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Online Appendix

Itskhoki and Mukhin: "Exchange Rate Disconnect in General Equilibrium"

A.1 Data appendix

As explained in Section 4, for comparability, we estimate most empirical moments in Tables 1 and 3 following CKM. In particular, we use their quarterly data from 1973–94 available on Ellen McGrattan's website and estimate the moments for the US against the PPP-weighted sum of France, Germany, Italy and the UK. While we use the first differences of the variables, rather than the HP-filtered series, our estimates are very close to the ones in CKM.

As a robustness exercise, we also collected quarterly data for a longer time period from 1981–2017 from FRED database. In particular, we use seasonally-adjusted GDP, consumption and gross capital formation in constant prices and aggregate them across the same countries using PPP-adjusted nominal GDP in 2000 (from OECD database). Table A2 shows that the estimates for the two periods are very similar. The only exception are the moments for net exports in Table 3: in contrast to CKM, who focus on bilateral trade between the US and the European countries, we use total imports and exports of the US, consistent with our model. Indeed, the former includes only a part of international trade of the US and underestimates the openness of the US economy.

Table A2: Empirical Moments

Moments	CKM	IM	Moments	CKM	IM
A. Exchange rate disconnect:			D. International business cycle moments:		
$\rho(\Delta e)$	0.3	0.3	$\sigma(\Delta c)/\sigma(\Delta gdp)$	0.82	0.81
$\sigma(\Delta e)/\sigma(\Delta gdp)$	5.2	6.5	$\mathrm{corr}(\Delta c, \Delta gdp)$	0.64	0.63
$\sigma(\Delta e)/\sigma(\Delta c)$	6.3	8.0	$\mathrm{corr}(\Delta z, \Delta gdp)$	0.81	0.75
B. Real exchange rate and the PPP:			$\mathrm{corr}(\Delta gdp, \Delta gdp^*)$	0.35	0.42
$\rho(q)$	0.96	0.94	$\mathrm{corr}(\Delta c, \Delta c^*)$	0.30	0.40
$\sigma(\Delta q)/\sigma(\Delta e)$	0.99	0.97	$\mathrm{corr}(\Delta z, \Delta z^*)$	0.27	0.32
$\mathrm{corr}(\Delta q, \Delta e)$	0.99	0.99	E. Terms of trade and net exports moments:		
C. Backus-Smith correlation:			$\sigma(\Delta nx)/\sigma(\Delta q)$	0.01	0.09
$\mathrm{corr}(\Delta q, \Delta c - \Delta c^*)$	-0.20	-0.17	$\mathrm{corr}(\Delta nx, \Delta q)$	-0.01	0.35

Note: CKM and IM correspond respectively to the estimates obtained for the periods 1973–94 and our estimates for 1981–2017.

We use seasonally-adjusted hourly earnings in manufacturing as a proxy for nominal wages to compute the wage-based real exchange rate. Given the limited availability of the terms-of-trade data at the quarterly frequency, we use the estimates for terms-of-trade moments from Obstfeld and Rogoff (2001) and Gopinath et al. 2020. We also compute the same moments using FRED annual data on import and export price indices for the US. For our baseline period from 1973–1994, we obtain $\sigma(\Delta s)/\sigma(\Delta e) = 0.29$ and $\mathrm{corr}(\Delta s, \Delta e) = 0.20$, in line with the estimates in the literature (see Table 3).

Finally, we borrow several financial moments from the previous literature. The slope coefficient β and R^2 in the Fama regression are from the survey by Engel (1996) and recent estimates by Burnside, Han, Hirshleifer, and Wang (2011, Table 1) and Valchev (2020, Table B.1). The estimates for the Sharpe ratio correspond to the forward premium trade from Hassan and Mano (2014, Table 2). We estimate the volatility and persistence of the interest rates using quarterly data for the US versus the UK, France, Germany and Japan from 1979–2009.

We calculate additional moments for the six countries in Table 4 against the rest of the world using quarterly data from 1981–2017 and focus on a sample of developed economies (Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Japan, Korea, Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, the UK and the US). After computing the log changes of variables for each individual country, we aggregate series into the RoW using the average import and export shares for 1991-2017 from the Worldbank WITS database. Despite the fact that our sample does not include such import trade partners as Mexico and China, and we use time-invariant weights, the resulting exchange rates mimic very closely the "narrow effect" exchange rates calculated by the BIS.

A.2 Full quantitative model

We focus on home households and firms with the understanding that the problem of foreign agents is symmetric. The equation below generalize the equations given in the text in the context of a simplified baseline model.

Households A representative home household maximizes the expected utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} C_t^{1-\sigma} - \frac{1}{1+1/\nu} L_t^{1+1/\nu} \right), \tag{A1}$$

where $\nu \equiv 1/\varphi$ is the Frisch elasticity, subject to the flow budget constraint:

$$P_t C_t + P_t Z_t + \frac{B_{t+1}}{R_t} \le W_t L_t + R_t^K K_t + B_t + \Pi_t, \tag{A2}$$

where R_t^K is the nominal rental rate of capital and Z_t is the gross investment into the domestic capital stock K_t, which accumulates according to a standard rule with depreciation δ and quadratic capital adjustment costs:

$$K_{t+1} = (1-\delta) K_t + \big[Z_t - \frac{\kappa}{2} \frac{(\Delta K_{t+1})^2}{K_t} \big]. \tag{A3}$$

The domestic households allocate their within-period consumption expenditure $P_t C_t$ between home and foreign varieties of the goods

$$P_t C_t = P_{Ht} C_{Ht} + P_{Ft} C_{Ft} = \int_0^1 \Big[P_{Ht}(i) C_{Ht}(i) + P_{Ft}(i) C_{Ft}(i) \Big] \mathrm{d}i \tag{A4}$$

to minimize expenditure on aggregate consumption, defined implicitly by a Kimball (1995) aggregator:[42]

$$\int_0^1 \left[(1-\gamma) g\left(\frac{C_{Ht}(i)}{(1-\gamma) C_t} \right) + \gamma g\left(\frac{C_{Ft}(i)}{\gamma C_t} \right) \right] \mathrm{d}i = 1, \tag{A5}$$

[42] The CES demand is nested as a special case of the Kimball aggregator (A5) with $g(z) = 1 + \frac{\theta}{\theta-1}\big(z^{1-1/\theta} - 1\big)$, resulting in the demand schedule $h(x) = x^{-\theta}$ and price index $P_t = \mathcal{P}_t = \big(\int_0^1 \big[(1-\gamma) P_{Ht}(i)^{1-\theta} + \gamma P_{Ft}(i)^{1-\theta} \big] \mathrm{d}i \big)^{1/(1-\theta)}$.

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where the aggregator function $g(\cdot)$ in (A5) has the following properties: $g'(\cdot) > 0$, $g''(\cdot) < 0$ and $-g''(1) \in (0,1)$, and two normalizations: $g(1) = g'(1) = 1$. The solution to the optimal expenditure allocation results in the following homothetic demand schedules:

$$C_{Ht}(i) = (1-\gamma)h\left(\frac{P_{Ht}(i)}{\mathcal{P}_t}\right)C_t \qquad \text{and} \qquad C_{Ft}(j) = \gamma h\left(\frac{P_{Ft}(j)}{\mathcal{P}_t}\right)C_t, \tag{A6}$$

where $h(\cdot) = g'^{-1}(\cdot) > 0$ and satisfies $h(1) = 1$ and $h'(\cdot) < 0$. The function $h(\cdot)$ controls the curvature of the demand schedule, and we denote its point elasticity with $\theta \equiv -\frac{\partial \log h(x)}{\partial \log x}\big|_{x=1} = -h'(1) > 1$. The consumer price level P_t and the auxiliary variable $\mathcal{P}_t$ in (A6) are two alternative measures of average prices in the home market, which are defined implicitly by (A4) and (A5) after substituting in the demand schedules (A6).

Production Home output is produced according to a Cobb-Douglas technology in labor L_t, capital K_t and intermediate inputs X_t:

$$Y_t = \left(e^{a_t}K_t^{\vartheta}L_t^{1-\vartheta}\right)^{1-\phi}X_t^{\phi}, \tag{A7}$$

where ϑ is the elasticity of the value added with respect to capital and ϕ is the elasticity of output with respect to intermediates. Intermediates are the same bundle of home and foreign varieties as the final consumption bundle (A5). The marginal cost of production is thus:

$$MC_t = \tfrac{1}{\varpi}\left[e^{-a_t}(R_t^K)^{\vartheta}W_t^{1-\vartheta}\right]^{1-\phi}P_t^{\phi}, \quad \text{where } \varpi \equiv \phi^{\phi}\left[(1-\phi)\vartheta^{\vartheta}(1-\vartheta)^{1-\vartheta}\right]^{1-\phi}. \tag{A8}$$

The aggregate *value-added productivity* follows an AR(1) process in logs:

$$a_t = \rho_a a_{t-1} + \sigma_a \varepsilon_t^a, \qquad \varepsilon_t^a \sim iid(0,1), \tag{A9}$$

where $\rho_a \in [0,1]$ is the persistence parameter and $\sigma_a \geq 0$ is the volatility of the innovation.

Profits and price setting The firm maximizes profits from serving the home and foreign markets:

$$\mathbb{E}_0 \sum\nolimits_{t=0}^{\infty}\Theta_t\Pi_t(i), \quad \text{where} \quad \Pi_t(i) = \left(P_{Ht}(i) - MC_t\right)Y_{Ht}(i) + \left(P_{Ht}^*(i)\mathcal{E}_t - MC_t\right)Y_{Ht}^*(i), \tag{A10}$$

where $\Theta_t \equiv \beta^t \frac{C_t^{-\sigma}}{P_t}$ is the nominal stochastic discount factor. In the absence of nominal frictions, this results in the markup pricing rules, with a common price across all domestic firms $i \in [0,1]$ in a given destination market and expressed in the destination currency:

$$P_{Ht}(i) = P_{Ht} = \mu\left(\frac{P_{Ht}}{\mathcal{P}_t}\right)\cdot MC_t \qquad \text{and} \qquad P_{Ht}^*(i) = P_{Ht}^* = \mu\left(\frac{P_{Ht}^*}{\mathcal{P}_t^*}\right)\cdot \frac{MC_t}{\mathcal{E}_t},$$

where $\mu(x) \equiv \frac{\tilde{\theta}(x)}{\tilde{\theta}(x)-1}$ is the markup function and $\tilde{\theta}(x) = -\frac{\partial \log h(x)}{\partial \log x}$ is the demand elasticity schedule derived from demand (A6) (see also (A11) below).

Nominal rigidities We introduce Calvo sticky prices and wages in a conventional way (see e.g. Clarida, Galí, and Gertler 2002, Galí 2008, as we further discuss in Appendix A.7 below). Denote with ϵ the elasticity of substitution between varieties of labor, and let λ_p and λ_w be the Calvo probability of price and wage non-adjustment. Then the resulting New Keynesian Phillips Curves (NKPC) for nominal wages inflation and domestic prices inflation can be written respectively as:

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$$\pi_t^w = k_w \left[\sigma c_t + \frac{1}{\nu}\ell_t + p_t - w_t\right] + \beta \mathbb{E}_t \pi_{t+1}^w, \quad \text{where} \quad k_w = \frac{(1-\beta\lambda_w)(1-\lambda_w)}{\lambda_w(1+\epsilon/\nu)},$$

$$\pi_{Ht} = k_p\Big[(1-\alpha)mc_t + \alpha p_t - p_{Ht}\Big] + \beta \mathbb{E}_t \pi_{Ht+1}, \quad \text{where} \quad k_p = \frac{(1-\beta\lambda_p)(1-\lambda_p)}{\lambda_p}.$$

The NKPC for export prices depends on the currency of invoicing and is given by:

$$\pi_{Ht}^* = k_p\Big[(1-\alpha)(mc_t - e_t) + \alpha p_t^* - p_{Ht}^*\Big] + \beta \mathbb{E}_t \pi_{Ht+1}^* \quad \text{under LCP},$$

$$(\pi_{Ht}^* + \Delta e_t) = k_p\Big[(1-\alpha)mc_t + \alpha(p_t^* + e_t) - (p_{Ht}^* + e_t)\Big] + \beta \mathbb{E}_t\left(\pi_{Ht+1}^* + \Delta e_{t+1}\right) \quad \text{under PCP}.$$

Notice that the DCP case with all international trade invoiced in Foreign currency can be expressed as a mix of the two other regimes — Home exporters use LCP and Foreign exporters use PCP.

Good and factor market clearing The labor market clearing requires that L_t equals simultaneosly the labor supply of the households and the labor demand of the firms, and equivalently for L_t^* in foreign. Similarly, equilibrium in the capital market requires that K_t (and K_t^*) equals simultaneously the capital supply of the households and the capital demand of the local firms. The goods market clearing requires that the total production by the home firms is split between supply to the home and foreign markets respectively, $Y_t = Y_{Ht} + Y_{Ht}^*$, and satisfies the local demand in each market for the final, intermediate and capital goods:

$$Y_{Ht} = C_{Ht} + X_{Ht} + Z_{Ht} = (1-\gamma)h\left(\frac{P_{Ht}}{\mathcal{P}_t}\right)\big[C_t + X_t + Z_t\big], \tag{A11}$$

$$Y_{Ht}^* = C_{Ht}^* + X_{Ht}^* + Z_{Ht}^* = \gamma h\left(\frac{P_{Ht}^*}{\mathcal{P}_t^*}\right)\big[C_t^* + X_t^* + Z_t^*\big]. \tag{A12}$$

Lastly, we combine the household budget constraint (A2) with profits (A10), aggregated across all home firms, as well as the market clearing conditions above to obtain the home country budget constraint:

$$\frac{B_{t+1}}{R_t} - B_t = NX_t \qquad \text{with} \qquad NX_t = \mathcal{E}_t P_{Ht}^* Y_{Ht}^* - P_{Ft} Y_{Ft}, \tag{A14}$$

where NX_t denotes net exports expressed in units of the home currency.

Financial sector The structure of the financial markets is exactly the same as described in the text for the baseline model (see Section 2.2).

Monetary policy rule The monetary policy is implemented by means of a conventional Taylor rule:

$$i_t = \rho_m i_{t-1} + (1-\rho_m)\phi_\pi \pi_t + \sigma_m \varepsilon_t^m, \tag{A15}$$

where $i_t \equiv \log R_t$ is the log nominal interest rate, $\pi_t = \Delta \log P_t$ is the inflation rate, and $\varepsilon_t^m \sim iid(0,1)$ is the monetary policy shock with volatility parameter $\sigma_m \geq 0$; parameter ρ_m captures the persistence of monetary policy.

A.3 Derivation of the equilibrium conditions

This section derives the optimality conditions of firms and households in goods and asset markets. We focus on a generalized version of the model with Kimball demand and capital and intermediate goods in production, but keep flexible prices as in the baseline model (see Appendix A.7).

