Appendix C: Estimation Details for "Housing Market Spillovers: Evidence from an Estimated DSGE Model"

1 Estimation Strategy

The parameters of the model are estimated using Bayesian methods. We use Bayesian methods because they allow incorporating a priori information on the parameters of the model and also because pure maximum likelihood tends to produce fragile results, particularly in situations in which some parameters are weakly identified.

2 Estimation of the model

Before estimating the model a transformation of the data that is consistent with the balanced-growth path assumption must be taken. Let a sans-serif denote the detrended variables, that is the variables scaled by their deterministic trend. Therefore: $\mathsf{C}_t = C_t/G_C^t$, $\mathsf{IH}_t = IH_t/G_{IH}^t$, $\mathsf{IK}_t = IK_t/G_{IK}^t$, $\mathsf{q}_t = q_t/G_q^t$. Let a superscript d denote the data (see Appendix A for data sources). The measurement equations are:

$$
\begin{aligned}
\log C_t^d - \log C_{1965:1}^d &= \widehat{\mathsf{C}}_t + (G_C - 1)\, t \\
\log IK_t^d - \log IK_{1965:1}^d &= \widehat{\mathsf{IK}}_t + (G_{KC} - 1)\, t \\
\log IH_t^d - \log IH_{1965:1}^d &= \widehat{\mathsf{IH}}_t + (G_H - 1)\, t \\
\log q_t^d - \log q_{1965:1}^d &= \widehat{\mathsf{q}}_t + (G_Q - 1)\, t \\
\log N_{ct}^d &= \alpha \widehat{n}_{ct} + (1-\alpha)\, \widehat{n}'_{ct} \\
\log N_{ht}^d &= \alpha \widehat{n}_{ht} + (1-\alpha)\, \widehat{n}'_{ht} \\
\pi_t^d &= \widehat{\pi}_t \\
R_t^d &= \widehat{R}_t \\
\omega_{ct}^d &= \frac{w_c}{w_c + w'_c}\widehat{\omega}_{ct} + \frac{w'_c}{w_c + w'_c}\widehat{\omega}'_{ct} \\
\omega_{ht}^d &= \frac{w_h}{w_h + w'_h}\widehat{\omega}_{ht} + \frac{w'_h}{w_h + w'_h}\widehat{\omega}'_{ht}
\end{aligned}
$$

where real consumption, real business fixed investment and real residential investment are divided by the civilian non-institutional population over 16 (CNP16OV), and a hat over a variable denotes its percentage deviation from the steady state, detrended value. The first observation is taken away from the trending series since we do not use information on the long-run averages of the detrended data.

3 The simulation of the posterior with the Metropolis algorithm

In the Bayesian framework both the data Y and of the parameters Θ are random variables. Starting from their joint probability distribution $P(Y, \Theta)$ one can derive the relationship between their marginal and conditional distributions, i.e. the Bayes theorem:

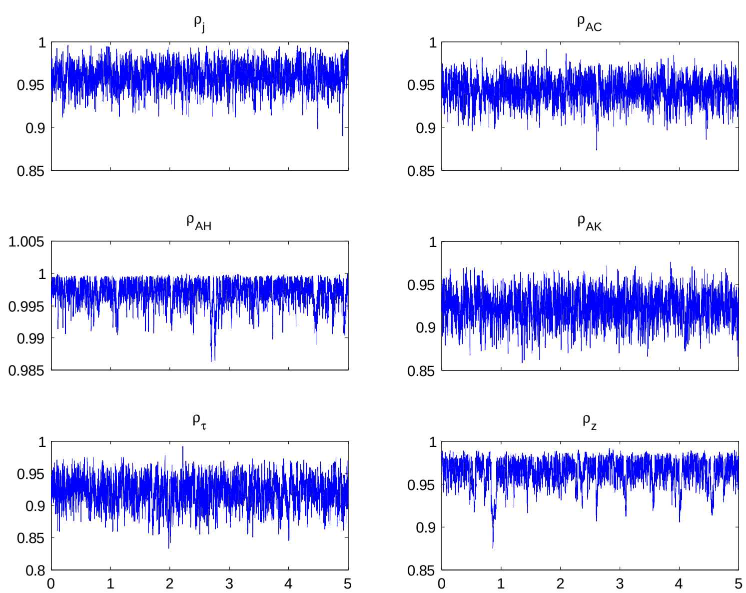
$$P(\Theta|Y) \propto P(Y|\Theta) * P(\Theta)$$

The information contained in the prior distribution $P(\Theta)$ is updated with the likelihood, $P(Y|\Theta)$, of the observed data to deliver the posterior distribution of the parameters $P(\Theta|Y)$. The posterior density can then be used to perform statistical inference either on the parameters themselves or on any function of them.

However, since the posterior distribution of the parameters does not belong to any known family of distributions we need to build our inference on a (Monte Carlo) simulation algorithm that generates a vector of draws from an unknown distribution using a known distribution. As the length of the simulation increases the Markow chain produced by the algorithm converges to the true unknown "target" distribution. The most commonly used algorithm for this purpose is the Metropolis one.[1] As in Schorfheide (2000)[2] and Smets and Wouters (2007), inference is done in two steps. First we maximize the log of the posterior density and compute an approximation of the inverse of the Hessian at the mode. Second, we generate 200,000 draws from the posterior distribution of the parameters using a multivariate normal (the so-called "jump" distibution) with covariance matrix proportional to the inverse of the Hessian. The constant of proportionality is called "scaling" factor. This factor is set at 0.2 and it results in an acceptance rate of 27 percent in 200,000 draws. The first 50,000 draws are used as burn-in sample. The inference described in the paper is based on a total of 150,000 draws from the posterior distribution.

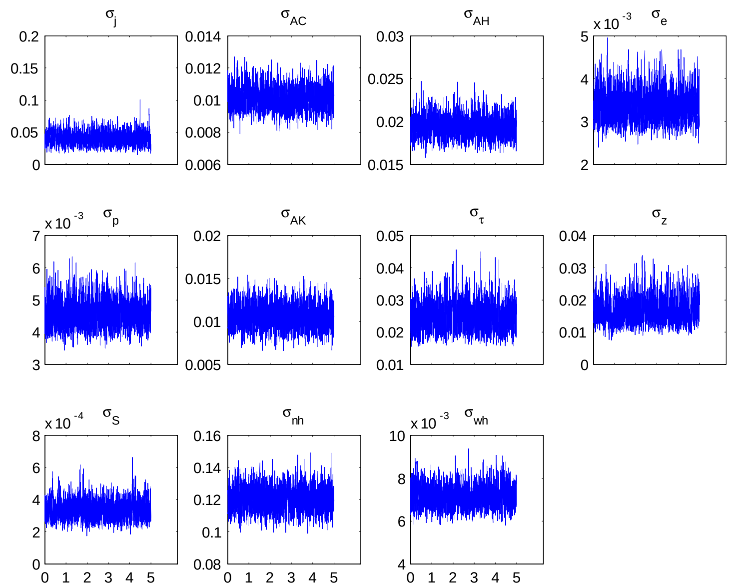
3.1 The output of the Metropolis

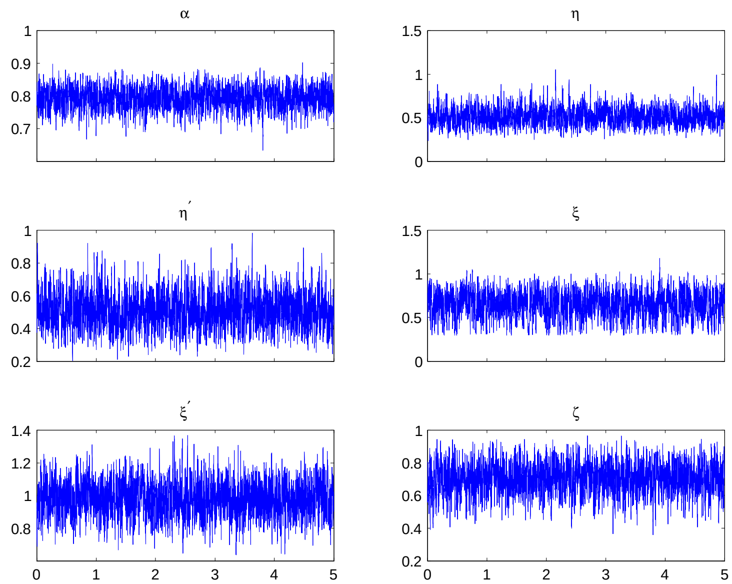
The following graphs report the time series of the draws from the posterior distribution generated by the Metropolis algorithm. On the horizontal axis each tick denotes 1,600 draws.

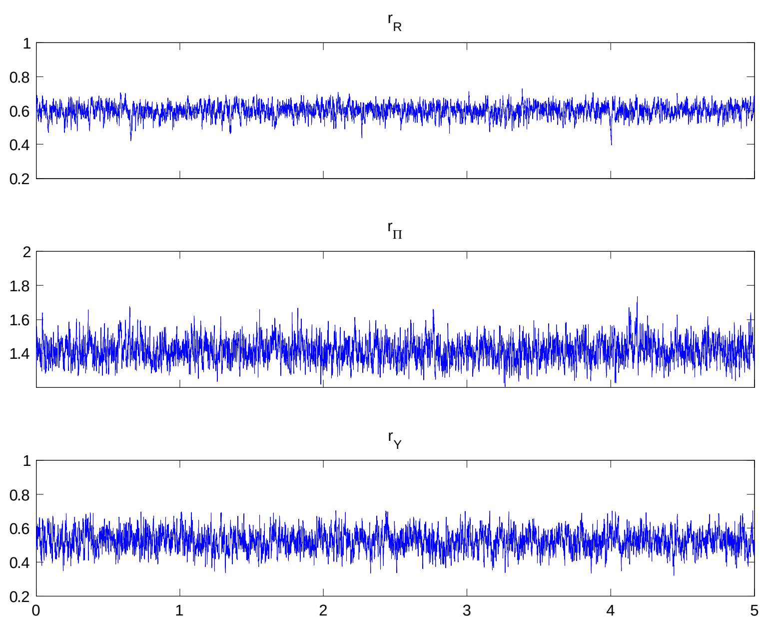


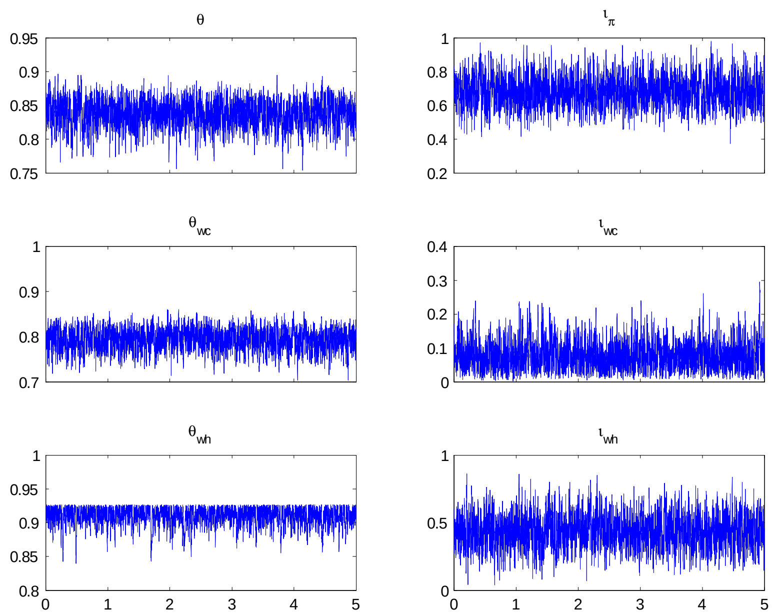
[1] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller, and E. Teller, "Equations of State Calculations by Fast Computing Machines", *Journal of Chemical Physics*, 21(6):1087-1092, 1953.

