\left(\bar{\mathcal{S}}^A \otimes \bar{\mathcal{S}}^A\right)$$

where

$$\bar{\mathcal{S}}^A = \left\{\bar{x}^A - x_{ss}^A, 0, 1\right\}.$$

Now, if a crisis occurs, there is no chance of intervention, since $p_b = 0$, so now $s_t = 2$, and the value of being in this economy is hence

$$V^A = V_{ss}^A + H^A\{2\}\bar{\mathcal{S}}^A + \frac{1}{2} H_2^A\{2\}\left(\bar{\mathcal{S}}^A \otimes \bar{\mathcal{S}}^A\right)$$

In economy B, there is a chance p_b of being in regime $s_t = 3$, and a chance $(1 - p_b)$ of being in regime $s_t = 1$. The values are

$$V^B(s_t = 2) = V_{ss}^B + H^B\{2\}\bar{\mathcal{S}}^B + \frac{1}{2}H_2^B\{2\}\left(\bar{\mathcal{S}}^B \otimes \bar{\mathcal{S}}^B\right)$$

and

$$V^B(s_t = 3) = V_{ss}^B + H^B\{3\}\bar{\mathcal{S}}^B + \frac{1}{2}H_2^B\{3\}\left(\bar{\mathcal{S}}^B \otimes \bar{\mathcal{S}}^B\right)$$

where again

$$\bar{\mathcal{S}}^B = \left\{\bar{x}^A - x_{ss}^B, 0, 1\right\}.$$

So the total value of being in economy B is

$$V^B = (1 - p_b)V^B(s_t = 2) + p_b V^B(s_t = 3)$$

and so the *ex-post* welfare measure is

$$\begin{aligned}
\Upsilon &= \exp\left\{(1-\beta)\left(V_t^B - V_t^A\right)\right\} - 1 \\
&\simeq \exp\left\{(1-\beta)\left[\begin{array}{c} (1-p_b)\left(V_{ss}^B + H^B\{2\}\bar{\mathcal{S}}^B + \frac{1}{2}H_2^B\{2\}\left(\bar{\mathcal{S}}^B \otimes \bar{\mathcal{S}}^B\right)\right) \\ +p_b\left(V_{ss}^B + H^B\{3\}\bar{\mathcal{S}}^B + \frac{1}{2}H_2^B\{3\}\left(\bar{\mathcal{S}}^B \otimes \bar{\mathcal{S}}^B\right)\right) \\ -\left(V_{ss}^A + H^A\{1\}\bar{\mathcal{S}}^A + \frac{1}{2}H_2^A\{1\}\left(\bar{\mathcal{S}}^A \otimes \bar{\mathcal{S}}^A\right)\right) \end{array}\right]\right\} - 1
\end{aligned}$$