Household optimization Substituting the capital accumulation equation into the budget constraint (A2), the Lagrangian for household utility maximization (A1) is:

$$\max_{\{C_t,L_t,B_{t+1},K_{t+1}\}} \sum_{t,s^t} \beta^t \pi(s^t) \bigg\{ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+1/\nu}}{1+1/\nu} \\ + \lambda_t \left[W_t L_t - P_t C_t + B_t - \frac{B_{t+1}}{R_t} + P_t K_t \left(\frac{R_t^K}{P_t} - \delta - \frac{\Delta K_{t+1}}{K_t} - \frac{\kappa}{2}\left(\frac{\Delta K_{t+1}}{K_t}\right)^2 \right) \right] \bigg\},$$

where $\pi(s^t)$ is probability of state $s^t = (s_0, s_1, \ldots, s_t)$ at time t and $\beta^t \pi(s^t)\lambda_t$ is the Lagrange multiplier on the flow budget constraint in state s^t at time t (note that we suppress the dependence of variables on s^t for brevity). The optimality conditions are:

$$\begin{aligned} C_t^{-\sigma} &= \lambda_t P_t, \\ L_t^{1/\nu} &= \lambda_t W_t, \\ \lambda_t &= \beta R_t \mathbb{E}_t \lambda_{t+1}, \\ 1 + \kappa \frac{\Delta K_{t+1}}{K_t} &= \beta \mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left[\frac{R_{t+1}^K}{P_{t+1}} + (1-\delta) + \kappa \frac{\Delta K_{t+2}}{K_{t+1}} + \frac{\kappa}{2}\left(\frac{\Delta K_{t+2}}{K_{t+1}}\right)^2 \right]. \end{aligned}$$

Solving out $\lambda_t = C_t^{-\sigma}/P_t$, we arrive at the optimality conditions (2)–(3) in the text, as well as the Euler equation for capital:

$$\eta_t = \beta \mathbb{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left[\frac{R_{t+1}^K}{P_{t+1}} - \delta + \eta_{t+1} + \frac{(\eta_{t+1}-1)^2}{2\kappa} \right] \right\}, \tag{A16}$$

where $\eta_t \equiv 1 + \kappa \frac{\Delta K_{t+1}}{K_t}$ is the (q-theory) market price of one unit of capital in units of the home consumption good, with the last term in (A16) arising from the quadratic adjustment costs.

Expenditure minimization and the price level The household expenditure minimization problem for a given consumption level C_t is given by

$$\min_{\{C_{Ht}(i),C_{Ft}(i)\}} P_t C_t = \int_0^1 \Big[P_{Ht}(i) C_{Ht}(i) + P_{Ft}(i) C_{Ft}(i) \Big] \mathrm{d}i$$

subject to (A5). The optimality conditions are:

$$P_{Jt}(i) = \mathcal{R}_t g' \left(\frac{C_{Jt}(i)}{\gamma_J C_t} \right) \frac{1}{C_t}, \qquad J \in \{H, F\},$$

where $\gamma_H \equiv 1-\gamma$ and $\gamma_F \equiv \gamma$, and $\mathcal{R}_t$ is the Lagrange multiplier on the consumption aggregator constraint.
Denoting $\mathcal{P}_t = \mathcal{R}_t / C_t$ and $h(\cdot) \equiv g'^{-1}(\cdot)$, we obtain the demand schedules (A6):

$$C_{Jt}(i) = \gamma_J \, h \left(\frac{P_{Jt}(i)}{\mathcal{P}_t} \right) C_t, \tag{A17}$$

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where $\mathcal{P}_t$ and P_t are defined implicitly by the following system:

$$1 = \int_0^1 \left[(1-\gamma)g\left(h\left(\frac{P_{Ht}(i)}{\mathcal{P}_t}\right)\right) + \gamma g\left(h\left(\frac{P_{Ft}(i)}{\mathcal{P}_t}\right)\right)\right] \mathrm{d}i, \tag{A18}$$

$$P_t = \int_0^1 \left[(1-\gamma)P_{Ht}(i)h\left(\frac{P_{Ht}(i)}{\mathcal{P}_t}\right) + \gamma P_{Ft}(i)h\left(\frac{P_{Ft}(i)}{\mathcal{P}_t}\right)\right] \mathrm{d}i. \tag{A19}$$

Equation (A18) arises from the definition of the Kimball consumption aggregator (A5), and uniquely determines $\mathcal{P}_t$, as $h'(\cdot) < 0$. Given $\mathcal{P}_t$, equation (A19) determines the value of P_t, which ensures that the sum of all market shares is one, given that P_tC_t is total expenditure. Indeed,

$$S_{Jt}(i) = \frac{C_{Jt}(i)P_{Jt}(i)}{P_tC_t} = \gamma_J \frac{P_{Jt}(i)}{P_t} h\left(\frac{P_{Jt}(i)}{\mathcal{P}_t}\right)$$

is the market share of domestic (for $J = H$) or foreign (for $J = F$) variety i in the home market.

When all $P_{Ht}(i) = P_{Ft}(j) = \mathcal{P}_t$ for all i, j and for some $\mathcal{P}_t$, then $P_t = \mathcal{P}_t$, given our normalization that $g(1) = g'(1) = 1$, which implies $h(1) = 1$. More generally, P_t and $\mathcal{P}_t$ offer two alternative generalized averages of prices of the varieties in the domestic market, and the difference between P_t and $\mathcal{P}_t$ is second order in the dispersion of prices. Indeed, taking the first order approximation to (A18)–(A19) around a symmetric equilibrium described above, we have:

$$p_t \equiv \mathrm{d}\log P_t = \int_0^1 [(1-\gamma)p_{Ht}(i) + \gamma p_{Ft}(i)]\,\mathrm{d}i \quad \text{and} \quad \mathrm{d}\log \mathcal{P}_t = \int_0^1 [(1-\gamma)p_{Ht}(i) + \gamma p_{Ft}(i)]\,\mathrm{d}i = p_t,$$

where $\mathrm{d}\log P_t$ and $\mathrm{d}\log \mathcal{P}_t$ denote the log-deviations from some symmetric steady state. In the derivations, we used the facts that $g'(1) = h(1) = 1$ and $h'(1) = -\theta$. This confirms that P_t and $\mathcal{P}_t$ differ at most by a second-order term in the dispersion of the vector $\big(\{p_{Ht}(i) - p_t\}_i, \{p_{Ft}(j) - p_t\}_j\big)$, which is an identical zero in a symmetric steady state. Lastly, note that the expenditure share on foreign goods in a symmetric equilibrium is given by $\int_0^1 S_{Ft}(i)\mathrm{d}i = \gamma$.

Price setting Monopolistically competitive firms set prices flexibly to maximize profits (A10) subject to the demand schedule (A6). The price setting problem of a representative home firm i partitions into price setting in the home and foreign markets separately:

$$P_{Ht}(i) = \arg\max_{P_{Ht}(i)} \left\{(P_{Ht}(i) - MC_t)(1-\gamma)h\left(\frac{P_{Ht}(i)}{\mathcal{P}_t}\right)C_t\right\} = \mu\left(\frac{P_{Ht}(i)}{\mathcal{P}_t}\right)\cdot MC_t, \tag{A20}$$

$$P^*_{Ht}(i) = \arg\max_{P^*_{Ht}(i)} \left\{(P^*_{Ft}(i)\mathcal{E}_t - MC_t)\gamma h\left(\frac{P^*_{Ht}(i)}{\mathcal{P}^*_t}\right)C^*_t\right\} = \mu\left(\frac{P^*_{Ht}(i)}{\mathcal{P}^*_t}\right)\cdot \frac{MC_t}{\mathcal{E}_t}, \tag{A21}$$

where $\mu(x) \equiv \frac{\tilde{\theta}(x)}{\tilde{\theta}(x)-1}$ is the optimal markup function and $\tilde{\theta}(x) = -\frac{\partial \log h(x)}{\partial \log x}$ is the elasticity of the demand schedule, and $\mathcal{P}^*_t$ is the auxiliary average price in the foreign market. Note that all home firms charge the same price in each of the markets, P_{Ht} and P^*_{Ht} repsectively, yet the prices may differ across markets, $P_{Ht} \neq P^*_{Ht}\mathcal{E}_t$, violating the law of one price. This happens *iff* $\mathcal{P}_t \neq \mathcal{P}^*_t$.

Note that P_{Ht} and P^*_{Ht} also correspond to the price indexes of the home good aggregator in the home and foreign markets respectively, defined in parallel with the overall price index P_t in (A19). Given the same prices, we also have the same quantities across domestic firms, in particular $P_{Ht}C_{Ht} = P_{Ht}(i)C_{Ht}(i)$ for all i, where C_{Ht} is the aggregate consumption index of all home goods in the home market defined analogously to the aggregate consumption C_t in (A5). The same property applies in the foreign market, and for foreign goods in both markets. Finally, we have the aggregate expenditure in the home and foreign markets given by $P_tC_t = P_{Ht}C_{Ht} + P_{Ft}C_{Ft}$ and $P_t^*C_t^* = P_{Ht}^*C_{Ht}^* + P_{Ft}^*C_{Ft}^*$ respectively.

Next, consider the full log-differential of the optimal price setting equations around a symmetric equilibrium in both markets:

$$p_{Ht} = -\Gamma(p_{Ht} - p_t) + mc_t,$$
$$p_{Ht}^* = -\Gamma(p_{Ht}^* - p_t^*) + (mc_t - e_t),$$

where small letters denote log-deviations from the symmetric equilibrium and we used the fact that $\mathrm{d}\log\mathcal{P}_t = p_t$ (and same in foreign) and that

$$\Gamma \equiv -\frac{\partial \log \mu(x)}{\partial \log x}\Big|_{x=1} = \frac{\epsilon}{\theta-1},$$

where we used the properties $\mu(x) = \frac{\tilde\theta(x)}{\tilde\theta(x)-1}$ and $\theta = \tilde\theta(1)$, and we defined the super elasticity of demand $\epsilon = \frac{\partial \log\tilde\theta(x)}{\partial\log x}\big|_{x=1}$. From the definition of $\tilde\theta(x)$, it follows that

$$\epsilon = \left[1 - \frac{h'(x)x}{h(x)} + \frac{h''(x)x}{h'(x)}\right]\Big|_{x=1} = 1 + \theta + \frac{h''(x)x}{h'(x)}\Big|_{x=1} = 1 + \theta - h''(1)/\theta,$$

and therefore, given the slope of demand θ, ϵ characterizes the curvature (i.e. the second derivative, $h''(1)$) of the demand schedule. We assume that the demand schedule $h(\cdot)$ is log-concave, that is $\epsilon \geq 0$, and therefore $\Gamma \geq 0$, that is the markup decreases with the relative price of the firm, and hence increases with its market share. Note that the $\theta > 1$ requirement corresponds to the second-order condition for the optimal price.

Solving the equations above for p_{Ht} and p_{Ht}^*, we arrive at

$$p_{Ht}(i) \equiv p_{Ht} = (1-\alpha)mc_t + \alpha p_t, \tag{A22}$$
$$p_{Ht}^*(i) \equiv p_{Ht}^* = (1-\alpha)(mc_t - e_t) + \alpha p_t^*, \tag{A23}$$

where the coefficient $\alpha = \frac{\Gamma}{1+\Gamma} = \frac{\epsilon}{\epsilon+\theta-1} \in [0,1)$, as $\theta > 1$ and $\epsilon \geq 0$. In particular, $\alpha = 0$ *iff* $\epsilon = 0$, which also implies $\tilde\theta(x) \equiv \theta = const$, that is a constant elasticity (CES) demand.

Example 1: CES Consider the case of CES demand, which obtains when $g(z) = 1 + \frac{\theta}{\theta-1}\left(z^{\frac{\theta-1}{\theta}} - 1\right)$, which is normalized to satisfy $g(1) = g'(1) = 1$. In this case, $g'(z) = z^{-1/\theta}$ and $h(x) = x^{-\theta}$, so that $g(h(x)) = -\frac{1}{\theta-1} + \frac{\theta}{\theta-1}x^{1-\theta}$. As a result, equations (A18)–(A19) can be solved to yield:

$$P_t = \mathcal{P}_t = \left[\int_0^1 \left[(1-\gamma)P_{Ht}(i)^{1-\theta} + \gamma P_{Ft}(i)^{1-\theta}\right]\mathrm{d}i\right]^{1/(1-\theta)}.$$

Note that in the CES model $\tilde\theta(x) = -\frac{\partial\log h(x)}{\partial\log x} \equiv \theta = const$, implying $\epsilon = 0$, and therefore $\mu(x) \equiv \frac{\theta}{\theta-1} = const$ and $\Gamma = \alpha = 0$.

Example 2: Klenow and Willis (2016) Consider the demand structure implicitly defined by the demand schedule $h(x) = [1 - \epsilon\log(x)]^{\theta/\epsilon}$ for some elasticity parameter $\theta > 1$ and super-elasticity parameter $\epsilon > 0$. This demand structure has been originally developed by Klenow and Willis (2016) and was later used in Gopinath and Itskhoki (2010) in the context of exchange rate transmission. Note that it is indeed the case that $h(1) = 1$, $h'(1) = -\theta$, $\tilde\theta(x) = -\frac{\partial\log h(x)}{\partial\log x} = \frac{\theta}{1-\epsilon\log x}$ and

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$\frac{\partial \log \tilde{\theta}(x)}{\partial \log x}\big|_{x=1} = \frac{\epsilon}{1-\epsilon \log x}\big|_{x=1} = \epsilon$. Therefore, parameter θ controls the local slope of the demand schedule, while parameter ϵ controls its local curvature, and the two can be chosen independently. As $\varepsilon \to 0$, the demand schedule converges to CES demand, $h(x) \to x^{-\theta}$. The preference aggregator $g(\cdot)$ corresponding to the Klenow-Willis demand schedule $h(\cdot)$ is well-defined, but is not an analytical function. Therefore, there is no analytical characterization of $\mathcal{P}_t$ and P_t in the general case, yet the general result that P_t and $\mathcal{P}_t$ are both first-order equivalent to the sales-weighted average industry price still holds.[43] With this demand structure, we have $\mu(x) = \frac{\tilde{\theta}(x)}{\tilde{\theta}(x)-1} = \frac{\theta}{\theta-1+\epsilon \log x}$, which results in $\Gamma = -\frac{\partial \log \mu(x)}{\partial \log x}\big|_{x=1} = \frac{\epsilon}{\theta-1} > 0$. Therefore, indeed, Γ and α are shaped by the primitive parameters of demand ϵ and θ, and any value of $\alpha = \frac{\epsilon}{\epsilon+\theta-1} > 0$ can be obtained independently of the value of θ by setting $\epsilon = \frac{\alpha}{1-\alpha}(\theta - 1) > 0$. In the CES limit, as $\epsilon \to 0$, we have $\alpha \to 0$.

Country budget constraint and Walras law. Aggregating firm profits (A10) across all domestic firms:

$$\begin{aligned}\Pi_t &= (P_{Ht} - MC_t)\, Y_{Ht} + (P^*_{Ht}\mathcal{E}_t - MC_t)\, Y^*_{Ht} \\ &= P_{Ht}Y_{Ht} + \mathcal{E}_t P^*_{Ht} Y^*_{Ht} - MC_t Y_t \\ &= P_{Ht}Y_{Ht} + \mathcal{E}_t P^*_{Ht} Y^*_{Ht} - W_t L_t - R^K_t K_t - P_t X_t,\end{aligned}$$

where we used in the second line the fact that total output $Y_t = Y_{Ht} + Y^*_{Ht}$ and in the third line the expressions for the aggregate production demand for labor

$$W_t L_t = (1-\vartheta)(1-\phi) MC_t\, Y_t, \tag{A24}$$

and analogous conditions for capital and intermediates:

$$R^K_t K_t = (1-\phi)\vartheta MC_t\, Y_t \qquad \text{and} \qquad P_t X_t = \phi MC_t\, Y_t. \tag{A25}$$

We next substitute Π_t into the household budget constraint (1), resulting after rearranging in:

$$\frac{B_{t+1}}{R_t} - B_t = P_{Ht}Y_{Ht} + \mathcal{E}_t P^*_{Ht} Y^*_{Ht} - P_t(C_t + X_t + Z_t).$$

Finally, we use the fact that total domestic expenditure can be split into the expenditure on the home and the foreign goods, $P_t(C_t+X_t+Z_t) = P_{Ht}Y_{Ht}+P_{Ft}Y_{Ft}$, as ensured by expenditure minimization (A4) and market clearing (A11) and the foreign counterpart to (A12), which implies $Y_{Ft} = C_{Ft}+X_{Ft}+Z_{Ft}$. As a result, we can rewrite:

$$\frac{B_{t+1}}{R_t} - B_t = \mathcal{E}_t P^*_{Ht} Y^*_{Ht} - P_{Ft}Y_{Ft} \equiv NX_t, \tag{A26}$$

which yields (11) in the text.