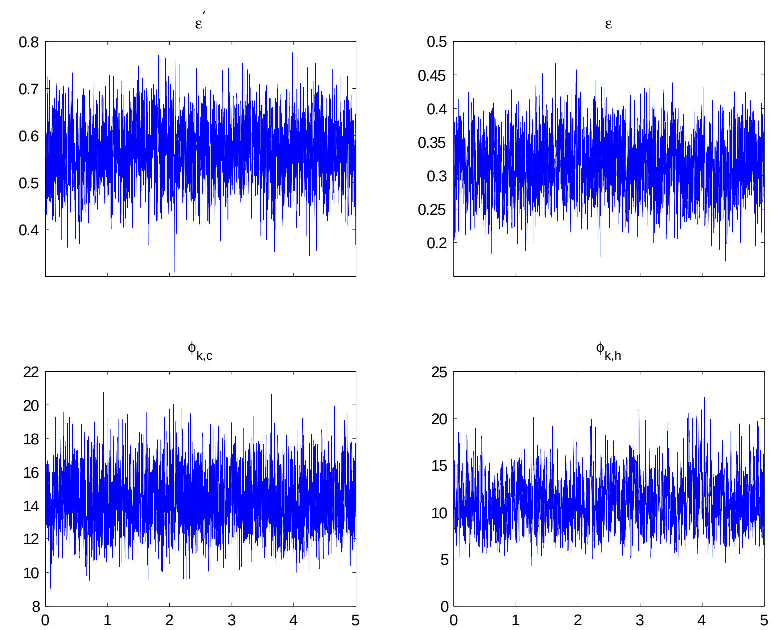
[2] Frank Schorfheide, 2000. "Loss function-based evaluation of DSGE models," Journal of Applied Econometrics, vol. 15(6), pages 645-670.

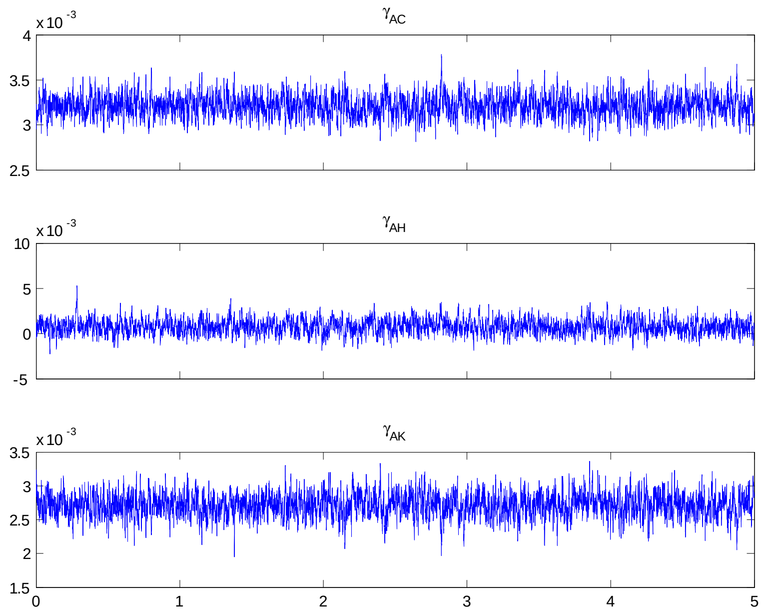

σ_j
σ_AC
σ_AH
σ_e
σ_p
σ_AK
σ_τ
σ_z
σ_S
σ_nh
σ_wh


α
η
η´
ξ
ξ´
ζ


r_R
r_Π
r_Y


θ
0.95
0.9
0.85
0.8
0.75
ι_π
1
0.8
0.6
0.4
0.2
θ_wc
1
0.9
0.8
0.7
ι_wc
0.4
0.3
0.2
0.1
0
θ_wh
1
0.95
0.9
0.85
0.8
ι_wh
1
0.5
0
0 1 2 3 4 5
0 1 2 3 4 5


ε′
0.8
0.7
0.6
0.5
0.4
ε
0.5
0.45
0.4
0.35
0.3
0.25
0.2
ϕ_k,c
22
20
18
16
14
12
10
8
ϕ_k,h
25
20
15
10
5
0
0 1 2 3 4 5
0 1 2 3 4 5


γAC
x 10^-3
4
3.5
3
2.5
γAH
x 10^-3
10
5
0
-5
γAK
x 10^-3
3.5
3
2.5
2
1.5
0
1
2
3
4
5

3.2 Convergence of the algorithm

Convergence of the algorithm is assessed by looking at the plots of the draws, the first four moments (mean, standard deviation, skewness and kurtosis) obtained by splitting the draws into two samples (first and second half) and by computing recursively the first four moments of the marginal posterior distribution of each parameter. Table C.1 reports the first and second moments of the posterior marginal distributions based on 200,000 draws. Table C.2 reports the moments based on 500,000 draws.

Table C.1. Posterior mean and standard deviation: 200,000 draws

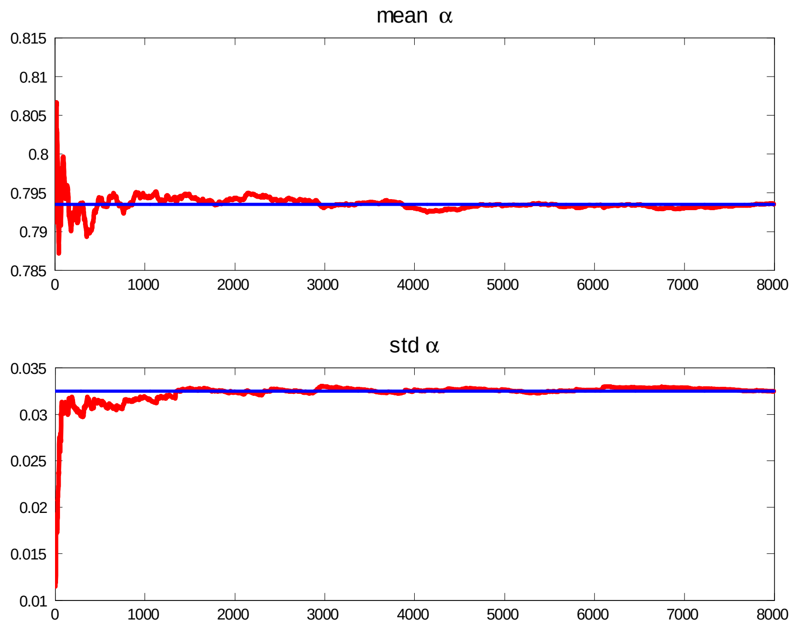
	mean		standard deviation	
parameter	first half	second half	first half	second half
ϵ	0.3170	0.3142	0.0409	0.0396
ϵ'	0.5659	0.5680	0.0620	0.0607
η	0.5169	0.5179	0.1025	0.0928
η'	0.5106	0.5112	0.1020	0.1021
ξ	0.6521	0.6474	0.1434	0.1443
ξ'	0.9821	0.9776	0.1018	0.0996
$\phi_{k,c}$	14.3652	14.2922	1.6103	1.5578
$\phi_{k,h}$	10.9398	11.4178	2.4066	2.6513
α	0.7930	0.7940	0.0326	0.0323
r_R	0.5996	0.5999	0.0393	0.0376
r_π	1.4175	1.4161	0.0661	0.0673
r_Y	0.5281	0.5222	0.0602	0.0585
θ_π	0.8381	0.8363	0.0202	0.0193
ι_π	0.6814	0.6847	0.0868	0.0860
$\theta_{w,c}$	0.7951	0.7927	0.0238	0.0234
$\iota_{w,c}$	0.0796	0.0793	0.0414	0.0400
$\theta_{w,h}$	0.9091	0.9091	0.0139	0.0126
$\iota_{w,h}$	0.4360	0.4348	0.1210	0.1173
ζ	0.7021	0.7039	0.0915	0.0884
γ_{AC}	0.0032	0.0032	0.0001	0.0001
γ_{AH}	0.0008	0.0008	0.0008	0.0008
γ_{AK}	0.0027	0.0027	0.0002	0.0002
ρ_{AC}	0.9436	0.9427	0.0141	0.0144
ρ_{AH}	0.9970	0.9968	0.0017	0.0020
ρ_{AK}	0.9224	0.9231	0.0178	0.0164
ρ_j	0.9594	0.9599	0.0138	0.0144
ρ_z	0.9624	0.9648	0.0163	0.0149
ρ_τ	0.9213	0.9204	0.0218	0.0216
σ_{AC}	0.0102	0.0101	0.0007	0.0006
σ_{AH}	0.0195	0.0194	0.0011	0.0011
σ_{AK}	0.0106	0.0105	0.0013	0.0012
σ_j	0.0408	0.0404	0.0097	0.0100
σ_R	0.0034	0.0034	0.0003	0.0003
σ_z	0.0168	0.0173	0.0041	0.0040
σ_τ	0.0255	0.0252	0.0045	0.0043
σ_p	0.0046	0.0046	0.0004	0.0004
σ_s	0.0003	0.0003	0.0001	0.0001
$\sigma_{n,h}$	0.1207	0.1206	0.0068	0.0068
$\sigma_{w,h}$	0.0072	0.0072	0.0005	0.0005

Table C.2. Posterior mean and standard deviation: 500,000 draws

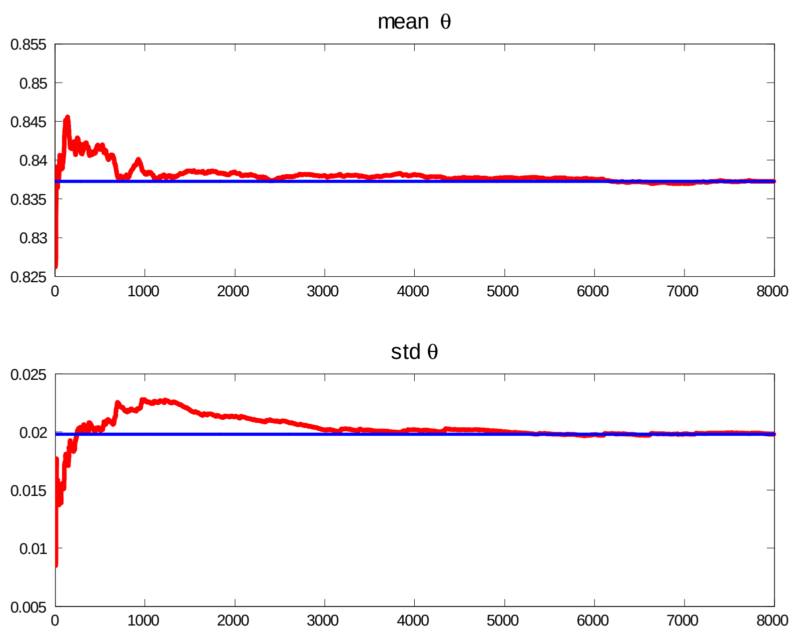
	mean		standard deviation	
parameter	first half	second half	first half	second half
ϵ	0.3198	0.3181	0.0410	0.0407
ϵ'	0.5672	0.5676	0.0620	0.0630
η	0.5149	0.5155	0.0980	0.1000
η'	0.5105	0.5138	0.1020	0.1015
ξ	0.6453	0.6403	0.1416	0.1385
ξ'	0.9773	0.9823	0.1022	0.1012
$\phi_{k,c}$	14.3400	14.2892	1.5737	1.5672
$\phi_{k,h}$	11.1922	11.1316	2.5818	2.5239
α	0.7929	0.7928	0.0321	0.0329
r_R	0.6012	0.5993	0.0389	0.0377
r_π	1.4173	1.4192	0.0669	0.0676
r_Y	0.6012	0.5993	0.0608	0.0622
θ_π	0.8371	0.8379	0.0193	0.0188
ι_π	0.6845	0.6779	0.0844	0.0859
$\theta_{w,c}$	0.7954	0.7939	0.0254	0.0256
$\iota_{w,c}$	0.0802	0.0831	0.0400	0.0417
$\theta_{w,h}$	0.9092	0.9091	0.0128	0.0125
$\iota_{w,h}$	0.4355	0.4312	0.1204	0.1224
ζ	0.7007	0.6941	0.0901	0.0960
γ_{AC}	0.0032	0.0032	0.0001	0.0001
γ_{AH}	0.0008	0.0008	0.0008	0.0008
γ_{AK}	0.0027	0.0027	0.0002	0.0002
ρ_{AC}	0.9423	0.9427	0.0147	0.0144
ρ_{AH}	0.9968	0.9969	0.0020	0.0021
ρ_{AK}	0.9229	0.9230	0.0173	0.0169
ρ_j	0.9597	0.9601	0.0143	0.0137
ρ_z	0.9629	0.9643	0.0176	0.0151
ρ_τ	0.9185	0.9208	0.0229	0.0226
σ_{AC}	0.0101	0.0101	0.0006	0.0006
σ_{AH}	0.0194	0.0194	0.0011	0.0011
σ_{AK}	0.0106	0.0105	0.0013	0.0012
σ_j	0.0405	0.0400	0.0098	0.0093
σ_R	0.0034	0.0034	0.0034	0.0034
σ_z	0.0173	0.0174	0.0042	0.0043
σ_τ	0.0259	0.0256	0.0052	0.0049
σ_p	0.0046	0.0046	0.0004	0.0004
σ_s	0.0003	0.0003	0.0001	0.0001
$\sigma_{n,h}$	0.1208	0.1210	0.0068	0.0068
$\sigma_{w,h}$	0.0071	0.0072	0.0005	0.0005

Table C.3. Posterior mean and standard deviation: 200,000 and 500,000 draws

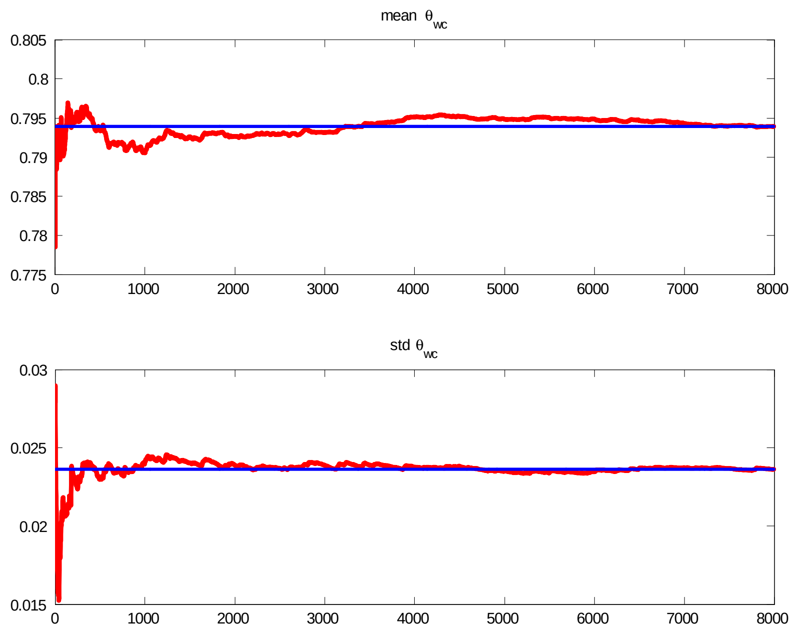
	mean		standard deviation	
	200,000		500,000	
parameter	second half	second half	first half	second half
ϵ	0.3170	0.3198	0.0409	0.0407
ϵ'	0.5659	0.5672	0.0620	0.0630
η	0.5169	0.5149	0.1025	0.1000
η'	0.5106	0.5105	0.1020	0.1015
ξ	0.6521	0.6453	0.1434	0.1385
ξ'	0.9821	0.9773	0.1018	0.1012
$\phi_{k,c}$	14.3652	14.3400	1.6103	1.5672
$\phi_{k,h}$	10.9398	11.1922	2.4066	2.5239
α	0.7930	0.7929	0.0326	0.0329
r_R	0.5996	0.6012	0.0393	0.0377
r_π	1.4175	1.4173	0.0661	0.0676
r_Y	0.5281	0.6012	0.0602	0.0622
θ_π	0.8381	0.8371	0.0202	0.0188
ι_π	0.6814	0.6845	0.0868	0.0859
$\theta_{w,c}$	0.7951	0.7954	0.0238	0.0256
$\iota_{w,c}$	0.0796	0.0802	0.0414	0.0417
$\theta_{w,h}$	0.9091	0.9092	0.0139	0.0125
$\iota_{w,h}$	0.4360	0.4355	0.1210	0.1224
ζ	0.7021	0.7007	0.0915	0.0960
γ_{AC}	0.0032	0.0032	0.0001	0.0001
γ_{AH}	0.0008	0.0008	0.0008	0.0008
γ_{AK}	0.0027	0.0027	0.0002	0.0002
ρ_{AC}	0.9436	0.9423	0.0141	0.0144
ρ_{AH}	0.9970	0.9968	0.0017	0.0021
ρ_{AK}	0.9224	0.9229	0.0178	0.0169
ρ_j	0.9594	0.9597	0.0138	0.0137
ρ_z	0.9624	0.9629	0.0163	0.0151
ρ_τ	0.9213	0.9185	0.0218	0.0226
σ_{AC}	0.0102	0.0101	0.0007	0.0006
σ_{AH}	0.0195	0.0194	0.0011	0.0011
σ_{AK}	0.0106	0.0106	0.0013	0.0012
σ_j	0.0408	0.0405	0.0097	0.0093
σ_R	0.0034	0.0034	0.0003	0.0034
σ_z	0.0168	0.0173	0.0041	0.0043
σ_τ	0.0255	0.0259	0.0045	0.0049
σ_p	0.0046	0.0046	0.0004	0.0004
σ_s	0.0003	0.0003	0.0001	0.0001
$\sigma_{n,h}$	0.1207	0.1208	0.0068	0.0068
$\sigma_{w,h}$	0.0072	0.0071	0.0005	0.0005