A parallel expression to (A26) for foreign yields:

$$\frac{B^*_{t+1}}{R^*_t} - B^*_t = NX^*_t + \tilde{R}^*_t \frac{D^*_t + N^*_t}{R^*_{t-1}},$$

where $NX^*_t = -\frac{NX_t}{\mathcal{E}_t} = \frac{P_{Ft}}{\mathcal{E}_t} Y_{Ft} - P^*_{Ht} Y^*_{Ht}$ and the last term is the period t realized income (or loss)

[43] A special case with a tractable analytical solution obtains when $\theta = \epsilon$. In this case, $\mathcal{P}_t$ is a simple weighted average of prices $\bar{P}_t$, while P_t equals to $\bar{P}_t$ adjusted downwards by ϵ times the measure of price dispersion (namely, the Theil index), as consumers benefit from a greater price dispersion holding the average price constant.

from the aggregate carry trade position of the financial sector, i.e. the intermediaries and the noise traders combined, transferred lump-sum to the foreign households. Indeed, note that $\frac{D_t^*+N_t^*}{R_{t-1}^*}$ is the dollar exposure of the financial sector from $t-1$ to t and $\tilde{R}_t^* = R_{t-1}^* - R_{t-1}\frac{\mathcal{E}_{t-1}}{\mathcal{E}_t}$ is the realized return at t per one dollar invested in a carry trade at $t-1$.

Using the market clearing condition in the financial sector (15), we have $D_t^* + N_t^* = -B_t^*$, and therefore we can rewrite the foreign country budget constraint as

$$NX_t^* = \frac{B_{t+1}^*}{R_t^*} - B_t^* + \frac{B_t^*}{R_{t-1}^*}\tilde{R}_t^* = \frac{B_{t+1}^*}{R_t^*} - B_t^*\frac{R_{t-1}}{R_{t-1}^*}\frac{\mathcal{E}_{t-1}}{\mathcal{E}_t},$$

where the second equality substitutes in the definitions of the carry trade return $\tilde{R}_t^*$ from (13).

Lastly, since the financial sector holds a zero-capital position, this implies a zero-capital position for the home and foreign households combined, that is $\frac{B_{t+1}}{R_t} + \mathcal{E}_t\frac{B_{t+1}^*}{R_t^*} = 0$ at all t. Applying this market clearing condition at $t-1$ and t to the foreign budget constraint yields after rearranging:

$$\frac{B_{t+1}}{R_t} - B_t = -\mathcal{E}_t NX_t^* = NX_t,$$

which is exactly equivalent to the home country budget constraint (11). Note that this represents a version of *Walras Law* in our economy with the financial sector, making the foreign budget constraint a redundant equation in the equilibrium system.

A.4 Financial sector

This appendix proves the results in Section 2.2; we make use of equations (12)–(16) displayed in the text.

Proof of Lemma 1 The proof of the lemma follows two steps. First, it characterizes the solution to the portfolio problem (14) of the intermediaries. Second, it combines this solution with the financial market clearing (15) to derive the equilibrium condition (16).

(a) Portfolio choice: *The solution to the portfolio choice problem* (14) *when the time periods are short is given by:*

$$\frac{d_{t+1}^*}{P_t^*} = -\frac{i_t - i_t^* - \mathbb{E}_t\Delta e_{t+1} + \frac{1}{2}\sigma_e^2 + \sigma_{e\pi^*}}{\omega\sigma_e^2}, \tag{A27}$$

where $i_t - i_t^* \equiv \log(R_t/R_t^*)$, $\sigma_e^2 \equiv \mathrm{var}_t(\Delta e_{t+1})$ *and* $\sigma_{e\pi^*} = \mathrm{cov}_t(\Delta e_{t+1}, \Delta p_{t+1}^*)$.

Proof: The proof follows Campbell and Viceira (2002, Chapter 3 and Appendix 2.1.1). Consider the objective in the intermediary problem (14) and rewrite it as:

$$\max_{d_{t+1}^*} \mathbb{E}_t\left\{-\frac{1}{\omega}\exp\left(-\omega\big(1-e^{x_{t+1}^*}\big)e^{-\pi_{t+1}^*}\frac{d_{t+1}^*}{P_t^*}\right)\right\}, \tag{A28}$$

where we used the definition of $\tilde{R}_{t+1}^*$ in (13) and the following algebraic manipulation:

$$\frac{\tilde{R}_{t+1}^*}{P_{t+1}^*}\frac{d_{t+1}^*}{R_t^*} = \frac{\tilde{R}_{t+1}^*/R_t^*}{P_{t+1}^*/P_t^*}\frac{d_{t+1}^*}{P_t^*} = \frac{1-\frac{R_{t+1}}{R_t^*}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}}{e^{\pi_{t+1}^*}}\frac{d_{t+1}^*}{P_t^*} = \left(1-e^{x_{t+1}^*}\right)e^{-\pi_{t+1}^*}\frac{d_{t+1}^*}{P_t^*}$$

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and defined the log Carry trade return and foreign inflation rate as

$$x^*_{t+1} \equiv i_t - i^*_t - \Delta e_{t+1} = \log(R_t/R^*_t) - \Delta \log \mathcal{E}_{t+1} \qquad \text{and} \qquad \pi^*_{t+1} \equiv \Delta \log P^*_{t+1}.$$

When time periods are short, (x^*_{t+1}, π^*_{t+1}) correspond to the increments of a vector normal diffusion process $(\mathrm{d}\mathcal{X}^*_t, \mathrm{d}\mathcal{P}^*_t)$ with time-varying drift $\boldsymbol{\mu}_t$ and time-invariant conditional variance matrix $\boldsymbol{\sigma}$:

$$\begin{pmatrix} \mathrm{d}\mathcal{X}^*_t \\ \mathrm{d}\mathcal{P}^*_t \end{pmatrix} = \boldsymbol{\mu}_t \mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}B_t, \tag{A29}$$

where B_t is a standard two-dimensional Brownian motion. Indeed, as we show below, in equilibrium x^*_{t+1} and π^*_{t+1} follow stationary linear stochastic processes (ARMAs) with correlated innovations, and therefore

$$(x^*_{t+1}, \pi^*_{t+1}) \,\big|\, \mathcal{I}_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\sigma}^2),$$

where $\mathcal{I}_t$ is the information set at time t, and the drift and variance matrix are given by:

$$\boldsymbol{\mu}_t = \mathbb{E}_t \begin{pmatrix} x^*_{t+1} \\ \pi^*_{t+1} \end{pmatrix} = \begin{pmatrix} i_t - i^*_t - \mathbb{E}_t \Delta e_{t+1} \\ \mathbb{E}_t \pi^*_{t+1} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\sigma}^2 = \mathrm{var}_t \begin{pmatrix} x^*_{t+1} \\ \pi^*_{t+1} \end{pmatrix} = \begin{pmatrix} \sigma_e^2 & -\sigma_{e\pi^*} \\ -\sigma_{e\pi^*} & \sigma^2_{\pi^*} \end{pmatrix},$$

where $\sigma_e^2 \equiv \mathrm{var}_t(\Delta e_{t+1})$, $\sigma^2_{\pi^*} \equiv \mathrm{var}_t(\Delta p^*_{t+1})$ and $\sigma_{e\pi^*} \equiv \mathrm{cov}_t(\Delta e_{t+1}, \Delta p^*_{t+1})$ are time-invariant (annualized) conditional second moments. Following Campbell and Viceira (2002), we treat (x^*_{t+1}, π^*_{t+1}) as discrete-interval differences of the continuous process, $(\mathcal{X}^*_{t+1} - \mathcal{X}^*_t, \mathcal{P}^*_{t+1} - \mathcal{P}^*_t)$. With short time periods, the solution to (A28) is equivalent to

$$\max_{d^*} \mathbb{E}_t \left\{ -\frac{1}{\omega} \exp\left(-\omega \big(1 - e^{\mathrm{d}\mathcal{X}^*_t}\big) e^{-\mathrm{d}\mathcal{P}^*_t} \frac{d^*}{P^*_t} \right) \right\}, \tag{A30}$$

where $(\mathrm{d}\mathcal{X}^*_t, \mathrm{d}\mathcal{P}^*_t)$ follow (A29). Using Ito's Lemma, we rewrite the objective as:

$$\begin{aligned}
&\mathbb{E}_t \left\{ -\frac{1}{\omega} \exp\left(-\omega \Big(-\mathrm{d}\mathcal{X}^*_t - \tfrac{1}{2}(\mathrm{d}\mathcal{X}^*_t)^2 \Big) \Big(1 - \mathrm{d}\mathcal{P}^*_t + \tfrac{1}{2}(\mathrm{d}\mathcal{P}^*_t)^2 \Big) \frac{d^*}{P^*_t} \right) \right\} \\
&= \mathbb{E}_t \left\{ -\frac{1}{\omega} \exp\left(-\omega \Big(-\mathrm{d}\mathcal{X}^*_t - \tfrac{1}{2}(\mathrm{d}\mathcal{X}^*_t)^2 + \mathrm{d}\mathcal{X}^*_t \mathrm{d}\mathcal{P}^*_t \Big) \frac{d^*}{P^*_t} \right) \right\} \\
&= -\frac{1}{\omega} \exp\left(\left[\omega \big(\mu_{1,t} + \tfrac{1}{2}\sigma_e^2 + \sigma_{e\pi^*} \big) \frac{d^*}{P^*_t} + \frac{\omega^2 \sigma_e^2}{2} \left(\frac{d^*}{P^*_t} \right)^2 \right] \mathrm{d}t \right),
\end{aligned}$$

where the last line uses the facts that $(\mathrm{d}\mathcal{X}^*_t)^2 = \sigma_e^2 \mathrm{d}t$ and $\mathrm{d}\mathcal{X}^*_t \mathrm{d}\mathcal{P}^*_t = -\sigma_{e\pi^*}\mathrm{d}t$, as well as the property of the expectation of an exponent of a normally distributed random variable; $\mu_{1,t}$ denotes the first component of the drift vector $\boldsymbol{\mu}_t$. Therefore, maximization in (A30) is equivalent to:

$$\max_{d^*} \left\{ -\omega \big(\mu_{1,t} + \tfrac{1}{2}\sigma_e^2 + \sigma_{e\pi^*} \big) \frac{d^*}{P^*_t} - \tfrac{1}{2}\omega^2 \sigma_e^2 \left(\frac{d^*}{P^*_t} \right)^2 \right\} \quad \text{w/solution} \quad \frac{d^*}{P^*_t} = -\frac{\mu_{1,t} + \frac{1}{2}\sigma_e^2 + \sigma_{e\pi^*}}{\omega \sigma_e^2}.$$

This is the portfolio choice equation (A27), which obtains under CARA utility in the limit of short time periods, but note is also equivalent to the exact solution under mean-variance preferences. The extra terms in the numerator correspond to Jensen's Inequality corrections to the expected real log return on the carry trade. ■

(b) Equilibrium condition: To derive the modified UIP condition (16), we combine the portfolio choice solution (A27) with the market clearing condition (15) and the noise-trader currency demand (12) to obtain:

$$B^*_{t+1} + R^*_t n\big(e^{\psi_t} - 1\big) - mP^*_t \frac{i_t - i^*_t - \mathbb{E}_t\Delta e_{t+1} + \frac{1}{2}\sigma_e^2 + \sigma_{e\pi^*}}{\omega\sigma_e^2} = 0. \tag{A31}$$

The market clearing conditions in (15) together with the fact that both intermediaries and noise traders take zero capital positions, that is $\frac{D_{t+1}+N_{t+1}}{R_t} = -\mathcal{E}_t \frac{D^*_{t+1}+N^*_{t+1}}{R^*_t}$, results in the equilibrium balance between home and foreign household asset positions, $\frac{B_{t+1}}{R_t} = -\mathcal{E}_t \frac{B^*_{t+1}}{R^*_t}$. Therefore, we can rewrite (A31) as:

$$\frac{i_t - i^*_t - \mathbb{E}_t\Delta e_{t+1} + \frac{1}{2}\sigma_e^2 + \sigma_{e\pi^*}}{\omega\sigma_e^2/m} = \frac{R^*_t}{P^*_t} n\big(e^{\psi_t} - 1\big) - \frac{R^*_t}{R_t}\frac{Y_t}{\mathcal{Q}_t}\frac{B_{t+1}}{P_t Y_t},$$

where we normalized net foreign assets by nominal output $P_t Y_t$ and used the definition of the real exchange rate $\mathcal{Q}_t$. We next log-linearize this equilibrium condition around a symmetric equilibrium with $\bar{R} = \bar{R}^* = 1/\beta$, $\bar{B} = \bar{B}^* = 0$, $\mathcal{Q} = 1$, and $\bar{P} = \bar{P}^* = 1$ and some $\bar{Y}$. As shocks become small, the (co)variances σ_e^2 and $\sigma_{e\pi^*}$ become second order and drop out from the log-linearization. We adopt the asymptotics in which as σ_e^2 shrinks ω/m increases proportionally leaving the risk premium term $\omega\sigma_e^2/m$ constant, finite and nonzero in the limit. As a result, the log-linearized equilibrium condition is:

$$\frac{1}{\omega\sigma_e^2/m}\Big(i_t - i^*_t - \mathbb{E}_t\Delta e_{t+1}\Big) = \frac{n}{\beta}\psi_t - \bar{Y} b_{t+1}, \tag{A32}$$

where $b_{t+1} = \frac{1}{\bar{P}\bar{Y}} B_{t+1} = -\frac{1}{\bar{P}\bar{Y}} B^*_{t+1}$. This corresponds to the modified UIP condition (16) in Lemma 1, which completes the proof of the lemma.[44] ■

Income and losses in the financial market Consider the income and losses of the non-household participants in the financial market — the intermediaries and the noise traders:

$$\frac{D^*_{t+1} + N^*_{t+1}}{R^*_t}\tilde{R}^*_{t+1} = \Big(md^*_{t+1} + R^*_t n(e^{\psi_t} - 1)\Big)\big(1 - e^{x_{t+1}}\big),$$

where we used the definition of $\tilde{R}^*_{t+1}$ in (13) and the log Carry trade return $x_{t+1} \equiv i_t - i^*_t - \mathbb{E}_t\Delta e_{t+1} = \log(R_t/R^*_t) - \Delta\log\mathcal{E}_{t+1}$. Using the same steps as in the proof of Lemma 1, we can approximate this

[44] Note that σ_e^2/m is the quantity of risk per intermediary and ω is their aversion to risk; alternatively, ω/m can be viewed as the effective risk aversion of the whole sector of intermediaries who jointly hold all exchange rate risk. Our approach follows Hansen and Sargent (2011) and Hansen and Miao (2018), who consider the continuous-time limit in the models with ambiguity aversion. The economic rationale of this asymptotics is not that second moments are zero and effective risk aversion ω/m is infinite, but rather that risk premia terms, which are proportional to $\omega\sigma_e^2/m$, are finite and nonzero. Indeed, the first-order dynamics of the equilibrium system results in well-defined second moments of the variables, including σ_e^2, as in Devereux and Sutherland (2011) and Tille and van Wincoop (2010); an important difference of our solution concept is that it allows for a non-zero first-order component of the return differential, namely a non-zero expected Carry trade return. We characterize the equilibrium σ_e^2 in Appendix A.5.

income as:

$$\left(-m\frac{\mathbb{E}_t x_{t+1}}{\omega\sigma_e^2}+\frac{n}{\beta}\psi_t\right)\big(-x_{t+1}\big)=m\left(\frac{\mathbb{E}_t x_{t+1}}{\omega\sigma_e^2}-\frac{n}{\beta m}\psi_t\right)x_{t+1}=-\bar{Y}b_{t+1}x_{t+1},$$

where the last equality uses (A32). Therefore, while the UIP deviations (realized x_{t+1} and expected $\mathbb{E}_t x_{t+1}$) are first order, the income and losses in the financial markets are only second order, as $B_{t+1}=\bar{P}\bar{Y}b_{t+1}$ is first order around $\bar{B}=0$. Intuitively, the income and losses in the financial market are equal to the realized UIP deviation times the gross portfolio position — while both are first order, their product is second order, and hence negligible from the point of view of the country budget constraint.