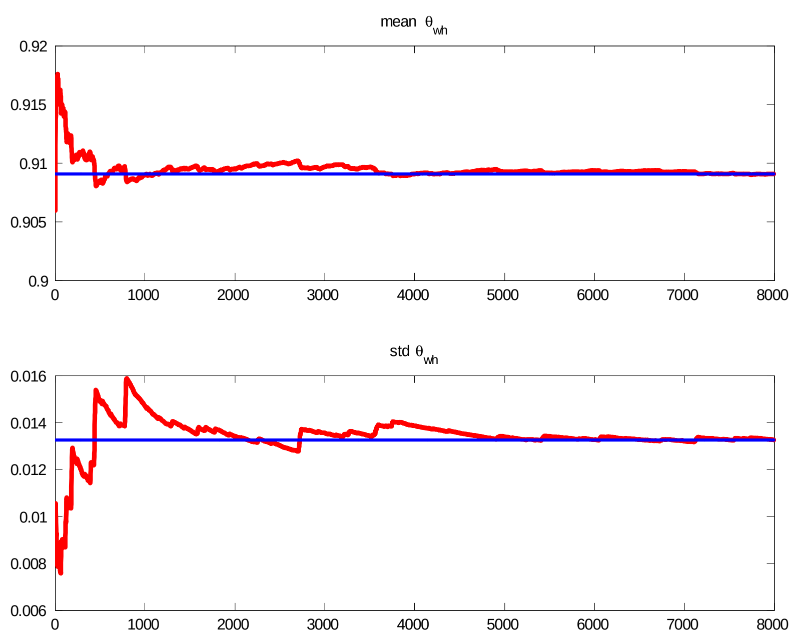
α



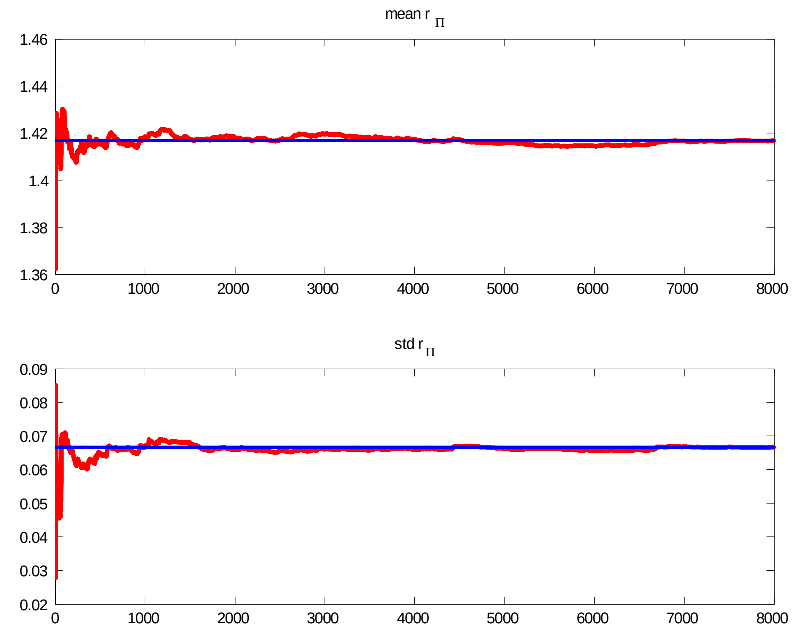
θ_π



θ_{wc}



θ_{wh}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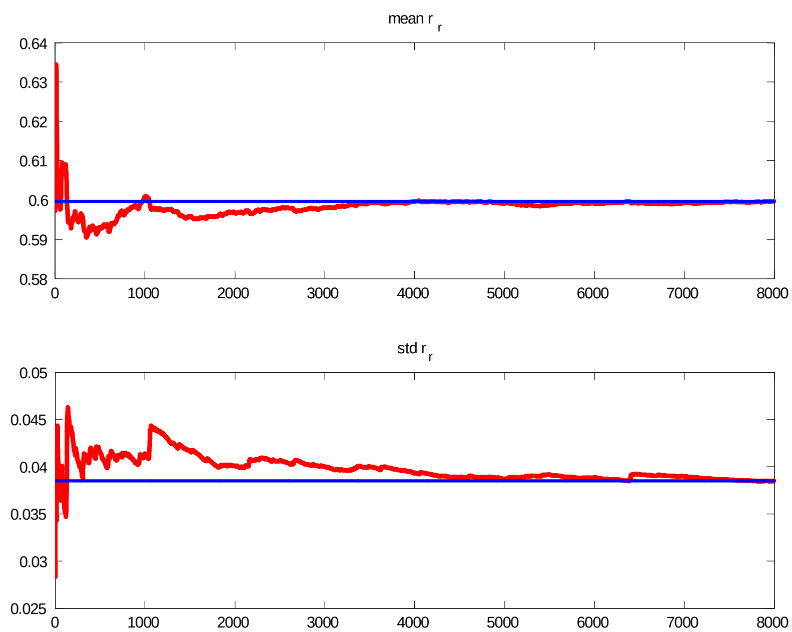
1.46
1.44
1.42
1.4
1.38
1.36
0
1000
2000
3000
4000
5000
6000
7000
8000
std r Π
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02

r_π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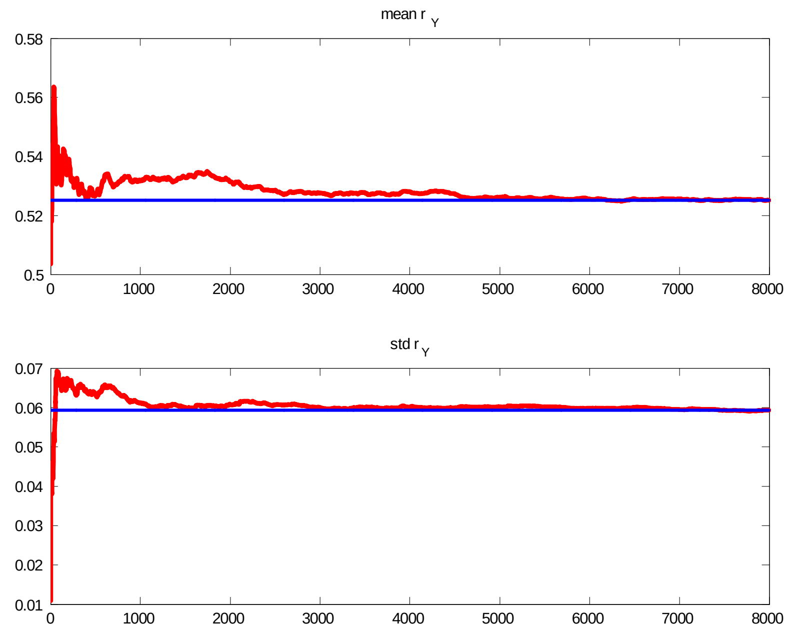
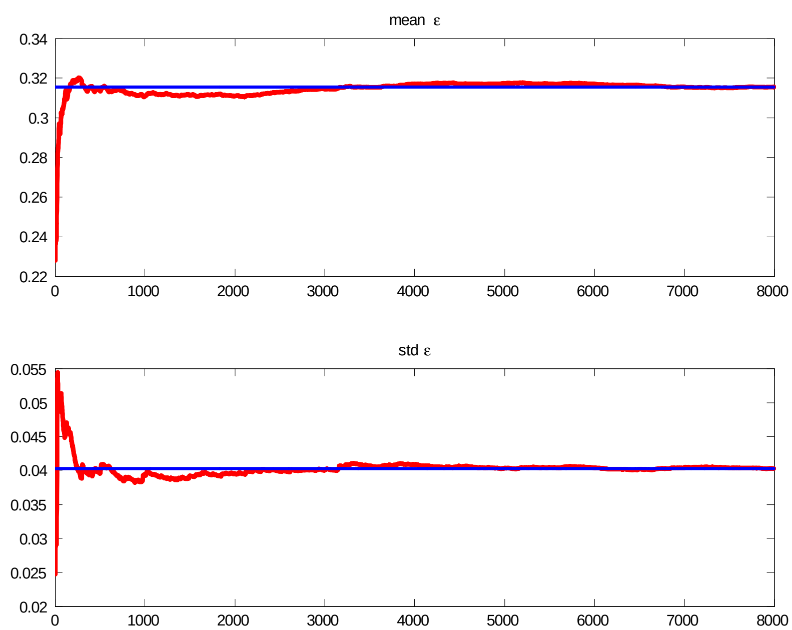
0.64
0.63
0.62
0.61
0.6
0.59
0.58
0
1000
2000
3000
4000
5000
6000
7000
8000
std r r
0.05
0.045
0.04
0.035
0.03
0.025

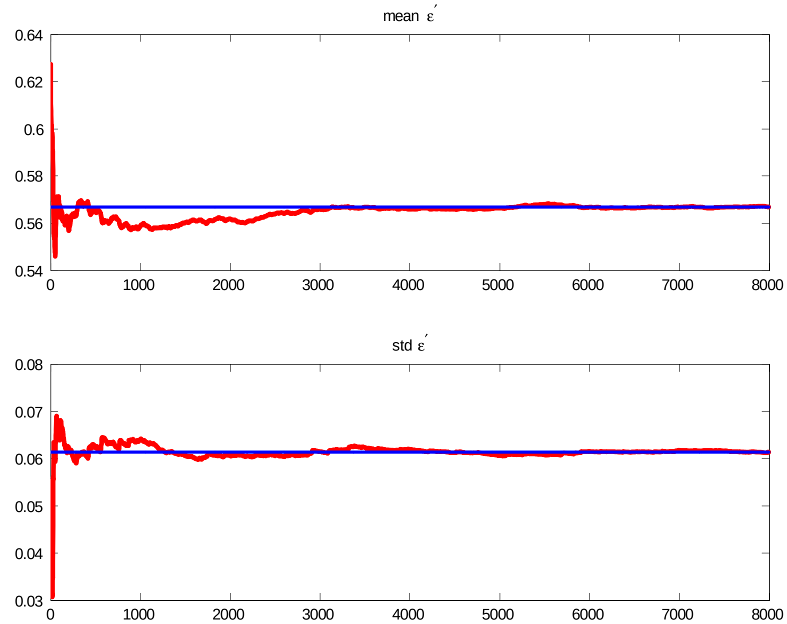
r_R


mean r Y
0.58
0.56
0.54
0.52
0.5
0
1000
2000
3000
4000
5000
6000
7000
8000
std r Y
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
1000
2000
3000
4000
5000
6000
7000
8000

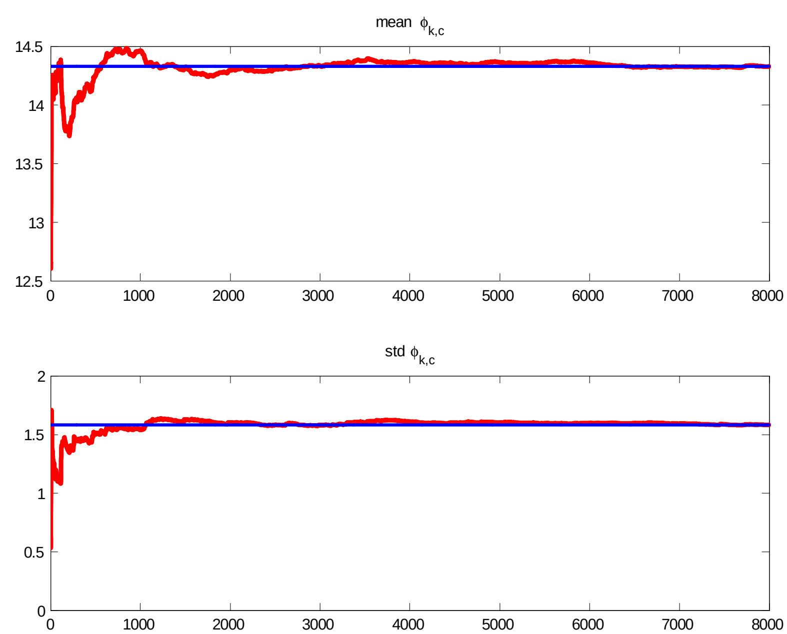
r_Y


mean ε
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0
1000
2000
3000
4000
5000
6000
7000
8000
std ε
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0
1000
2000
3000
4000
5000
6000
7000
8000