Covered interest parity Consider the extension of the portfolio choice problem (14) of the intermediaries with the additional option to invest in the CIP deviations:

$$\max_{d^*_{t+1},d^{F*}_{t+1}}\ \mathbb{E}_t\left\{-\frac{1}{\omega}\exp\left(-\omega\left[\frac{\tilde{R}^*_{t+1}}{P^*_{t+1}}\frac{d^*_{t+1}}{R^*_t}+\frac{R^{F*}_t}{P^*_{t+1}}\frac{d^{F*}_{t+1}}{R^*_t}+\frac{R^*_t}{P^*_{t+1}}\mathcal{W}^*_t\right]\right)\right\},$$

where the return on one dollar invested in the CIP deviation (long foreign-currency bond, short foreign-currency bond, plus a forward) is:

$$R^{F*}_t=R^*_t-\frac{\mathcal{E}_t}{\mathcal{F}_t}R_t,$$

since 1 dollar at t buys R^*_t units of foreign-currency bonds and $\mathcal{E}_tR_t$ units of home-currency bonds, and hence $\frac{d^{F*}_{t+1}}{R^*_t}\gtrless 0$ is the period-t dollar size of this position. Note that we also allowed for nonzero dollar wealth $\mathcal{W}^*_t$ of the intermediaries, which is by default invested into the 'riskless' foreign-currency bond. Both CIP investment and wealth investment are subject to the foreign inflation risk only, but no risk of nominal return, unlike the carry trade d^*_{t+1}. Note that the CIP investment, just like the carry trade, requires no capital at time t. Lastly, note that intermediaries may be pricing the forward without trading it, or trading it with the noise traders; as long as the households have access to the home-currency bond only, and not the forward, this does not change the macro equilibrium outcomes of the model.

The first order optimality condition of the intermediaries with respect to the CIP investment is:

$$R^{F*}_t\cdot\mathbb{E}_t\left\{\frac{1}{P^*_{t+1}R^*_t}\exp\left(-\omega\left[\frac{\tilde{R}^*_{t+1}}{P^*_{t+1}}\frac{d^*_{t+1}}{R^*_t}+\frac{R^{F*}_t}{P^*_{t+1}}\frac{d^{F*}_{t+1}}{R^*_t}+\frac{R^*_t}{P^*_{t+1}}\mathcal{W}^*_t\right]\right)\right\}=0.$$

However, since the expectation term is strictly positive for any $d^{F*}_{t+1}\in(-\infty,\infty)$, this condition can be satisfied only if $R^{F*}_t=0$. If $R^{F*}_t>0$, the intermediaries will take an unbounded position in the CIP trade, $d^{F*}_{t+1}=\infty$, and vice versa.

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A.5 Equilibrium system

We summarize here the equilibrium system of the full flexible-price model by breaking it into blocks. The version of the model with sticky prices and wages is described in Appendix A.7.

1. **Labor market**: Labor supply (2) and its exact foreign counterpart. Labor demand in (A24), used together with the definition of the marginal cost (A8), and its exact foreign counterparts. Labor market clearing ensures that L_t (L_t^* respectively) satisfies simultaneously labor demand and labor supply at the equilibrium wage rate W_t (W_t^* respectively).

2. **Capital market**: Euler equation for capital (A16) determines supply of capital and the firm capital demand is given by the first-order condition:

$$R_t^K K_t = (1-\phi)\vartheta MC_t Y_t, \tag{A33}$$

where marginal cost MC_t is defined in (A8). The equilibrium rental rate of capital R_t^K ensures that K_t satisfies simultaneously the demand and supply of capital. Identical equations characterize equilibrium in the foreign capital market. The home gross investment Z_t obtains from the capital dynamics equation, which we rewrite here as:

$$Z_t = \Big[K_{t+1} - (1-\delta)K_t\Big] + \frac{\kappa}{2}\frac{(\Delta K_{t+1})^2}{K_t}, \tag{A34}$$

where the first term is net investment and the second term is adjustment cost. The foreign investment Z_t^* satisfies a symmetric equation.

3. **Goods prices**: Price setting is characterized by (A20) and (A21) for home firms in the two markets, and symmetric equations characterize price setting by foreign firms. The price indexes P_t and $\mathcal{P}_t$ are defined implicitly by (A19)–(A18) respectively. As a result, equilibrium prices of all varieties supplied from a given country to a given market are the same: $P_{Jt}(i) = P_{Jt}$ and $P_{Jt}^*(i) = P_{Jt}^*$ for all $i \in [0,1]$ and $J \in \{H, F\}$.

4. **Goods market**: As a result of price setting, the quantities supplied by all firms from a given country to a given market are also the same: $Y_{Jt}(i) = Y_{Jt}$ and $Y_{Jt}^*(i) = Y_{Jt}^*$ for all $i \in [0,1]$ and $J \in \{H, F\}$. The total demand for home and foreign goods satisfies:

$$Y_t = Y_{Ht} + Y_{Ht}^* \qquad \text{and} \qquad Y_t^* = Y_{Ft} + Y_{Ft}^*, \tag{A35}$$

where the sources of demand for home goods are given in (A11) and (A12), and the counterpart sources of demand for foreign goods are given by:

$$Y_{Ft} = C_{Ft} + X_{Ft} + Z_{Ft} = \gamma h\left(\frac{P_{Ft}}{\mathcal{P}_t}\right)\big[C_t + X_t + Z_t\big], \tag{A36}$$

$$Y_{Ft}^* = C_{Ft}^* + X_{Ft}^* + Z_{Ft}^* = \gamma h\left(\frac{P_{Ft}^*}{\mathcal{P}_t^*}\right)\big[C_t^* + X_t^* + Z_t^*\big], \tag{A37}$$

where X_t is the intermediate good demand by the home firms:

$$P_t X_t = \phi MC_t Y_t, \tag{A38}$$

with MC_t defined in (A8), and a symmetric equation characterizes the intermediate good demand by the foreign firms X_t^*. The total supply (production) of the home goods Y_t satisfies the production function (A7), with log productivity a_t that follows an exogenous shock process (A9).[45] A symmetric equation and foreign productivity process a_t^* characterize foreign production Y_t^*.

5. **Asset market**: The only traded assets are home- and foreign-currency bonds, which are in zero net supply according to market clearing (15). The demand for home-currency bonds by home households B_{t+1} satisfies the Euler equation (3) given the nominal interest rate R_t. Similarly, the demand for foreign-currency bonds by foreign households B_{t+1}^* satisfies a symmetric Euler equation given foreign nominal interest rate R_t^*. The demand for bonds by noise traders and arbitrageurs are characterized by Lemma 1 respectively. The noise trader shock follows an exogenous process (12). No other assets are traded.

   Nominal interest rates are set by the monetary authorities according to the Taylor rule (A15) — where $i_t \equiv \log R_t$, $\pi_t \equiv \Delta \log P_t$ and an exogenous random shock ε_t^m — and its foreign counterpart.

6. **Country budget constraint**: The home-country flow budget constraint (A26) derives from the combination of the household budget constraint and firm profits. The flow budget constraint (A26), together with the household Euler equation (3) and its foreign counterpart, establishes a condition on the path of consumption and nominal exchange rate, $\{C_t, C_t^*, \mathcal{E}_t\}$. The foreign flow-budget constraint is redundant by Walras Law (see Appendix A.3).

Symmetric steady state In a symmetric steady state, exogenous shocks $a_t = a_t^* = \varepsilon_t^m = \varepsilon_t^{m*} = \psi_t \equiv 0$, and state variables $\bar{B} = \bar{B}^* = \bar{NX} = 0$. This is the unique steady state in a model with $\chi_2 > 0$ in (16), which also ensures stationarity of the model around this steady state. We also for concreteness normalize $\bar{P} = \bar{P}^* = 1$. Then, from Euler equations (3) and (A16) and their foreign counterparts, we have:

$$\bar{R} = \bar{R}^* = \bar{R}^K + 1 - \delta = \bar{R}^{K*} + 1 - \delta = \frac{1}{\beta}.$$

By symmetry, the exchange rates and terms of trade satisfy

$$\bar{\mathcal{E}} = \bar{\mathcal{Q}} = \bar{\mathcal{S}} = 1,$$

and all individual prices are equal 1 (the price level). Denote the steady state markup with $\bar{\mu} \geq 1$, so that the steady state marginal costs $\bar{MC} = \bar{MC}^* = 1/\bar{\mu}$, which allows to solve for $\bar{W} = \bar{W}^*$ given $\bar{R} = 1/\beta$ and $\bar{P} = 1$ from (A8) as a function of model parameters.

Next, product and factor market clearing in a symmetric steady state requires:

$$\bar{Y} = \bar{C} + \bar{X} + \delta \bar{K},$$
$$\bar{Y} = \left(\bar{K}^{\vartheta} \bar{L}^{1-\vartheta}\right)^{1-\phi} \bar{X}^{\phi},$$

$$\bar{X} = \tfrac{\phi}{\bar{\mu}} \bar{Y},$$
$$\left(\frac{1-\beta}{\beta} + \delta\right) \bar{K} = \tfrac{(1-\phi)\vartheta}{\bar{\mu}} \bar{Y},$$
$$\bar{C}^{\sigma} \bar{L}^{1/\nu} = \tfrac{(1-\phi)(1-\vartheta)}{\bar{\mu}} \frac{\bar{Y}}{\bar{L}},$$

[45] Note that the input demand equations (A24), (A33) and (A38) together with the definition of the marginal cost (A8) imply the production function in (A7).

since $\bar{Z} = \delta\bar{K}$. These equations allow to solve for $(\bar{Y}, \bar{C}, \bar{L}, \bar{K}, \bar{X})$ and their symmetric foreign counterparts as a function of the model parameters.

Lastly, we define the following useful ratios in a symmetric steady state:

$$\zeta \equiv \frac{\text{GDP}}{\text{Output}} = \frac{P(C+Z)}{PY} = \frac{\bar{C}+\delta\bar{K}}{\bar{Y}} = \frac{\bar{Y}-\bar{X}}{\bar{Y}} = 1 - \frac{\phi}{\bar{\mu}}, \tag{A39}$$

$$\gamma \equiv \frac{\text{Import}}{\text{Expenditure}} = \frac{P_F Y_F}{P_H Y_H + P_F Y_F} = \frac{\bar{Y}_F}{\bar{Y}} = \gamma, \tag{A40}$$

$$\frac{\text{Import+Export}}{\text{GDP}} = \frac{\mathcal{E} P_H^* Y_H^* + P_F Y_F}{P(C+Z)} = \frac{2\bar{Y}_F}{\bar{Y}-\bar{X}} = \frac{2\gamma}{\zeta}. \tag{A41}$$

The steady state markup is $\bar{\mu} = \frac{\theta}{\theta-1}(1-\varsigma)$, where ς is the subsidy that offsets the markup distortion, conventional in the normative macro literature. To avoid the need to calibrate an extra parameter, we assume $\varsigma = 1/\theta$, so that $\bar{\mu} = 1$ and $\zeta = 1 - \phi$, or in words the share of intermediates in output equals the elasticity of the production function with respect to intermediate inputs. The qualitative and quantitative results in our analysis are not sensitive to the departures from $\zeta = 1 - \phi$.

Log-linearized system

We describe here the log-linearized equilibrium system in the model without capital or nominal rigidities and in the limiting case of monetary policy fully stabilizing the price levels, but allowing for Kimball demand and intermediate inputs. The simplified model studied in Sections 2–3 of the paper is the special case with $\alpha = \phi = 0$ (and recall that $\nu \equiv 1/\varphi$). We log-linearize the equilibrium system around the symmetric steady state. We take advantage of the block-recursive structure of the equilibrium system, and characterize the solution in blocks.

Exchange rates and prices The price block contains definitions of the price index at home and abroad:

$$p_t = (1-\gamma)p_{Ht} + \gamma p_{Ft}, \tag{A42}$$

$$p_t^* = \gamma p_{Ht}^* + (1-\gamma)p_{Ft}^*, \tag{A43}$$

as well as the price setting equations:

$$p_{Ht} = (1-\alpha)(1-\phi)(w_t - p_t - a_t) + p_t, \tag{A44}$$

$$p_{Ht}^* = (1-\alpha)[(1-\phi)(w_t - p_t - a_t) + p_t - e_t] + \alpha p_t^*, \tag{A45}$$

$$p_{Ft}^* = (1-\alpha)(1-\phi)(w_t^* - p_t^* - a_t^*) + p_t^*, \tag{A46}$$

$$p_{Ft} = (1-\alpha)[(1-\phi)(w_t^* - p_t^* - a_t^*) + p_t^* + e_t] + \alpha p_t. \tag{A47}$$

Note that small letters denote log-deviations from steady state, and therefore constant terms drop out from equations (A44)–(A47). In addition, we use the logs of the real exchange rate (RER) and the terms of trade (ToT):

$$q_t = p_t^* + e_t - p_t, \tag{A48}$$

$$s_t = p_{Ft} - p_{Ht}^* - e_t, \tag{A49}$$

as well as the wage-based and PPI-based real exchange rates:

$$q_t^W = w_t^* + e_t - w_t, \tag{A50}$$
$$q_t^P = p_{Ft}^* + e_t - p_{Ht}. \tag{A51}$$

We solve (A42)–(A51) for equilibrium prices and exchange rates. In particular, we have:

$$s_t = q_t^P - 2\alpha q_t \qquad \text{and} \qquad q_t = (1-\gamma)q_t^P - \gamma s_t,$$

where αq_t equals the equilibrium LOP deviation for both home- and foreign-produced goods:

$$\alpha q_t = p_{Ht}^* + e_t - p_{Ht} = p_{Ft}^* + e_t - p_{Ft},$$

as follows from (A44)–(A47). Intuitively, ToT equals PPI-RER adjusted for LOP deviations; and CPI-RER equals PPI-RER adjusted for ToT. Using these relationships, we solve for s_t and q_t^P as a function of q_t:

$$s_t = \frac{1-2\alpha(1-\gamma)}{1-2\gamma} q_t, \tag{A52}$$
$$q_t^P = \left[1 + (1-\alpha)\frac{2\gamma}{1-2\gamma}\right] q_t. \tag{A53}$$

Finally, we combine (A42)–(A47) to derive the relationship between q_t^W and q_t:

$$q_t^W = \left[1 + \frac{1}{1-\phi}\frac{2\gamma}{1-2\gamma}\right] q_t - (a_t - a_t^*), \tag{A54}$$

and in addition we have the expressions for the equilibrium real wages:

$$w_t - p_t = a_t - \frac{1}{1-\phi}\frac{\gamma}{1-2\gamma} q_t \qquad \text{and} \qquad w_t^* - p_t^* = a_t^* + \frac{1}{1-\phi}\frac{\gamma}{1-2\gamma} q_t. \tag{A55}$$

Intuitively, the real wage reflects the country productivity level adjusted by the international purchasing power of the country, which is proportional to the strength of its RER.

Real exchange rate and quantities The labor supply (2) and labor demand (A24) equations (together with the marginal cost (A8)) can be written as:

$$\sigma c_t + \tfrac{1}{\nu}\ell_t = w_t - p_t,$$
$$\ell_t = -(1-\phi)a_t - \phi(w_t - p_t) + y_t.$$

Combining the two to solve out ℓ_t, and using (A55) to solve out $(w_t - p_t)$, we obtain:

$$\nu\sigma c_t + y_t = (1+\nu)a_t - \tfrac{\nu+\phi}{1-\phi}\tfrac{\gamma}{1-2\gamma} q_t.$$

Subtracting a symmetric equation for foreign yields:

$$\nu\sigma\tilde{c}_t + \tilde{y}_t = (1+\nu)\tilde{a}_t - \tfrac{\nu+\phi}{1-\phi}\tfrac{2\gamma}{1-2\gamma} q_t, \tag{A56}$$

where $\tilde{x}_t \equiv x_t - x_t^*$ for any pair of variables (x_t, x_t^*). This characterizes the supply side.

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The demand side is the goods market clearing (A35) together with (A11)–(A12), which log-linearize as:

$$\begin{aligned}
y_t &= (1-\gamma)y_{Ht} + \gamma y^*_{Ht},\\
y_{Ht} &= -\theta(p_{Ht}-p_t) + (1-\phi)c_t + \phi\big[(1-\phi)(w_t-p_t-a_t)+y_t\big],\\
y^*_{Ht} &= -\theta(p^*_{Ht}-p^*_t) + (1-\phi)c^*_t + \phi\big[(1-\phi)(w^*_t-p^*_t-a^*_t)+y^*_t\big],
\end{aligned}$$

where $\phi = 1-\zeta \equiv \bar{X}/\bar{Y}$, and we used expression (A38) and (A8) to substitute for X_t (and correspondingly for X^*_t). Combining together, we derive:

$$y_t = \phi[y_t - \gamma\tilde{y}_t] + (1-\phi)[c_t - \gamma\tilde{c}_t] + \gamma\left[\theta(1-\alpha)\tfrac{2(1-\gamma)}{1-2\gamma} - \phi\right]q_t$$

where we have solved out $(p_{Ht}-p_t)$ and $(p^*_{Ht}-p^*_t)$ using (A44)–(A47) and $(w_t-p_t-a_t)$ and $(w^*_t-p^*_t-a^*_t)$ using (A55). Adding and subtracting the foreign counterpart, we obtain:

$$[1-(1-2\gamma)\phi]\tilde{y}_t = (1-2\gamma)(1-\phi)\tilde{c}_t + 2\gamma\left[\theta(1-\alpha)\tfrac{2(1-\gamma)}{1-2\gamma} - \phi\right]q_t. \tag{A57}$$

Combining (A56) and (A57) we can solve for $\tilde{y}_t$ and $\tilde{c}_t$. For example, the expression for $\tilde{c}_t$ is:

$$\tilde{c}_t = \kappa_a\tilde{a}_t - \gamma\kappa_q q_t, \qquad \text{where} \tag{A58}$$

$$\kappa_a \equiv \frac{(1+\nu)(1+\varkappa)}{1+\nu\sigma(1+\varkappa)} \quad\text{and}\quad \kappa_q \equiv \frac{\varkappa}{\gamma}\frac{2\theta(1-\alpha)\frac{1-\gamma}{1-2\gamma}+\nu(1+\varkappa)+\phi\varkappa}{1+\nu\sigma(1+\varkappa)}, \quad\text{with}\quad \varkappa \equiv \frac{1}{1-\phi}\frac{2\gamma}{1-2\gamma}.$$

Note that $\kappa_a, \kappa_q > 0$ independently of the values of the parameters, and in the autarky limit as $\gamma \to 0$ we have $\kappa_a \to \frac{1+\nu}{1+\nu\sigma}$ and $\kappa_q \to \frac{2}{1-\phi}\frac{2\theta(1-\alpha)+\nu}{1+\nu\sigma}$, since $\frac{\varkappa}{\gamma} \to \frac{2}{1-\phi}$. Therefore, (κ_a, κ_q) are positive derived parameters separated from zero even as $\gamma \to 0$.

Lastly, we log-linearize the flow budget constraint (11) as:

$$\beta b_{t+1} - b_t = nx_t = \gamma\Big(y^*_{Ht} - y_{Ft} - s_t\Big), \tag{A59}$$

where $\beta = 1/\bar{R}$ and since $\bar{B} = \bar{NX} = 0$ in a symmetric steady state, we define $b_{t+1} = B_{t+1}/\bar{Y}$ and $nx_t = NX_t/\bar{Y}$, so that γ represents the steady-state share of imports (and also exports) in output, which is the relevant coefficient in the log-linearization (A59). Next we use the expression for export quantity y^*_{Ht} above and a symmetric counterpart for y_{Ft}, together with the solution for prices and quantities, to derive:[46]

$$nx_t = \gamma\big[\lambda_q q_t - \lambda_a\tilde{a}_t\big], \tag{A60}$$

[46]This is a rather tedious deviations, which relies on the previous equilibrium relationships. We start with:

$$\begin{aligned}
y^*_{Ht} - y_{Ft} - s_t &= -\theta(p^*_{Ht}-p^*) + \theta(p_{Ft}-p) - \phi(1-\phi)(\tilde{w}_t - \tilde{p}_t - \tilde{a}_t) - [\phi\tilde{y}_t + (1-\phi)\tilde{c}_t] - s_t\\
&= -[\theta(1-\alpha)+\phi](1-\phi)(\tilde{w}_t-\tilde{p}_t-\tilde{a}_t) + 2\theta(1-\alpha)q_t - \tfrac{1-2\alpha(1-\gamma)}{1-2\gamma}q_t - [\phi\tilde{y}_t + (1-\phi)\tilde{c}_t]\\
&= \left([\theta(1-\alpha)+\phi]\frac{2\gamma}{1-2\gamma} + 2\theta(1-\alpha) - \tfrac{1-2\alpha(1-\gamma)}{1-2\gamma}\right)q_t - [\phi\tilde{y}_t + (1-\phi)\tilde{c}_t],
\end{aligned}$$

where the first line substitutes the expressions for y^*_{Ht} and y_{Ft}, the second equality uses (A45), (A47) and (A52), where the second equality substitutes in the solution for $(\tilde{y}_t, \tilde{c}_t)$ from (A56)–(A57), and we define:

$$\lambda_a \equiv \frac{1}{1-2\gamma}\frac{1+\nu}{1+\nu\sigma(1+\varkappa)},$$
$$\lambda_q \equiv \frac{1}{1-2\gamma}\left(\frac{1+\nu\sigma}{1+\nu\sigma(1+\varkappa)}\left[2\theta(1-\alpha)\frac{1-\gamma}{1-2\gamma}+\phi\varkappa+\frac{\nu\varkappa}{1+\nu\sigma}\right]-[1-2\alpha(1-\gamma)]\right).$$

Note that $\lambda_a \equiv \frac{1}{1-2\gamma}\frac{\kappa_a}{1+\varkappa} > 0$, and as $\gamma \to 0$ we have $\lambda_a - \kappa_a \to 0$. Furthermore, $\lambda_q > 0$ is equivalent to the generalized *Marshall-Lerner condition* in our general-equilibrium model, and $\theta > 1$ is sufficient to ensure this independently of the values of other parameters $(\gamma, \alpha, \phi, \nu, \sigma)$. In the limit $\gamma \to 0$, we have $\lambda_q \to 1+2(\theta-1)(1-\alpha)$, which is in general different from κ_q.

Exchange rate and interest rates We log-linearize the household Euler equation (3) and its foreign counterpart:

$$i_t = \mathbb{E}_t\left\{\sigma\Delta c_{t+1}+\Delta p_{t+1}\right\} \quad \text{and} \quad i_t^* = \mathbb{E}_t\left\{\sigma\Delta c_{t+1}^*+\Delta p_{t+1}^*\right\},$$

where $i_t \equiv \log R_t - \log \bar{R}$ and similarly for i_t^*. Taking the difference, we can express the interest rate differential as:

$$i_t - i_t^* = \mathbb{E}_t\left\{\sigma\Delta\tilde{c}_{t+1}+\Delta\tilde{p}_{t+1}\right\}, \tag{A61}$$

Subtracting the expected inflation differential, $\mathbb{E}_t\Delta\tilde{p}_{t+1}$, on both sides allows to characterize the equilibrium real interest rate differential as

$$i_t - i_t^* - \mathbb{E}_t\Delta\tilde{p}_{t+1} = \sigma\mathbb{E}_t\Delta\tilde{c}_{t+1} = \sigma\kappa_a\mathbb{E}_t\Delta\tilde{a}_{t+1} - \gamma\sigma\kappa_q\mathbb{E}_t\Delta q_{t+1}, \tag{A62}$$

where we substituted the solution for $\Delta\tilde{c}_{t+1}$ from (A58). To solve for the equilibrium nominal interest rate differential we combine (A61) with the Taylor rule (A15) and its foreign counterpart. Since $\Delta\tilde{p}_{t+1} \equiv 0$, the nominal interest rate (differential) tracks the real interest rate (differential), equal σ times the expected (relative) consumption growth.

Next, subtracting $\mathbb{E}_t\Delta e_{t+1}$ on both sides of (A61) and combining with the modified UIP equation (16), we have:

$$\mathbb{E}_t\left\{\sigma\Delta\tilde{c}_{t+1}-\Delta q_{t+1}\right\} = i_t - i_t^* - \mathbb{E}_t\Delta e_{t+1} = \chi_1\psi_t - \chi_2 b_{t+1}, \tag{A63}$$

which amounts to the international risk-sharing condition in this economy. Combining (A63) with the solution for the equilibrium consumption differential (A58), we arrive at the condition for the expected change in the real exchange rate:

$$(1+\gamma\sigma\kappa_q)\mathbb{E}_t\Delta q_{t+1} = \chi_2 b_{t+1} - \chi_1\psi_t + \sigma\kappa_a\mathbb{E}_t\Delta\tilde{a}_{t+1}. \tag{A64}$$

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and the third equality uses (A55). Next we use (A57) to solve for:

$$\phi\tilde{y}_t+(1-\phi)\tilde{c}_t = \frac{1}{1-2\gamma}\tilde{y}_t - \frac{2\gamma}{1-2\gamma}\left[\theta(1-\alpha)\tfrac{2(1-\gamma)}{1-2\gamma}-\phi\right]q_t$$
$$= \frac{1}{1-2\gamma}\frac{1+\nu}{1+\nu\sigma(1+\varkappa)}\tilde{a}_t - \tfrac{2\gamma}{1-2\gamma}\left[\theta(1-\alpha)\tfrac{2(1-\gamma)}{1-2\gamma}-\phi+\tfrac{1}{1-2\gamma}\tfrac{\nu+\phi}{1-\phi}-\frac{\nu\sigma\kappa_q}{2}\right]q_t,$$

where the second line uses (A56) to solve out $\tilde{y}_t$ and then (A58) to solve out $\tilde{c}_t$. Finally, substituting in the expression for κ_q from (A58) and rearranging terms yields the resulting (A60).

Substituting this into (A62) yields the solution for the interest rate differential:

$$i_t - i_t^* - \mathbb{E}_t\Delta\tilde{p}_{t+1} = -\frac{\gamma\sigma\kappa_q}{1+\gamma\sigma\kappa_q}[\chi_2 b_{t+1} - \chi_1\psi_t] + \frac{\sigma\kappa_a}{1+\gamma\sigma\kappa_q}\mathbb{E}_t\Delta\tilde{a}_{t+1}. \tag{A65}$$

A.6 Derivation of the analytical results in Section 3

Equilibrium exchange rate dynamics (Lemma 3) We first characterize the generalized version of the exchange rate dynamics in (30) in corresponding to the modified UIP condition (16) with endogenous coefficients χ_1 and χ_2. We combine (A64) with (A59)–(A60) and the exogenous shock processes (6) and (12) assuming for concreteness $\rho_a = \rho_\psi = \rho$ (and analogous derivations apply in the general case with $\rho_a \neq \rho_\psi$). This system corresponds to the generalized versions of (17)—(28) in the text.[47] We write the equilibrium dynamic system in matrix form:

$$\begin{pmatrix} 1 & -\hat{\chi}_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{E}_t q_{t+1} \\ \hat{b}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1/\beta \end{pmatrix}\begin{pmatrix} q_t \\ \hat{b}_t \end{pmatrix} - \begin{pmatrix} \hat{\chi}_1 & (1-\rho)k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi_t \\ \hat{a}_t \end{pmatrix},$$

where for brevity we made the following substitution of variables:

$$\hat{b}_t \equiv \frac{\beta}{\gamma\lambda_q}b_t, \quad \hat{a}_t \equiv \frac{\lambda_a}{\lambda_q}\tilde{a}_t, \quad \hat{\chi}_1 \equiv \frac{\chi_1}{1+\gamma\sigma\kappa_q}, \quad \hat{\chi}_2 \equiv \frac{\gamma\lambda_q/\beta}{1+\gamma\sigma\kappa_q}\chi_2, \quad k \equiv \frac{\sigma\kappa_a}{1+\gamma\sigma\kappa_q}\frac{\lambda_q}{\lambda_a}. \tag{A66}$$

Diagonalizing the dynamic system, we have:

$$\mathbb{E}_t z_{t+1} = Bz_t - C\begin{pmatrix} \psi_t \\ \hat{a}_t \end{pmatrix}, \quad \text{where} \quad B \equiv \begin{pmatrix} 1+\hat{\chi}_2 & \hat{\chi}_2/\beta \\ 1 & 1/\beta \end{pmatrix}, \quad C \equiv \begin{pmatrix} \hat{\chi}_1 & (1-\rho)k + \hat{\chi}_2 \\ 0 & 1 \end{pmatrix},$$

and we denoted $z_t \equiv (q_t, \hat{b}_t)'$. The eigenvalues of B are:

$$\mu_{1,2} = \frac{1}{2}\left[(1+\hat{\chi}_2+1/\beta) \mp \sqrt{(1+\hat{\chi}_2+1/\beta)^2 - 4/\beta}\right] \qquad \text{such that} \qquad 0 < \mu_1 \leq 1 < \frac{1}{\beta} \leq \mu_2,$$

and $\mu_1 + \mu_2 = 1 + \hat{\chi}_2 + 1/\beta$ and $\mu_1 \cdot \mu_2 = 1/\beta$. Note that when $\chi_2 = 0$, and hence $\hat{\chi}_2 = 0$, the two roots are simply $\mu_1 = 1$ and $\mu_2 = 1/\beta$.

The left eigenvalue associated with $\mu_2 > 1$ is $v = (1, 1/\beta - \mu_1)$, such that $vB = \mu_2 v$. Therefore, we can pre-multiply the dynamic system by v and rearrange to obtain:

$$vz_t = \frac{1}{\mu_2}\mathbb{E}_t\{vz_{t+1}\} + \frac{1}{\mu_2}\hat{\chi}_1\psi_t + \left[\frac{(1-\rho)k + \hat{\chi}_2}{\mu_2} + \frac{1/\beta - \mu_1}{\mu_2}\right]\hat{a}_t.$$

Using the facts that $\hat{\chi}_2 + 1/\beta - \mu_1 = \mu_2 - 1$ and $1/\mu_2 = \beta\mu_1$, we solve this dynamic equation forward to obtain the equilibrium cointegration relationship:

$$vz_t = q_t + (1/\beta - \mu_1)\hat{b}_t = \frac{\beta\mu_1\hat{\chi}_1}{1-\beta\rho\mu_1}\psi_t + \frac{1-\beta\mu_1+\beta(1-\rho)k\mu_1}{1-\beta\rho\mu_1}\hat{a}_t. \tag{A67}$$

[47] Note that, while monetary policy results in $e_t = q_t$, it does not affect the equilibrium system for RER q_t in the flexible-price version of the model we consider here.