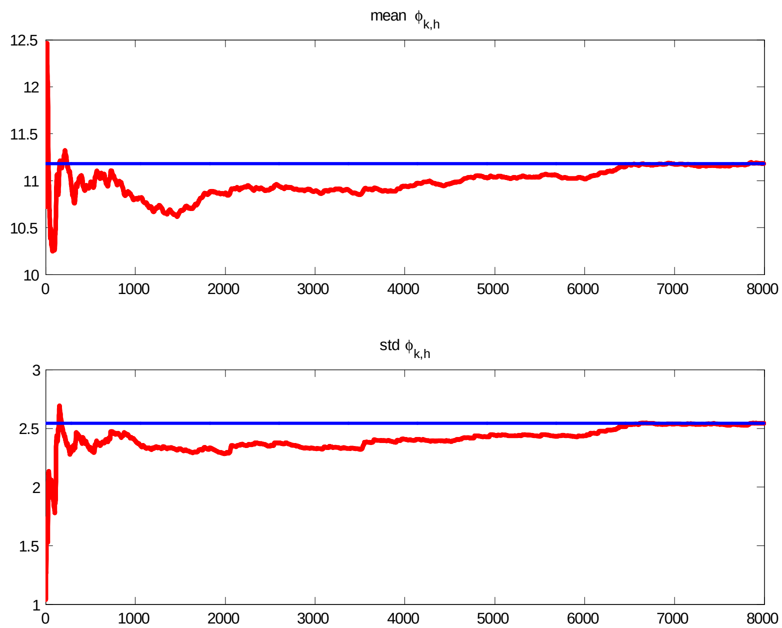
ϵ



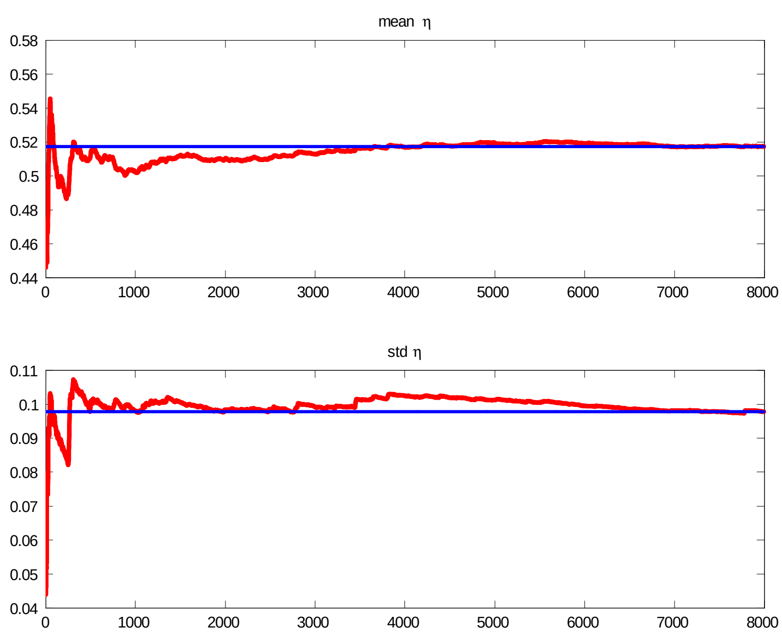
ϵ'



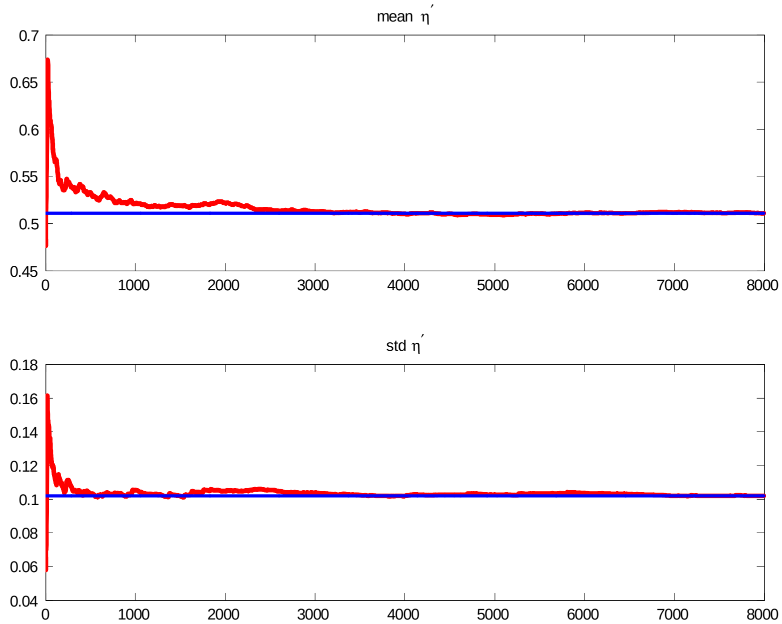
$\phi_{k,c}$



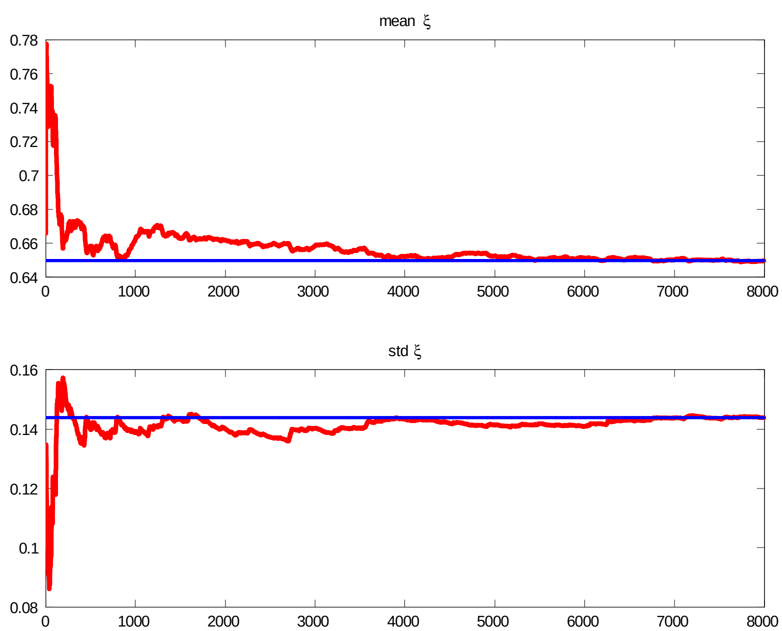
$\phi_{k,h}$



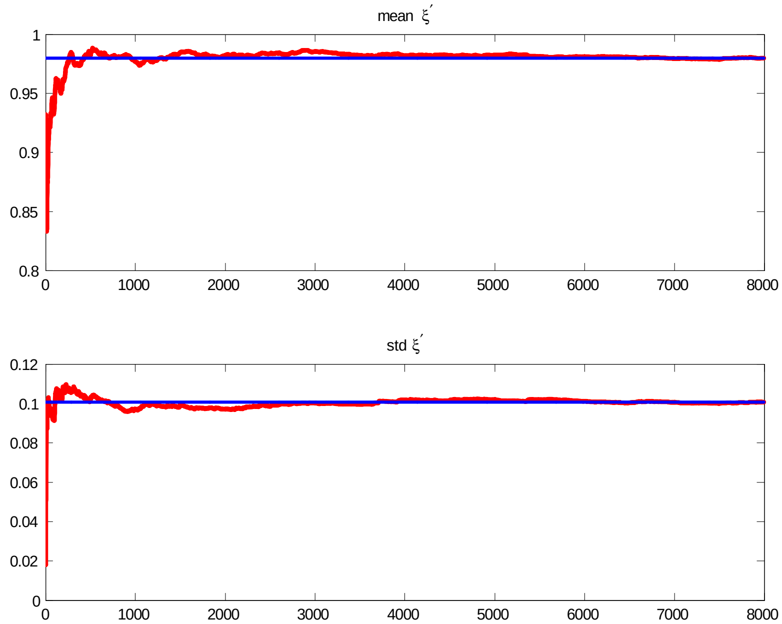
η



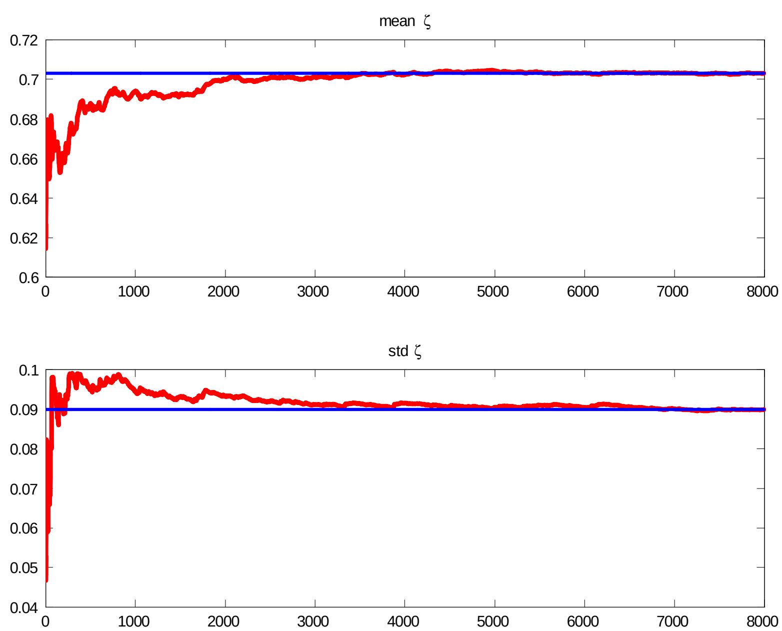
η'



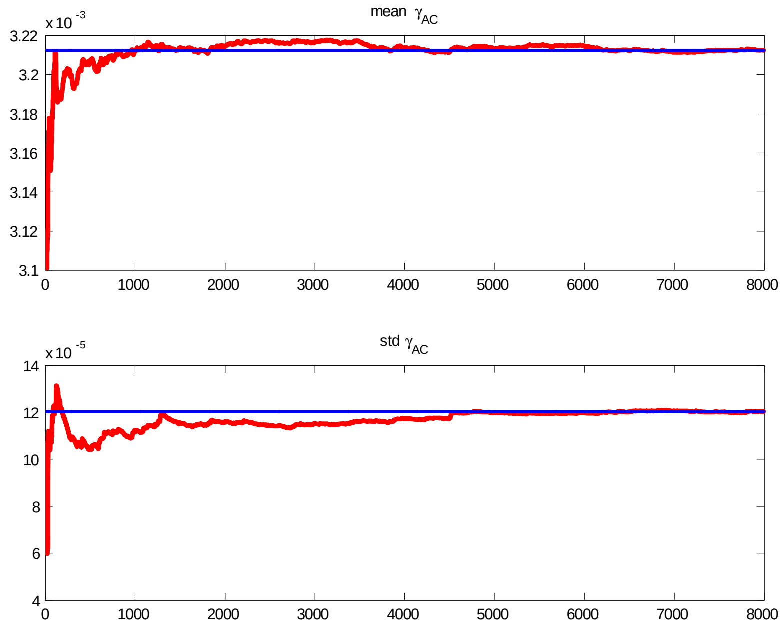
ξ


mean ξ´
1
0.95
0.9
0.85
0.8
0
1000
2000
3000
4000
5000
6000
7000
8000
std ξ´
0.12
0.1
0.08
0.06
0.04
0.02
0
0
1000
2000
3000
4000
5000
6000
7000
8000