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Combining this with the second dynamic equation for $\hat{b}_{t+1}$, we solve for:

$$\hat{b}_{t+1} - \mu_1 \hat{b}_t = \overbrace{q_t + \big(\tfrac{1}{\beta} - \mu_1\big)\hat{b}_t}^{=vz_t} - \hat{a}_t = \frac{\beta\mu_1\hat{\chi}_1}{1-\beta\rho\mu_1}\psi_t + \frac{\beta(1-\rho)(k-1)\mu_1}{1-\beta\rho\mu_1}\hat{a}_t, \tag{A68}$$

Note that $\hat{b}_{t+1}$ in (A68) follows a stationary AR(2) with roots ρ and μ_1. Recall that as $\chi_2 \to 0$, $\mu_1 \to 1$, and the process for $\hat{b}_{t+1}$ becomes an ARIMA(1,1,0), which corresponds to the solution in footnote 26 in the text (after reverse substitution of variables).

Finally, we apply lag operator $(1-\mu_1 L)$ to (A67) and use (A68) to solve for:

$$\begin{aligned}(1-\mu_1 L)q_t &= (1-\beta^{-1}L)\left[\frac{\beta\mu_1\hat{\chi}_1}{1-\beta\rho\mu_1}\psi_t + \frac{\beta(1-\rho)(k-1)\mu_1}{1-\beta\rho\mu_1}\hat{a}_t\right] + (1-\mu_1 L)\hat{a}_t \\ &= (1-\beta^{-1}L)\left[\frac{\beta\mu_1\hat{\chi}_1}{1-\beta\rho\mu_1}\psi_t + \frac{\beta(1-\rho)\mu_1}{1-\beta\rho\mu_1}k\hat{a}_t\right] + \frac{1-\beta\mu_1}{1-\beta\rho\mu_1}(1-\rho\mu_1 L)\hat{a}_t, \end{aligned} \tag{A69}$$

where L is the lag operator such that $Lq_t = q_{t-1}$. Therefore, equilibrium RER q_t follows a stationary ARMA(2,1) with autoregressive roots μ_1 and ρ. Again, in the limit $\chi_2 \to 0$, $\mu_1 \to 1$, and this process becomes an ARIMA(1,1,1), which corresponds to (30) in the text.[48] ■

Equilibrium variance of the exchange rate and Lemma 1 Solution (A69) characterizes the behavior of q_t for given values of χ_1 and χ_2 (and hence μ_1, μ_2), which from (16) themselves depend on $\sigma_e^2 = \mathrm{var}_t(\Delta e_{t+1})$. Since the monetary policy stabilizes inflation, ensuring $e_t = q_t$, we also have $\sigma_e^2 = \mathrm{var}_t(\Delta q_{t+1})$, and we now solve for the equilibrium value of σ_e^2, and hence $(\chi_1, \chi_2, \mu_1, \mu_2)$.

Using (A69), we calculate $\sigma_e^2 = \mathrm{var}_t(\Delta q_{t+1})$ for given χ_1 and χ_2:

$$\sigma_e^2 = \mathrm{var}_t(\Delta q_{t+1}) = \left(\frac{\beta\mu_1\hat{\chi}_1}{1-\beta\rho\mu_1}\right)^2\sigma_\psi^2 + \left(\frac{\beta(1-\rho)\mu_1 k + (1-\beta\mu_1)}{1-\beta\rho\mu_1}\right)^2\hat{\sigma}_a^2 = \frac{\hat{\chi}_1^2\sigma_\psi^2 + \big((1-\rho)k + (\mu_2-1)\big)^2\hat{\sigma}_a^2}{(\mu_2-\rho)^2},$$

where $\hat{\sigma}_a$ is the standard deviation of the innovation to $\hat{a}_t$, and the second line uses the fact that $\beta\mu_1 = 1/\mu_2$. In addition, recall that:

$$\hat{\chi}_1 = \frac{n/\beta}{1+\gamma\sigma\kappa_q}\frac{\omega\sigma_e^2}{m}, \qquad \hat{\chi}_2 \equiv \frac{\gamma\lambda_q\bar{Y}/\beta}{1+\gamma\sigma\kappa_q}\frac{\omega\sigma_e^2}{m} \quad \text{and} \quad \mu_2 = \frac{(1+\beta\hat{\chi}_2+\beta) + \sqrt{(1+\beta\hat{\chi}_2+\beta)^2 - 4\beta}}{2\beta}.$$

We therefore can rewrite the fixed point equation for $\sigma_e^2 > 0$ as follows:

$$F(x,\tilde{\omega}) = \big(\mu_2(\tilde{\omega}x) - \rho\big)^2 x - b(\tilde{\omega}x)^2 - c = 0, \tag{A70}$$

where we used the following notation:

$$x \equiv \sigma_e^2 \geq 0, \qquad \tilde{\omega} = \frac{\omega}{m}, \qquad b \equiv \left(\frac{n/\beta}{1+\gamma\sigma\kappa_q}\right)^2\sigma_\psi^2, \qquad c \equiv \big((1-\rho)k + (\mu_2-1)\big)^2\hat{\sigma}_a^2 \geq 0,$$

and $\mu_2(\cdot)$ is a function which gives the equilibrium values of μ_2 defined above as a function of $\tilde{\omega}\sigma_e^2$ for

[48] Note that we prefer the specification in the second line of (A69) since it separates the exchange rate effects of $\hat{a}_t$ via the budget constraint (latter term) and via the modified UIP condition (the former term with factor k in front of $\hat{a}_t$). In the limit of $\rho \to 1$, the effect through the UIP condition vanishes, while the effect through the budget constraint results in a random walk response of q_t to $\hat{a}_t$.

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given values of the model parameters. Note that for any given $\tilde{\omega} > 0$:

$$\lim_{x\to 0} F(x,\tilde{\omega}) = -c \leq 0,$$
$$\lim_{x\to\infty} \frac{F(x,\tilde{\omega})}{x^3} = \lim_{x\to\infty} \left(\frac{\mu_2(\tilde{\omega}x)}{x}\right)^2 = \left(\frac{\beta\hat{\chi}_2^2}{\sigma_e^2}\right) = \left(\frac{\gamma\lambda_q\bar{Y}}{1+\gamma\sigma\kappa_q}\tilde{\omega}\right)^2 > 0.$$

Therefore, by continuity at least one fixed-point $F(\sigma_e^2,\omega) = 0$ with $\sigma_e^2 \geq 0$ exists, and all such $\sigma_e^2 > 0$ whenever $c > 0$ (that is, when $\hat{\sigma}_a > 0$). One can further show that when $\hat{\sigma}_a/\sigma_\psi$ is not too small, this equilibrium is unique.[49]

Finally, we consider the limit of log-linearization in Lemma 1, where $(\hat{\sigma}_a,\sigma_\psi) = \sqrt{\xi}\cdot(\bar{\sigma}_a,\bar{\sigma}_\psi) = \mathcal{O}(\sqrt{\xi})$ as $\xi \to 0$, where $(\bar{\sigma}_a,\bar{\sigma}_\psi)$ are some fixed numbers. Then in (A70), $(b,c) = \mathcal{O}(\xi)$, as (b,c) are linear in $(\hat{\sigma}_a^2,\sigma_\psi^2)$. This implies that for any given fixed point $(\bar{\sigma}_e^2,\bar{\omega})$, with $F(\bar{\sigma}_e^2,\bar{\omega};\bar{\sigma}_a^2,\bar{\sigma}_\psi^2) = 0$, there exists a sequence of fixed points $F(\xi\bar{\sigma}_e^2,\bar{\omega}/\xi;\xi\bar{\sigma}_a^2,\xi\bar{\sigma}_\psi^2) = 0$ as $\xi\to 0$, for which $\sigma_e^2 = \xi\bar{\sigma}_e^2 = \mathcal{O}(\xi)$, $\tilde{\omega} = \bar{\omega}/\xi = \mathcal{O}(1/\xi)$ and $\tilde{\omega}\sigma_e^2 = \bar{\omega}\bar{\sigma}_e^2 = const$. To verify this, one can simply divide (A70) by ξ and note that, for a given $\tilde{\omega}x$, $F(x,\tilde{\omega})$ is linear in (x,b,c), which means that the fixed point x scales with (b,c) provided that $\tilde{\omega}x$ stays constant. This confirms the conjecture used in the proof of Lemma 1. ■

Proofs of Propositions In the remainder of the proofs, we specialize to the case of $\chi_2 = 0$ and normalization $\chi_1 = 1$, which we adopted in the text of Section 3. The results, however, obtain under the general case with endogenous nonzero χ_1 and χ_2, as we considered until now. Throughout the proofs we use the notation for $\hat{\chi}_1$, k and $\hat{a}_t$ introduced in (A66) above.

Proof of Lemma 3 The equilibrium process for the nominal exchange rate (30) follows directly from the solution (A69), as $\mu_1 = 1$ when $\chi_2 = 0$, and considering the fact that the assumed monetary policy ensures $e_t = q_t$ (see (17)). This is the unique path of the exchange rate that simultaneously satisfies the modified UIP condition (27) and the country budget constraint (28). Indeed, (27) determines e_t and $\mathbb{E}_t e_{t+j}$ for all $j > 0$ up to a long-run expectation $\mathbb{E}_t e_\infty \equiv \lim_{j\to\infty}\mathbb{E}_t e_{t+j}$, which in turn is uniquely pinned down by the budget constraint, as any departures from $\mathbb{E}_t e_\infty$ result in expected violations of the intertemporal budget of the country.

Proofs of Proposition 1 and 5 As $\beta,\rho\to 1$, we have $\big[(1-\beta^{-1}L)-(1-\rho L)\big]x_t \to 0$ for any stationary x_t, and therefore we can use (30) to show that in this limit:

$$\lim_{\beta,\rho\to 1}\left\{\Delta e_t - \Big[\overbrace{\tfrac{\beta\hat{\chi}_1}{1-\beta\rho}\sigma_\psi\varepsilon_t^\psi + \big(\tfrac{\beta(1-\rho)}{1-\beta\rho}k + \tfrac{1-\beta}{1-\beta\rho}\big)\hat{\sigma}_a\hat{\varepsilon}_t^a}^{\sim iid}\Big]\right\}$$
$$= \lim_{\beta,\rho\to 1}\left\{\big[(1-\beta^{-1}L)-(1-\rho L)\big]\Big(\tfrac{\beta\hat{\chi}_1}{1-\beta\rho}\psi_t + \tfrac{\beta(1-\rho)}{1-\beta\rho}k\hat{a}_t\Big)\right\} = 0,$$

where we used the notation $\hat{\chi}_1$, k and $\hat{a}_t$ from (A66) and denoted with $\hat{\sigma}_a\hat{\varepsilon}_t^a$ the innovation of the $\hat{a}_t$ shock process, so that $\hat{\sigma}_a\hat{\varepsilon}_t^a = (1-\rho L)\hat{a}_t$ and similarly $\sigma_\psi\varepsilon_t^\psi = (1-\rho L)\psi_t$. Therefore, in the limit Δe_t is iid, and hence e_t follows a random walk.[50]

[49]For $\hat{\sigma}_a/\sigma_\psi \approx 0$, there typically exist three equilibria. In particular, when $\hat{\sigma}_a = 0$, there always exists an equilibrium with $\sigma_e^2 = \chi_1 = 0$, in addition to two other potential equilibria with $\sigma_e^2 > 0$, which exist when σ_ψ is not too small (see Itskhoki and Mukhin 2017).

[50]Two remarks are in order. First, as $\beta\rho\to 1$, we should ensure that $\psi_t/(1-\beta\rho)$ does not explode in order to keep the volatility of Δe_t finite; a natural approach is to require that $\sigma_\psi/(1-\beta\rho)$ remains finite in this limit. Second, note that the limiting process for Δe_t depends on the relative speed of convergence of β and ρ to 1, as

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We next prove related limiting results characterizing the second moments of the exchange rate process, by rewriting (30) as follows:

$$\Delta e_t = \tfrac{\beta\hat{\chi}_1}{1-\beta\rho}\Big[\sigma_\psi\varepsilon_t^\psi - \tfrac{1-\beta\rho}{\beta}\psi_{t-1}\Big] + \tfrac{\beta(1-\rho)k}{1-\beta\rho}\Big[\hat{\sigma}_a\hat{\varepsilon}_t^a - \tfrac{1-\beta\rho}{\beta}\hat{a}_{t-1}\Big] + \tfrac{1-\beta}{1-\beta\rho}\hat{\sigma}_a\hat{\varepsilon}_t^a,$$

where again we use notation $\hat{\chi}_1$, k and $\hat{a}_t$ from (A66). We can then calculate:

$$\begin{aligned}
\mathrm{var}(\Delta e_t) &= \tfrac{1+\beta^2-2\beta\rho}{(1-\beta\rho)^2(1-\rho^2)}\big[\hat{\chi}_1^2\sigma_\psi^2 + (1-\rho)^2k^2\hat{\sigma}_a^2\big] + \Big(\tfrac{1-\beta}{1-\beta\rho}\Big)^2\Big[1+\tfrac{2\beta(1-\rho)k}{1-\beta}\Big]\hat{\sigma}_a^2, \\
\mathrm{cov}(\Delta e_t,\Delta e_{t-1}) &= -\tfrac{\beta-\rho}{(1-\beta\rho)(1-\rho^2)}\big[\hat{\chi}_1^2\sigma_\psi^2 + (1-\rho)^2k^2\hat{\sigma}_a^2\big] - \tfrac{(1-\rho)(1-\beta)}{1-\beta\rho}k\hat{\sigma}_a^2, \\
\mathrm{var}_t(\Delta e_{t+1}) &= \Big(\tfrac{\beta}{1-\beta\rho}\Big)^2\hat{\chi}_1^2\sigma_\psi^2 + \Big(\tfrac{\beta(1-\rho)k}{1-\beta\rho} + \tfrac{1-\beta}{1-\beta\rho}\Big)^2\hat{\sigma}_a^2.
\end{aligned}$$

Next, taking first the $\beta\to 1$ limit, we can characterize the following ratios for $\rho<1$ and as $\rho\to 1$:[51]

$$\begin{aligned}
\mathrm{corr}(\Delta e_t,\Delta e_{t-1}) = \frac{\mathrm{cov}(\Delta e_t,\Delta e_{t-1})}{\mathrm{var}(\Delta e_t)} &= -\frac{1-\rho}{2}\to 0, \\
\frac{\mathrm{var}_t(\Delta e_{t+1})}{\mathrm{var}(\Delta e_{t+1})} = \frac{\mathrm{var}(\Delta e_{t+1}-\mathbb{E}_t\Delta e_{t+1})}{\mathrm{var}(\Delta e_{t+1})} &= \frac{1+\rho}{2}\to 1, \\
\frac{\mathrm{var}(\Delta e_t)}{\mathrm{var}(\hat{\chi}_1\psi_t)} &= \frac{2}{1-\rho}\,\frac{\hat{\chi}_1^2\sigma_\psi^2+(1-\rho)^2k^2\hat{\sigma}_a^2}{\hat{\chi}_1^2\sigma_\psi^2}\to\infty, \\
\frac{\mathrm{var}(\Delta e_t|\psi_t)}{\mathrm{var}(\Delta e_t)} &= \frac{\hat{\chi}_1^2\sigma_\psi^2}{\hat{\chi}_1^2\sigma_\psi^2+k^2\hat{\sigma}_a^2(1-\rho)^2}\to 1,
\end{aligned}$$

where we used the fact that $\mathrm{var}(\psi_t)=\sigma_\psi^2/(1-\rho^2)$. These are the results summarized in Proposition 1 and in the text following it. The last result in particular confirms that as $\beta\rho\to 1$, the contribution of the ψ_t shock to the variance of Δe_t fully dominates the contribution of the productivity shock. The second-to-last result confirms that as $\beta\rho\to 1$, an arbitrarily small volatility of the UIP shock $\chi_1\psi_t$ results in an arbitrarily large volatility of Δe_t. The first two results are the confirmation of the limiting random-walk behavior of the exchange rate as $\beta\rho\to 1$.