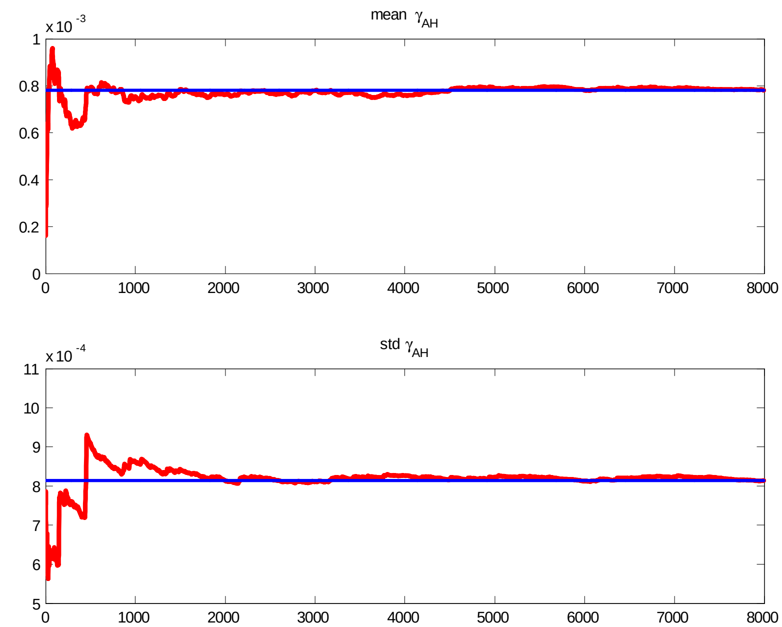
ξ'


mean ζ
0.72
0.7
0.68
0.66
0.64
0.62
0.6
0
1000
2000
3000
4000
5000
6000
7000
8000
std ζ
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0
1000
2000
3000
4000
5000
6000
7000
8000