Lastly, we prove that as γ *decreases* or $\beta\rho$ *increases*, the volatility of the exchange rate response to the UIP shock $\chi_1\psi_t$ *increases*. We denote $\sigma_{e|\psi}^2 \equiv \mathrm{var}(\Delta e_t|\psi_t) = \frac{1+\beta^2-2\beta\rho}{(1-\beta\rho)^2(1-\rho^2)}\hat{\chi}_1^2\sigma_\psi^2$ the variance of the exchange rate conditional on the financial shock ψ_t, or equivalently when we switch off the other shock, $\sigma_a=0$. We characterize:

$$\frac{\mathrm{var}(\Delta e_t|\psi_t)}{\mathrm{var}(\chi_1\psi_t)} = \frac{1+\beta^2-2\beta\rho}{(1-\beta\rho)^2}\,\frac{1}{(1+\gamma\sigma\kappa_q)^2},$$

where we used the notation for $\hat{\chi}_1$ in (A66) and the fact that $\mathrm{var}(\psi_t)=\sigma_\psi^2/(1-\rho^2)$, which already

depending on it $\frac{1-\rho}{1-\beta\rho},\frac{1-\beta}{1-\beta\rho}\in[0,1]$. For example, if we first take $\beta\to 1$, as will be our approach below, then $\Delta e_t\to\frac{\hat{\chi}_1}{1-\rho}\Delta\psi_t + k\Delta\hat{a}_t$, as the last term in (30) disappears already in this first limit, prior to taking $\rho\to 1$.

[51] We focus on this sequence of taking limits as the variance of Δe_t is not well-defined when $\rho=1$, as this results in a double-integrated process for the exchange rate. In contrast, all second moments are well-defined for $\beta=1$, and we can then study their properties as ρ increases towards 1. In addition, this sequence of limits provides a better approximation to our calibrated model, in which we have $\rho<\beta<1$, namely $\rho=0.97$ and $\beta=0.99$.

adjusts for the persistence of the shock.[52] It is immediate to see that the volatility of the exchange rate response always increases in β and increases in ρ iff $\rho < \beta$, which we take as the empirically relevant case. Note that the overall unscaled volatility of the exchange rate, $\sigma^2_{e|\psi} \equiv \text{var}(\Delta e_t|\psi_t)$, always increases in both β and ρ. Finally, we establish that $\gamma\kappa_q$ is increasing in γ, where κ_q is defined in (A58), reducing the volatility of the exchange rate response. We have:

$$\gamma\kappa_q = \varkappa \frac{2\theta(1-\alpha)\frac{1-\gamma}{1-2\gamma} + \nu(1+\varkappa) + \phi\varkappa}{1+\nu\sigma(1+\varkappa)}, \qquad \text{where} \qquad \varkappa \equiv \frac{1}{1-\phi}\frac{2\gamma}{1-2\gamma}.$$

Since $\varkappa$ is increasing in γ on its range $[0, 1/2)$, we only need to establish that the right-hand side is increasing in $\varkappa$, which can be immediately verified by differentiation. This completes the proof of Proposition 1.

To finish the proof of Proposition 5, we make use of (A57)–(A58), which in the limit of closed economy $\gamma \to 0$ becomes:

$$y_t = \frac{1+\nu}{1+\nu\sigma}a_t + \frac{2\gamma}{1-\phi}\frac{\nu\sigma 2\theta(1-\alpha) - \nu - (1+\nu\sigma)\phi}{1+\nu\sigma}q_t,$$

where the term in front of q_t may be positive or negative and is of the order of γ, that is vanishes in the limit if the volatility of q_t is finite. As a result, the volatility of y_t in this case is shaped by productivity shocks a_t alone. Using the results above, it is easy to verify that in the presence of the financial shock, as ρ increases towards 1, the volatility of Δe_t increases without bound relative to the volatility of Δa_t, completing the proof. ■

Proofs of Lemma 2 and Proposition 2 The relationships between nominal exchange rate, real exchange rates and terms of trade are derived above — see (A52)–(A54). The results follow immediately from these equations. In particular, in response to ψ_t, all real exchange rates — q_t, q_t^P and q_t^W — comove perfectly with the nominal exchange rate e_t, which under the consumer-price-stabilizing monetary policy equals RER, $e_t = q_t$. Smaller γ and larger α ensure that the ratios of volatility of q_t^W and q_t^P to that of q_t are closer to 1. In contrast, higher α reduces the relative volatility of the terms of trade s_t relative to q_t.

The proof of Proposition 1 already establishes that, as $\beta\rho \to 1$, the process for $q_t = e_t$ converges to a random walk with an arbitrarily large volatility of $\Delta q_t = \Delta e_t$ (when scaled by the exogenous volatility of the UIP shock ψ_t, that is arbitrarily small volatility of ψ_t can result in arbitrarily large volatility of Δq_t). We now show that the small-sample persistence of the real exchange also increases without bound, and hence so does the measured half-live of the RER process. Specifically, we calculate the finite-sample autocorrelation of the real exchange rate in levels, that is the coefficient from a regression of q_t on q_{t-1} (with a constant) in a sample with $T+1$ observations:

$$\hat{\rho}_q(T) = \frac{\frac{1}{T}\sum_{t=1}^T (q_t - \bar{q})(q_{t-1} - \bar{q})}{\frac{1}{T}\sum_{t=1}^T (q_{t-1} - \bar{q})^2} = 1 + \frac{\frac{1}{T}\sum_{t=1}^T \Delta q_t q_{t-1}}{\frac{1}{T}\sum_{t=1}^T (q_{t-1} - \bar{q})^2}.$$

Note that the denominator is positive and finite for any finite T, but diverges as $T \to \infty$, since q_t is an integrated process. The numerator, however, has a finite limit (conditional on a given initial value q_0,

[52]We could alternatively characterize the response to the primitive noise-trader shock ψ_t with an endogenous coefficient χ_1, rather than the overall endogenous UIP shock $\chi_1\psi_t$; this however simply introduces an amplification loop, by which an initial increase in $\sigma_{e|\psi}$ gets amplified by an endogenous increase in χ_1, leaving the qualitative result unchanged.

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and due to stationarity of Δq_t):

$$\underset{T\to\infty}{p\lim}\frac{1}{T}\sum_{t=1}^{T}\Delta q_t q_{t-1}=\text{cov}(\Delta q_t,q_{t-1})=\sum_{j=1}^{\infty}\text{cov}(\Delta q_t,\Delta q_{t-j})=-\frac{\beta-\rho}{(1-\rho)(1-\beta\rho)}\frac{\chi_1^2\sigma_\psi^2}{(1+\gamma\sigma\kappa_q)^2},$$

where we used the fact that for process (30) $\text{cov}(\Delta q_t,\Delta q_{t-j})=\rho^{j-1}\text{cov}(\Delta q_t,\Delta q_{t-1})$ for $j\geq 1$, and the expression for $\text{cov}(\Delta q_t,\Delta q_{t-1})=\text{cov}(\Delta e_t,\Delta e_{t-1})$ calculated above (conditional on ψ_t shocks, i.e. when $\sigma_a=0$). This implies that the finite sample autocorrelation of q_t: (a) tends to 1 asymptotically as samples size increases (and hence the associated half-life tends to infinity); and (b) is smaller than 1 in large but finite samples, provided that $\rho<\beta$. The associated half-live of q_t is given by $\log(0.5)/\log\hat{\rho}_q(T)$, is finite in finite samples, and increases unboundedly with T. This completes the proof of Proposition 2. ■

Proof of Proposition 3 The equilibrium condition (23) and the coefficients κ_a,κ_q are derived above: see (A58).

Consider a ψ_t shock first. From (30) and Proposition 1, it results in exchange rate depreciation, $\Delta q_t=\Delta e_t>0$. From (23), this depreciation is associated with a reduction in consumption, since $\gamma\kappa_q>0$. The Backus-Smith correlation in response to ψ_t derives from:

$$\text{cov}(\Delta c_t-\Delta c_t^*,\Delta q_t|\psi_t)=-\gamma\kappa_q\text{var}(\Delta q_t|\psi_t)<0,$$

which follows directly from (23). The relative volatility of consumption in response to ψ_t is:

$$\frac{\text{var}(\Delta\tilde{c}_t|\psi_t)}{\text{var}(\Delta q_t|\psi_t)}=(\gamma\kappa_q)^2,$$

which decreases as γ decreases (see the Proof of Proposition 1), and converges to zero in the limit $\gamma\to 0$ (as $\lim_{\gamma\to 0}\kappa_q=\frac{2}{1-\phi}\frac{2\theta(1-\alpha)+\nu}{1+\nu\sigma}$ if finite; see (A58)). ■

Proof of Proposition 4 (a) The proof of this part follows from the equilibrium relationship (A62) between expected (real) devaluation and the interest rate differential, which combines the household Euler equations with the market-clearing relationship between consumption and the real exchange rate (23). In view of perfect price level stabilization by the monetary authority, implying in particular $e_t=q_t$, we rewrite (A62) as:

$$i_t-i_t^*=-(1-\rho)\sigma\kappa_a\tilde{a}_t-\gamma\sigma\kappa_q\mathbb{E}_t\Delta e_{t+1},$$

where we also made use of the AR(1) assumption for $\tilde{a}_t$ to simplify the expression.

The Fama coefficient, β_F, is a projection coefficient of Δe_{t+1} on $i_t-i_t^*$, and since $i_t-i_t^*$ is known at time t, it is equivalent to the projection of $\text{E}_t\Delta e_{t+1}$ instead of Δe_{t+1}. Using the expression above, we have:

$$\beta_F\equiv\frac{\text{cov}(\mathbb{E}_t\Delta e_{t+1},i_t-i_t^*)}{\text{var}(i_t-i_t^*)}=-\frac{1}{\gamma\sigma\kappa_q}-(1-\rho)\frac{\sigma\kappa_a}{\gamma\sigma\kappa_q}\frac{\text{cov}(\tilde{a}_t,i_t-i_t^*)}{\text{var}(i_t-i_t^*)}.$$

In the absence of productivity shocks, $\sigma_a=0$, the second term is zero, and $\beta_F=-1/(\gamma\sigma\kappa_q)<0$. Note that this result does not rely on the assumption $\chi_2=0$ in (16), since it derives from equilibrium conditions other than (16) (akin to our resolution of the Backus-Smith puzzle in Proposition 3).

Furthermore, under the restriction that $\chi_2=0$, we can use the solution for the interest rate differential (26) to calculate:

$$\frac{\mathrm{cov}(\tilde{a}_t, i_t - i_t^*)}{\mathrm{var}(i_t - i_t^*)} = -\frac{1+\gamma\sigma\kappa_q}{(1-\rho)\sigma\kappa_a}\frac{(1-\rho)^2(\sigma\kappa_a)^2\sigma_a^2}{(\gamma\sigma\kappa_q)^2\chi_1^2\sigma_\psi^2 + (1-\rho)^2(\sigma\kappa_a)^2\sigma_a^2},$$

where the second term in the product on the right-hand side is the share of the productivity shock in the variance decomposition of $\mathrm{var}(i_t - i_t^*)$. When $\sigma_\psi = 0$, this share is one, and it follows that $\frac{\mathrm{cov}(\tilde{a}_t, i_t - i_t^*)}{\mathrm{var}(i_t - i_t^*)} = -\frac{1+\gamma\sigma\kappa_q}{(1-\rho)\sigma\kappa_a}$, resulting in $\beta_F = -\frac{1}{\gamma\sigma\kappa_q} + (1-\rho)\frac{\sigma\kappa_a}{\gamma\sigma\kappa_q}\cdot\frac{1+\gamma\sigma\kappa_q}{(1-\rho)\sigma\kappa_a} = 1$. $\beta_F = 1$ could also be obtained directly from (16) after imposing $\chi_2 = 0$ and $\sigma_\psi = 0$. This completes the proof of part (i).

We note, in addition, that the qualitative result still applies when $\chi_2 > 0$; in this case one can calculate β_F using (A65) and the equilibrium solution for the dynamics of b_{t+1} in (A68). As already shown above, this does not affect the value of $\beta_F = -1/(\gamma\sigma\kappa_q)$ conditional on ψ_t shocks. It does, however, affect the value of β_F conditional on productivity shocks, which can be below or above 1 depending on whether k defined in (A66) is above or below 1. Nevertheless, the departure of β_F from 1 in this case is quantitatively small, as it is of the order of γ^2. We omit the derivation for brevity.

(b) The proof here relies on the modified UIP condition (16) and the results in Proposition 1. For simplicity, we make use of $\chi_2 = 0$. Note from the solutions (27) and (26) that as $\rho \to 1$, the variance contribution of $\tilde{a}_t$ to both $\mathrm{var}(\mathbb{E}_t\Delta e_{t+1})$ and $\mathrm{var}(i_t - i_t^*)$ goes to zero, and the variance of both is fully shaped by the ψ_t shock, independently of the value of $\sigma_a, \sigma_\psi > 0$. Therefore, from our characterization in the proof of part (i), $\beta_F \to -1/(\gamma\sigma\kappa_q) < 0$ in this case. In order to characterize R^2 in the Fama regression, we additionally use the solution (30), which we rewrite as follows:

$$\Delta e_{t+1} = \mathbb{E}_t\Delta e_{t+1} + \frac{1}{1+\gamma\sigma\kappa_q}\frac{\beta\chi_1}{1-\beta\rho}\sigma_\psi\varepsilon_t^\psi + \left[\frac{\sigma\kappa_a}{1+\gamma\sigma\kappa_q}\frac{\beta(1-\rho)}{1-\beta\rho} + \frac{\lambda_a}{\lambda_q}\frac{1-\beta}{1-\beta\rho}\right]\tilde{\varepsilon}_t^a.$$

Since $i_t - i_t^*$ is orthogonal to the innovation $\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1}$, we can establish an upper bound:

$$R^2 \le \frac{\mathrm{var}(\mathbb{E}_t\Delta e_{t+1})}{\mathrm{var}(\Delta e_{t+1})} = 1 - \frac{\mathrm{var}_t(\Delta e_{t+1})}{\mathrm{var}(\Delta e_{t+1})} \to 0$$

as $\beta\rho \to 1$, since the proof of Proposition 1 establishes that in this case $\frac{\mathrm{var}_t(\Delta e_{t+1})}{\mathrm{var}(\Delta e_{t+1})} \to 1$.