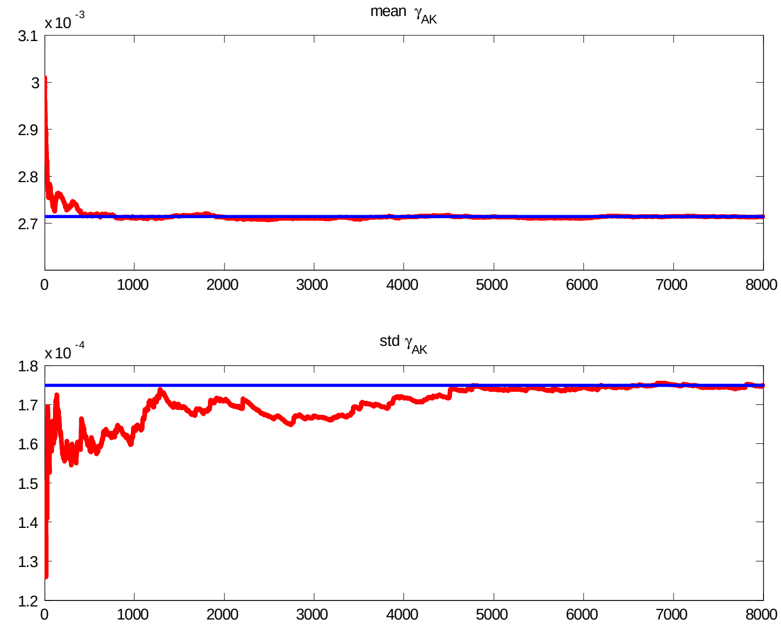
ζ



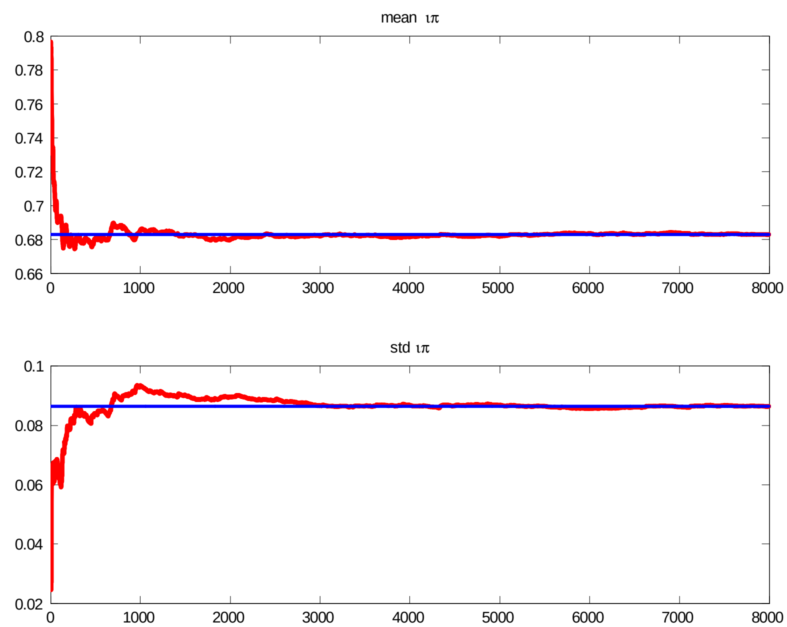
γ_{AC}



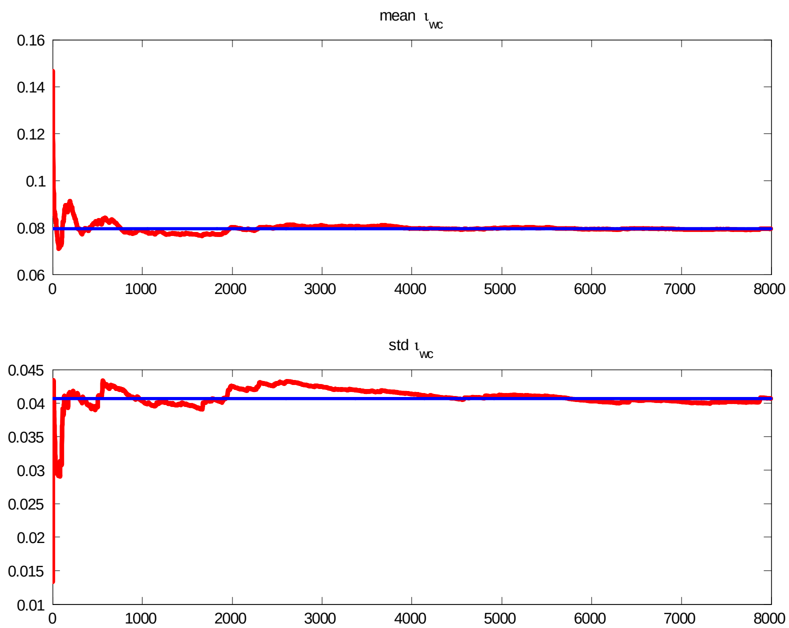
γ_{AH}



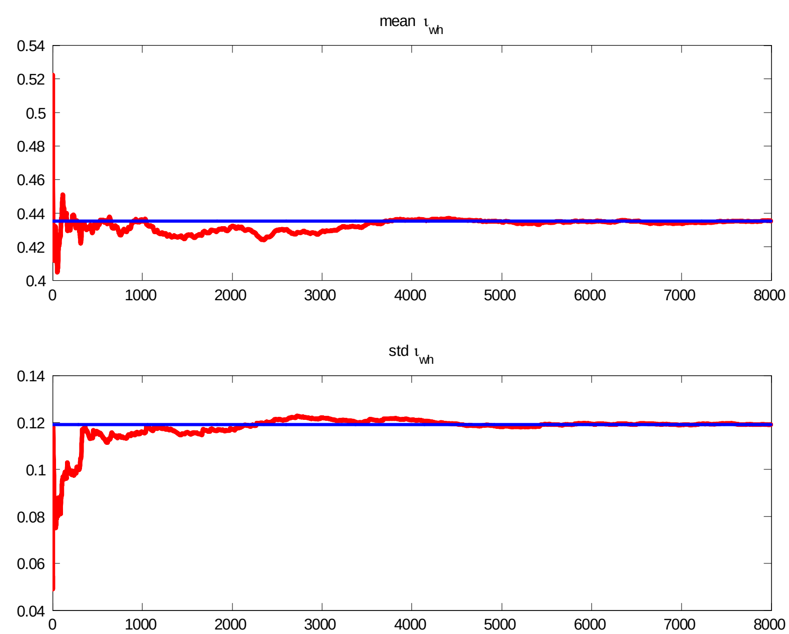
γ_{AK}



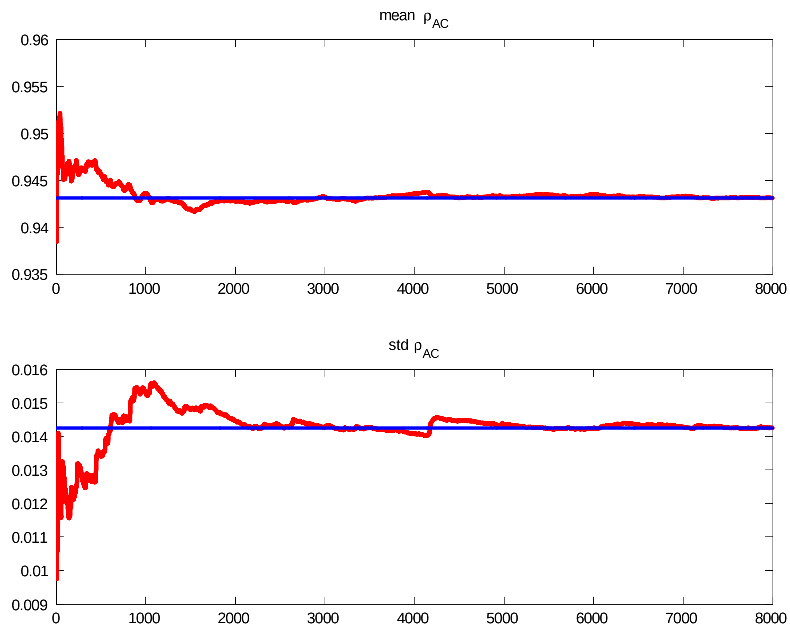
ι_π



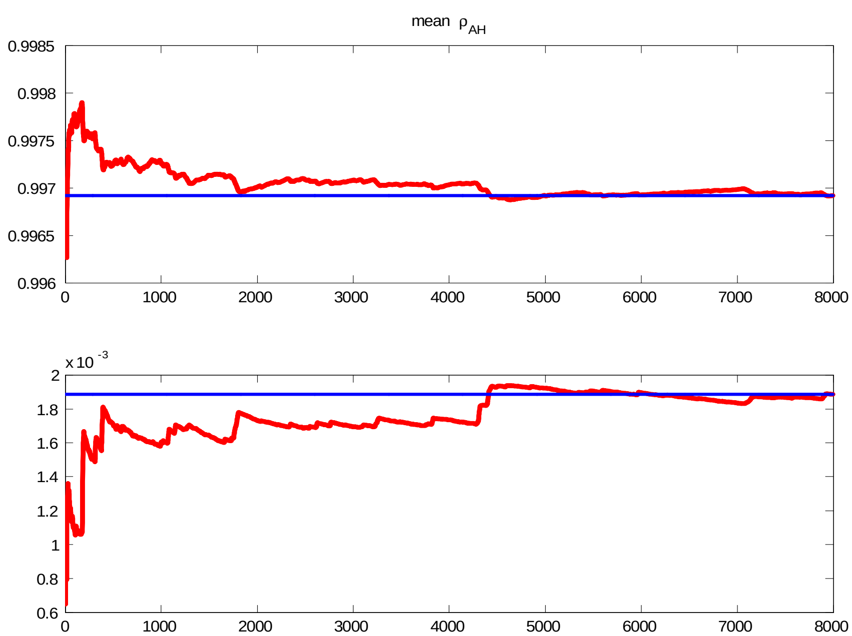
$\iota_{w,c}$



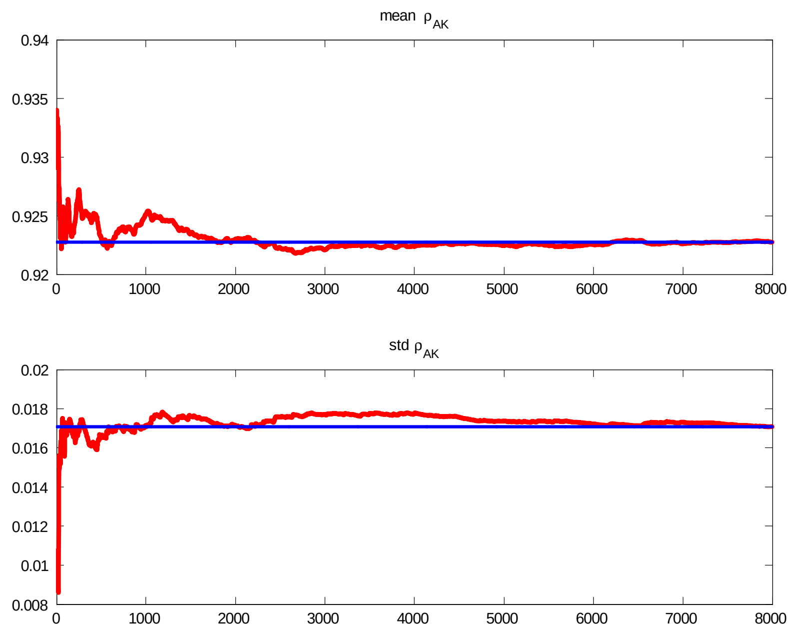
$\iota_{w,h}$



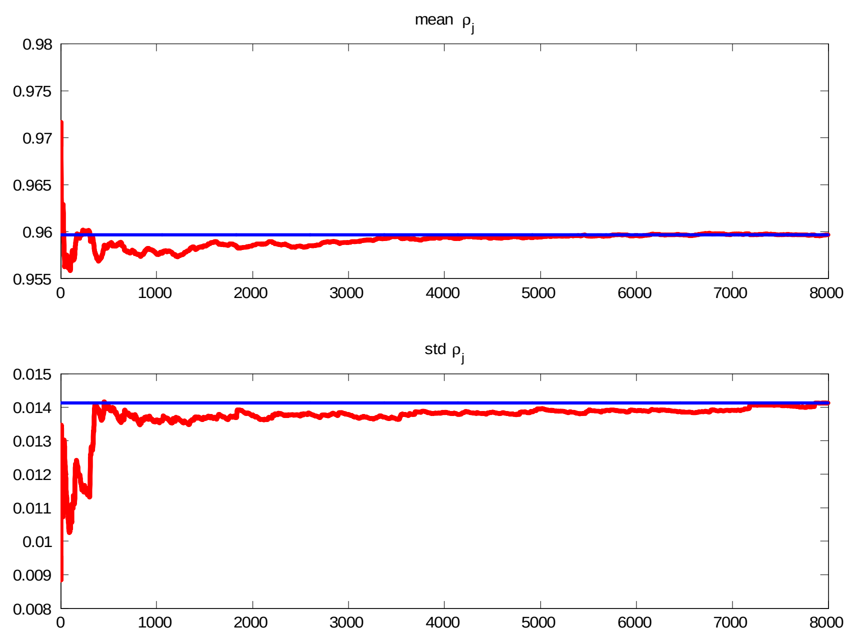
ρ_{AC}



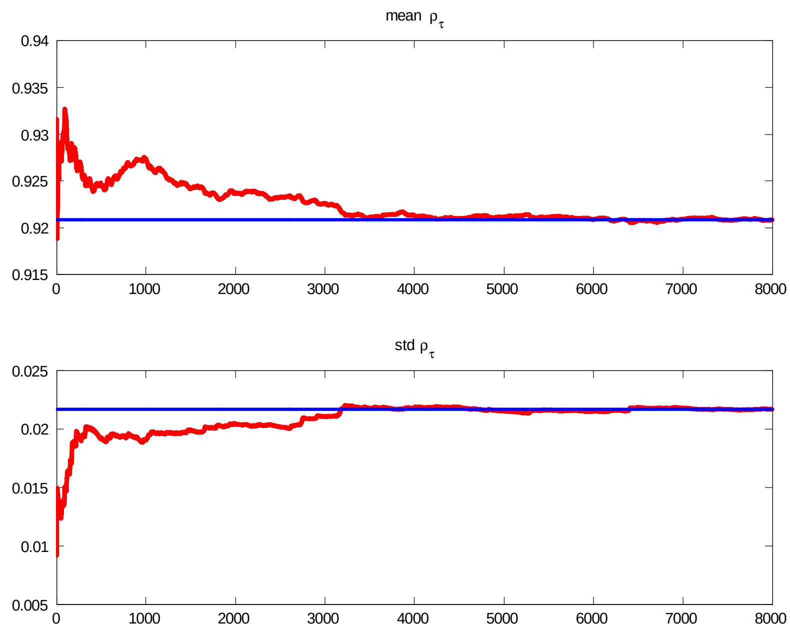
ρ_{AH}


mean ρ_AK
0.94
0.935
0.93
0.925
0.92
0
1000
2000
3000
4000
5000
6000
7000
8000
std ρ_AK
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0
1000
2000
3000
4000
5000
6000
7000
8000

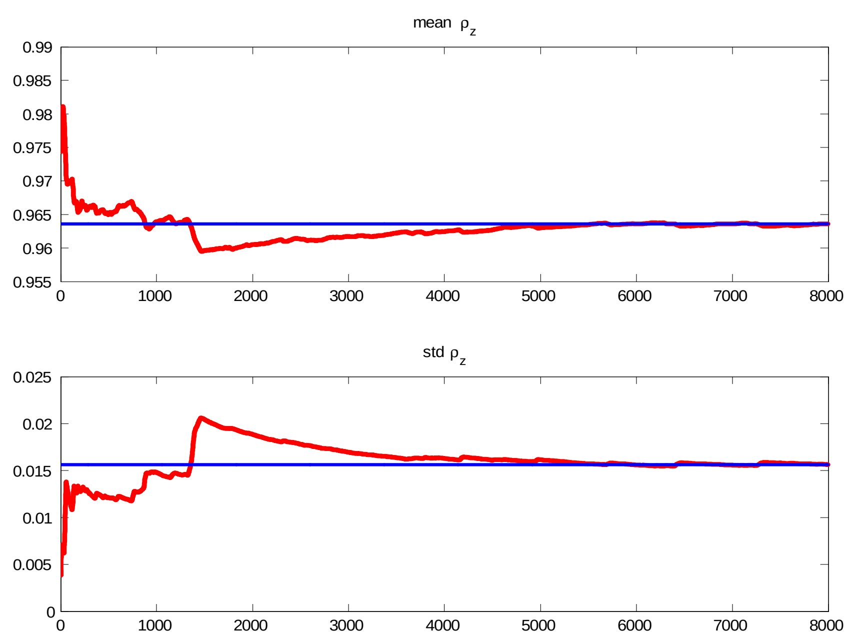
ρ_{AK}


mean ρ_j
0.98
0.975
0.97
0.965
0.96
0.955
0
1000
2000
3000
4000
5000
6000
7000
8000
std ρ_j
0.015
0.014
0.013
0.012
0.011
0.01
0.009
0.008
0
1000
2000
3000
4000
5000
6000
7000
8000

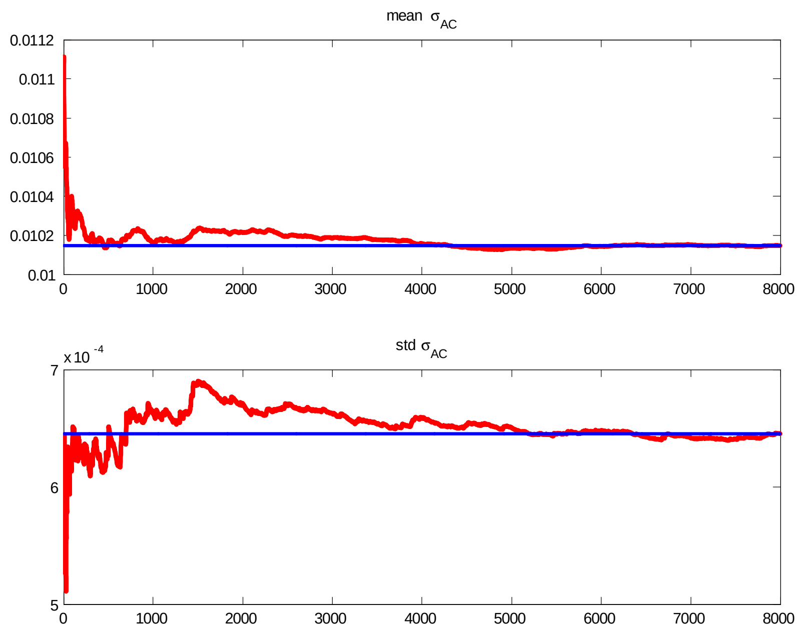
ρ_j


mean ρ_τ
0.94
0.935
0.93
0.925
0.92
0.915
0
1000
2000
3000
4000
5000
6000
7000
8000
std ρ_τ
0.025
0.02
0.015
0.01
0.005
0
1000
2000
3000
4000
5000
6000
7000
8000