By the same token, using the expression for $\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1}$ above and (26), we can evaluate $\frac{\mathrm{var}(i_t - i_t^*)}{\mathrm{var}_t(\Delta e_{t+1})}$. As in Proposition 1, we interpret the $\beta\rho \to 1$ limit sequentially with $\beta \to 1$ first and $\rho \to 1$ next (see the argument in footnote 51), and the results apply more generally with β and ρ converging to 1 simultaneously with $\rho \le \beta < 1$ along the limit sequence. We have:

$$\lim_{\beta\to 1}\frac{\mathrm{var}(i_t - i_t^*)}{\mathrm{var}_t(\Delta e_{t+1})} = \lim_{\beta\to 1}\frac{\left(\frac{\gamma\sigma\kappa_q}{1+\gamma\sigma\kappa_q}\right)^2\frac{\chi_1^2\sigma_\psi^2}{1-\rho^2} + (1-\rho)^2\left(\frac{\sigma\kappa_a}{1+\gamma\sigma\kappa_q}\right)^2\frac{\sigma_a^2}{1-\rho^2}}{\left(\frac{1}{1+\gamma\sigma\kappa_q}\frac{\beta}{1-\beta\rho}\right)^2\chi_1^2\sigma_\psi^2 + \left[\frac{\sigma\kappa_a}{1+\gamma\sigma\kappa_q}\frac{\beta(1-\rho)}{1-\beta\rho} + \frac{\lambda_a}{\lambda_q}\frac{1-\beta}{1-\beta\rho}\right]^2\sigma_a^2}$$
$$= \frac{1-\rho}{1+\rho}\frac{(\gamma\sigma\kappa_q)^2\chi_1^2\sigma_\psi^2 + (1-\rho)^2(\sigma\kappa_a)^2\sigma_a^2}{\chi_1^2\sigma_\psi^2 + (1-\rho)^2(\sigma\kappa_a)^2\sigma_a^2} \to 0,$$

as $\rho \to 1$. Since $\mathrm{var}_t(\Delta e_{t+1}) \le \mathrm{var}(\Delta e_{t+1})$, the variance of $i_t - i_t^*$ is indeed arbitrarily smaller than that of Δe_{t+1} in the limit as $\beta\rho \to 1$.

Proposition 1 has already established that the autocorrelation of Δe_{t+1} becomes arbitrarily close to zero as $\beta\rho \to 1$. At the same time, from (26), the autocorrelation of $i_t - i_t^*$ is equal to ρ, and hence it becomes arbitrarily close to 1 as $\beta\rho \to 1$.

Lastly, we characterize the Carry trade return and its Sharpe ratio. Consider a zero-capital strategy of buying a (temporarily) high-interest bond and selling short low-interest bond, with return $R_t\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}} - R_t^*$ per dollar of gross investment, and the size of the gross position determined by the expected return, $\mathbb{E}_t\big\{R_t\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}} - R_t^*\big\}$, which can be positive or negative.[53] Therefore, a log approximation to the return on this trade is:

$$r_{t+1}^C = (i_t - i_t^* - \mathbb{E}_t\Delta e_{t+1})(i_t - i_t^* - \Delta e_{t+1}) = \chi_1\psi_t\big[\chi_1\psi_t - (\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1})\big], \qquad \text{(A71)}$$

where the second equality uses (16) under $\chi_2 = 0$. The (unconditional) Sharp ratio associated with this trade is given by $SR^C = \frac{\mathbb{E}\, r_{t+1}^C}{\text{std}(r_{t+1}^C)}$. We calculate:

$$\mathbb{E}\, r_{t+1}^C = \mathbb{E}\big\{\chi_1\psi_t\,\mathbb{E}_t\{\chi_1\psi_t - (\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1})\}\big\} = \chi_1^2\mathbb{E}\psi_t^2 = \chi_1^2\text{var}(\psi_t),$$

$$\begin{aligned}\text{var}(r_{t+1}^C) &= \mathbb{E}(r_{t+1}^C)^2 - (\mathbb{E}r_{t+1}^C)^2 = \mathbb{E}\big\{\chi_1^2\psi_t^2\mathbb{E}_t\{\chi_1\psi_t - (\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1})\}^2\big\} - \big[\chi_1^2\text{var}(\psi_t)\big]^2 \\ &= \chi_1^4\mathbb{E}\psi_t^4 + \mathbb{E}\left\{\chi_1^2\psi_t^2\text{var}_t(\Delta e_{t+1})\right\} - \big[\chi_1^2\text{var}(\psi_t)\big]^2 = 2\big[\chi_1^2\text{var}(\psi_t)\big]^2 + \sigma_e^2\chi_1^2\text{var}(\psi_t),\end{aligned}$$

where the last line uses the fact that $\sigma_e^2 = \text{var}_t(\Delta e_{t+1})$ does not depend on t (and in particular on the realization of ψ_t, that is the unexpected component of Δe_{t+1} is homoskedastic, as we proof in Proposition 1), as well as the facts that $\mathbb{E}\{\psi_t^3(\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1})\} = \mathbb{E}\{\psi_t^3\mathbb{E}_t\{\Delta e_{t+1} - \mathbb{E}_t\Delta e_{t+1}\}\} = 0$ and $\mathbb{E}\psi_t^4 = 3(\mathbb{E}\psi_t^2)^2$ due to the normality of the shocks. With this, we calculate:

$$SR^C = \frac{\chi_1^2\text{var}(\psi_t)}{\sqrt{2[\chi_1^2\text{var}(\psi_t)]^2 + \sigma_e^2\chi_1^2\text{var}(\psi_t)}} = \left(2 + \frac{\sigma_e^2}{\chi_1^2\text{var}(\psi_t)}\right)^{-1/2},$$

which declines to zero as $\beta\rho \to 1$, since in this limit $\frac{\sigma_e^2}{\chi_1^2\text{var}(\psi_t)} \to \infty$, as we proved in Proposition 1. Indeed, our derivations show that the expected Carry trade return is proportional to the volatility of the UIP shock, $\chi_1\psi_t$, while the standard deviation of the return increases additionally with the volatility of the exchange rate, which in the limit becomes arbitrarily larger than the volatility of the UIP shock. ■

Engel decomposition and Balassa-Samuelson effect Engel (1999) decomposes $q_t = q_t^T + q_t^N$ into the tradable RER q_t^T and the relative price of non-tradables to tradables in the two countries q_t^N, and shows that q_t^T dominates the variance decomposition of q_t at all horizons. Our model reproduces this pattern. Since our model does not incorporate any asymmetry between domestically produced tradables and non-tradables (i.e., no Balassa-Samuelson force), the price index for non-tradables at home is $p_{Nt} = p_{Ht}$, i.e. the same as the price of domestically produced tradables shipped to the domestic market. Assuming ω is the share of tradable sectors in expenditure, we have:

$$p_t = \omega p_{Tt} + (1-\omega)p_{Nt} \qquad \Rightarrow \qquad p_{Tt} = \tfrac{1}{\omega}[p_t - (1-\omega)p_{Ht}] = \tfrac{\gamma}{\omega}p_{Ft} + \big(1 - \tfrac{\gamma}{\omega}\big)p_{Ht},$$

where we used the definition of p_t. Note that our model does not need to take a stand whether γ is due to home bias in tradables ($\omega = 1$), or due to non-tradables ($\omega = \gamma$), or a combination of the two ($\gamma < \omega < 1$, where $1 - \omega$ is the expenditure share of non-tradables and γ/ω is the home bias in the tradable sector). For concreteness, and in line with the empirical patterns in Engel (1999), we assume that $\omega \approx 1/2$, so that $\gamma \ll \omega$, and there exists a considerable home bias in tradables (on the role of the

[53] Note that in a symmetric steady state $\bar{R} = \bar{R}^* = 1/\beta$, and there is no low or high interest rate bond in the long run, yet over time the relative interest rates on bonds fluctuate according to (26), allowing for a temporary Carry trade (or 'forward premium trade' in the terminology of Hassan and Mano 2014).

local distribution margin in tradable prices, see e.g. Burstein, Eichenbaum, and Rebelo 2005).

The tradable RER is defined as $q_t^T = p_{Tt}^* + e_t - p_{Tt}$. We then calculate the non-tradable RER:[54]

$$\begin{aligned} q_t^N \equiv q_t - q_t^T &= (1-\omega)\big[(p_{Nt}^* - p_{Tt}^*) - (p_{Nt} - p_{Tt})\big] \\ &= \gamma\tfrac{1-\omega}{\omega}\big[(p_{Ft}^* - p_{Ht}^*) - (p_{Ht} - p_{Ft})\big] = \gamma\tfrac{1-\omega}{\omega}\big[q_t^P + s_t\big] = \tfrac{1-\omega}{\omega}(1-\alpha)\tfrac{2\gamma}{1-2\gamma}q_t \end{aligned}$$

Therefore, the contribution of the non-tradable component to the volatility of RER is:

$$\frac{\mathrm{cov}(\Delta q_t^N, \Delta q_t)}{\mathrm{var}(\Delta q_t)} = \frac{1-\omega}{\omega}(1-\alpha)\frac{2\gamma}{1-2\gamma},$$

which is small whenever γ is small, even if $\alpha = 0$. LOP deviations, due to PTM ($\alpha > 0$) or LCP, further reduce the contribution of the non-tradable component, but are conceptually not necessary to replicate the patterns documented in Engel (1999) and Betts and Kehoe (2008), as home bias in tradables is sufficient.

It is straightforward to extend the model and allow for the relative tradable-nontradable productivity shocks in order to address the evidence on the Balassa-Samuelson effect (see Rogoff 1996). In the data, Balassa-Samuelson forces are difficult to detect under a floating exchange rate regime, yet they become considerably more pronounced under a peg (see Mendoza 2005, Berka, Devereux, and Engel 2018). This is exactly in line with the prediction of the disconnect model, as we explore in Itskhoki and Mukhin (2019), where we use the model to study a switch from a peg to a float. Intuitively, to the extent that relative productivity shocks are considerably less volatile than the nominal exchange rate under a float, Balassa-Samuelson effect would be difficult to detect, yet when nominal exchange rate volatility is switched off under a peg, the relative productivity shocks become important drivers of the (much less volatile) real exchange rate. To summarize, the seminal evidence in Engel (1999) should not be interpreted as against the importance of nontradables in understanding the real exchange rate, but rather against the importance of (relative) productivity shocks as the key driver of the real exchange rate under a floating regime.

A.7 Nominal rigidities

This section outlines the details of the monetary model with nominal wage and price rigidities. As before, we focus on Home and symmetric relationships hold in Foreign. We consider a standard New Keynesian two-country model in a cashless limit, as described in Galí (2008).

Wages The aggregate labor input is a CES aggregate of individual varieties with elasticity of substitution ϵ, which results in labor demand:

$$L_{it} = \left(\frac{W_{it}}{W_t}\right)^{-\epsilon} L_t, \quad \text{where} \quad L_t = \left(\int L_{it}^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}i\right)^{\frac{\epsilon}{\epsilon-1}} \quad \text{and} \quad W_t = \left(\int W_{it}^{1-\epsilon} \mathrm{d}i\right)^{\frac{1}{1-\epsilon}},$$

and the rest of the model production structure is unchanged. Households set wages à la Calvo and supply as much labor as demanded at a given wage rate. The probability of changing wage in the next period is $1-\lambda_w$. The first-order condition for wage setting is:

$$\mathbb{E}_t \sum_{s=t}^{\infty} (\beta\lambda_w)^{s-t} \frac{C_s^{-\sigma}}{P_s} W_s^{\epsilon} L_s \left(\bar{W}_t^{1+\epsilon/\nu} - \frac{\kappa\epsilon}{\epsilon-1} P_s C_s^{\sigma} L_s^{1/\nu} W_s^{\epsilon/\nu}\right) = 0.$$

[54] We can also evaluate the tradable $q_t^T = p_{Tt}^* + e_t - p_{Tt} = \frac{\gamma}{\omega}p_{Ht}^* + \big(1-\frac{\gamma}{\omega}\big)p_{Ft}^* + e_t - \frac{\gamma}{\omega}p_{Ft} - \big(1-\frac{\gamma}{\omega}\big)p_{Ht} = \big(1-\frac{\gamma}{\omega}\big)q_t^P - \frac{\gamma}{\omega}s_t$.

Substituting in labor demand and log-linearizing, we obtain:

$$\hat{w}_t = \frac{1-\beta\lambda_w}{1+\epsilon/\nu}\left(\sigma c_t + \frac{1}{\nu}\ell_t + p_t + \frac{\epsilon}{\nu}w_t\right) + \beta\lambda_w \mathbb{E}_t \hat{w}_{t+1},$$

where $\hat{w}_t$ denotes the log deviation from the steady state of the wage rate reset at t. Note that the wage inflation can be expressed as $\pi_t^w \equiv \Delta w_t = (1-\lambda_w)(\hat{w}_t - w_{t-1})$. Aggregate wages using these equalities and express the wage process in terms of cross-country differences to obtain the NKPC for wages:

$$\tilde{\pi}_t^w = k_w\left[\sigma\tilde{c}_t + \frac{1}{\nu}\tilde{\ell}_t + \tilde{p}_t - \tilde{w}_t\right] + \beta\mathbb{E}_t\tilde{\pi}_{t+1}^w,$$

where $k_w = \frac{(1-\beta\lambda_w)(1-\lambda_w)}{\lambda_w(1+\epsilon/\nu)}$.

Prices We assume that firms set prices à la Calvo with a probability of changing price next period equal to $1-\lambda_p$. There are two Phillips curves, one for domestic sales and one for exports. The first-order conditions for reset prices in log-linearized form are

$$\hat{p}_{Ht} = (1-\beta\lambda_p)\,\mathbb{E}_t \sum\nolimits_{j=t}^{\infty} (\beta\lambda_p)^{j-t}\left[(1-\alpha)\,mc_j + \alpha p_j\right],$$
$$\hat{p}_{Ht}^* = (1-\beta\lambda_p)\,\mathbb{E}_t \sum\nolimits_{j=t}^{\infty} (\beta\lambda_p)^{j-t}\left[(1-\alpha)\left(mc_j - (1-\iota)e_j\right) + \alpha(p_j^* + \iota e_j)\right],$$

where $mc_t = (1-\phi)\left(\vartheta r_t^K + (1-\vartheta)w_t - a_t\right) + \phi p_t$ are the marginal costs of Home firms, and $\iota \in \{0,1\}$ with $\iota = 1$ corresponding to the case of PCP (producer currency pricing) and $\iota = 0$ to the case of LCP (local currency pricing). The optimal reset prices are equal to an expected discounted sum of future optimal static prices, which in turn are a weighted average of marginal costs and competitor prices. Given the law of motion for home prices, $\pi_{Ht} = (1-\lambda_p)(\hat{p}_{Ht} - p_{Ht-1}) = \frac{1-\lambda_p}{\lambda_p}(\hat{p}_{Ht} - p_{Ht})$, the resulting NKPC is:

$$\pi_{Ht} = k_p\Big[(1-\alpha)mc_t + \alpha p_t - p_{Ht}\Big] + \beta\mathbb{E}_t\pi_{Ht+1}, \quad \text{where} \quad k_p = \frac{(1-\beta\lambda_p)(1-\lambda_p)}{\lambda_p}.$$

The law of motion for export prices depends on the currency of invoicing. Under LCP, the law of motion for prices is, $\pi_{Ht}^* = \frac{1-\lambda_p}{\lambda_p}(\hat{p}_{Ht}^* - p_{Ht}^*)$, and the export NKPC can be expressed as:

$$\pi_{Ht}^* = k_p\Big[(1-\alpha)(mc_t - e_t) + \alpha p_t^* - p_{Ht}^*\Big] + \beta\mathbb{E}_t\pi_{Ht+1}^*.$$

Under PCP, the law of motion for the export price index, $\pi_{Ht}^* = \frac{1-\lambda_p}{\lambda_p}(\hat{p}_{Ht}^* - p_{Ht}^*) - \Delta e_t$, implies that the NKPC is

$$(\pi_{Ht}^* + \Delta e_t) = k_p\Big[(1-\alpha)mc_t + \alpha(p_t^* + e_t) - (p_{Ht}^* + e_t)\Big] + \beta\mathbb{E}_t\left(\pi_{Ht+1}^* + \Delta e_{t+1}\right).$$

Finally, notice that the DCP case with all international trade invoiced in Foreign currency can be expressed as a mix of the two other regimes — Home exporters use LCP and Foreign exporters use PCP. The rest of the model is as described in Section 2.1.

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