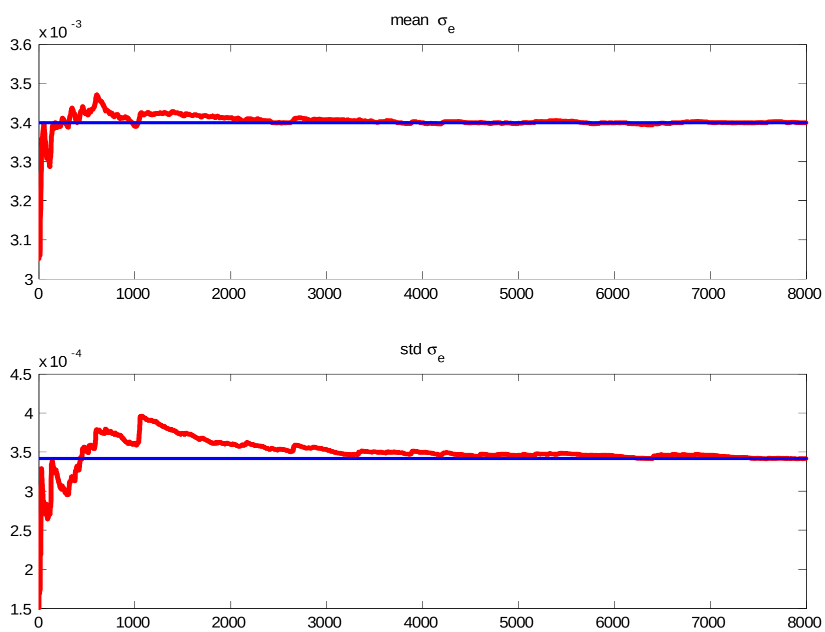
ρ_τ


mean ρ_z
0.99
0.985
0.98
0.975
0.97
0.965
0.96
0.955
0
1000
2000
3000
4000
5000
6000
7000
8000
std ρ_z
0.025
0.02
0.015
0.01
0.005
0
0
1000
2000
3000
4000
5000
6000
7000
8000

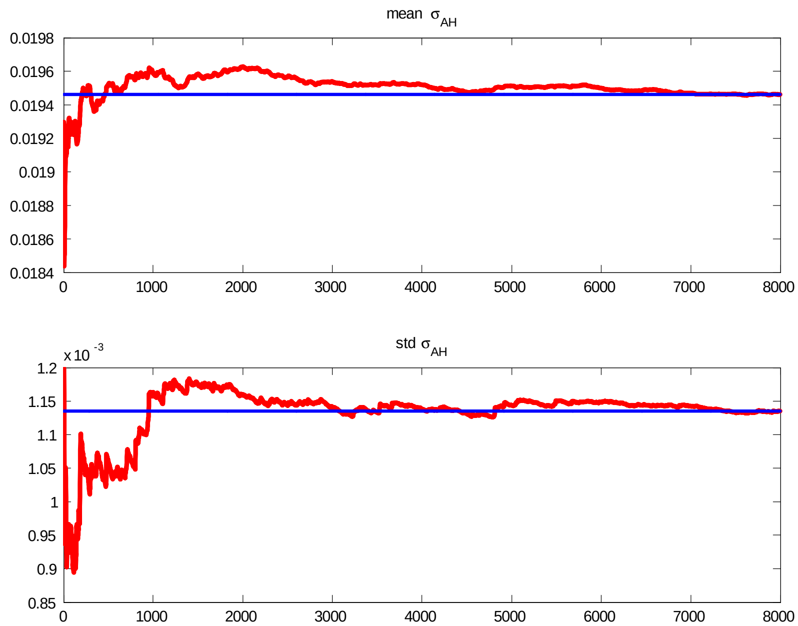
ρ_z



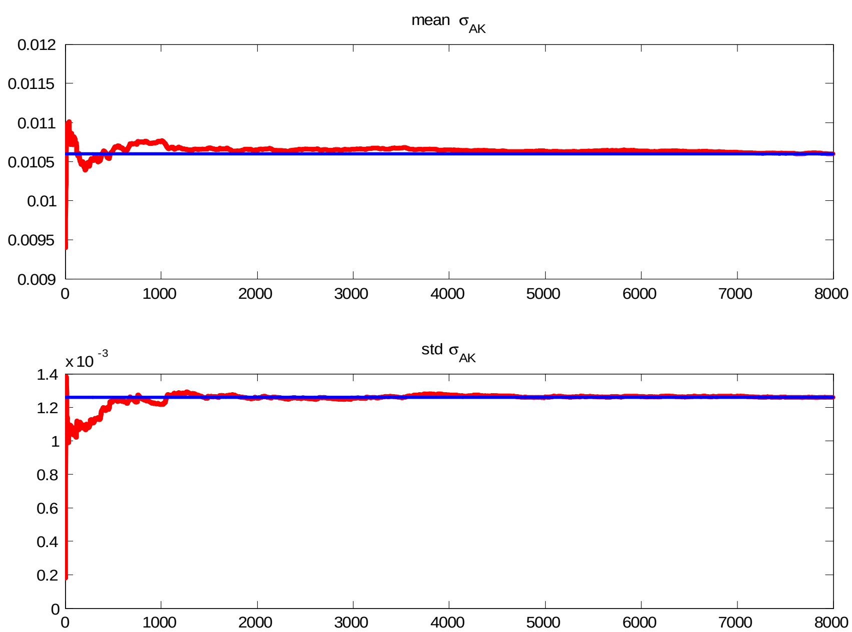
σ_{AC}



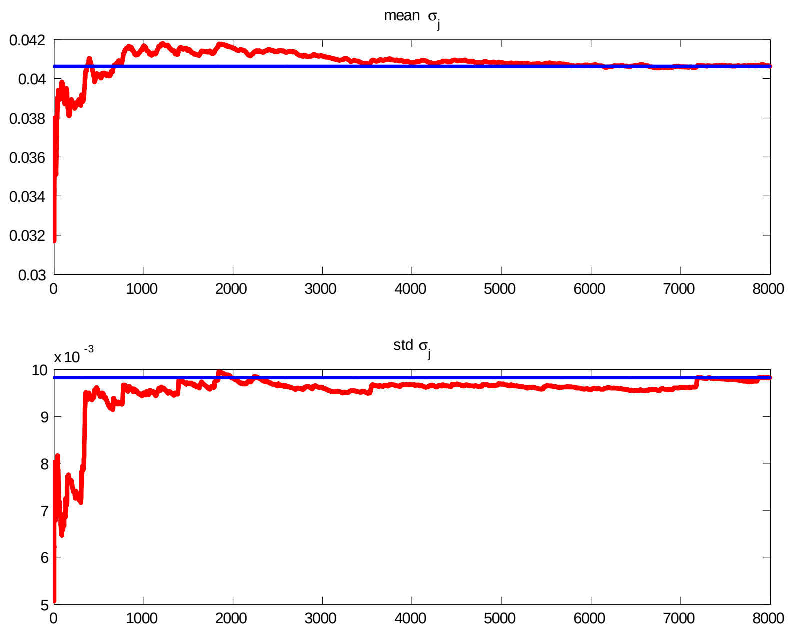
σ_e



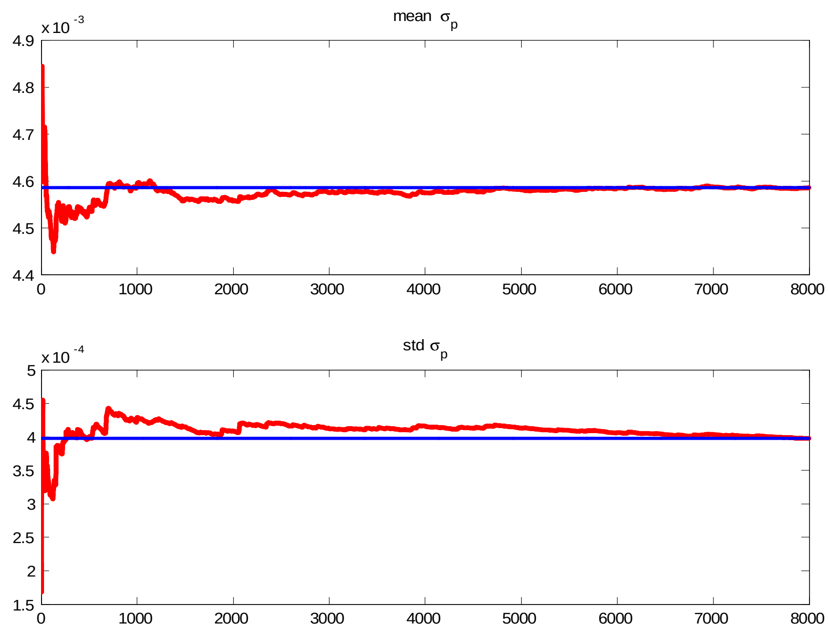
σ_{AH}



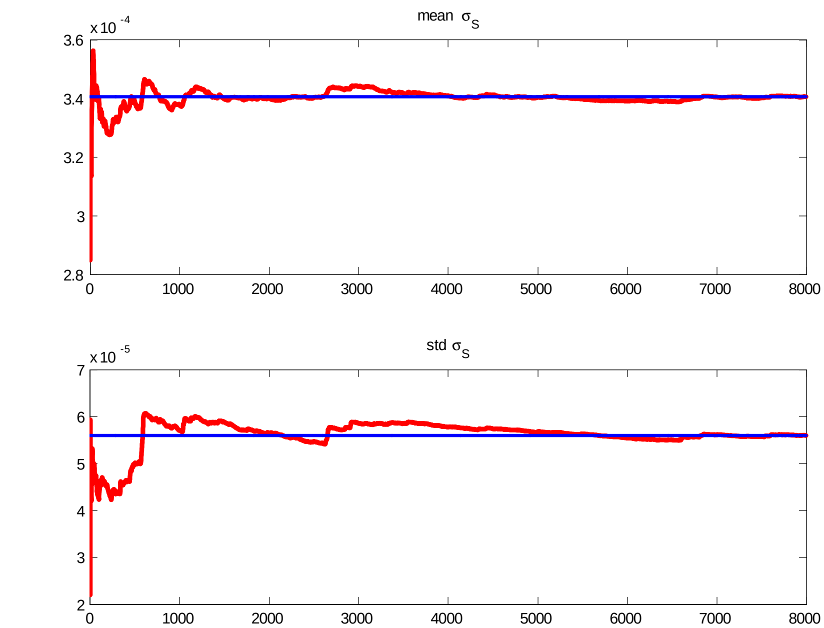
σ_{AK}



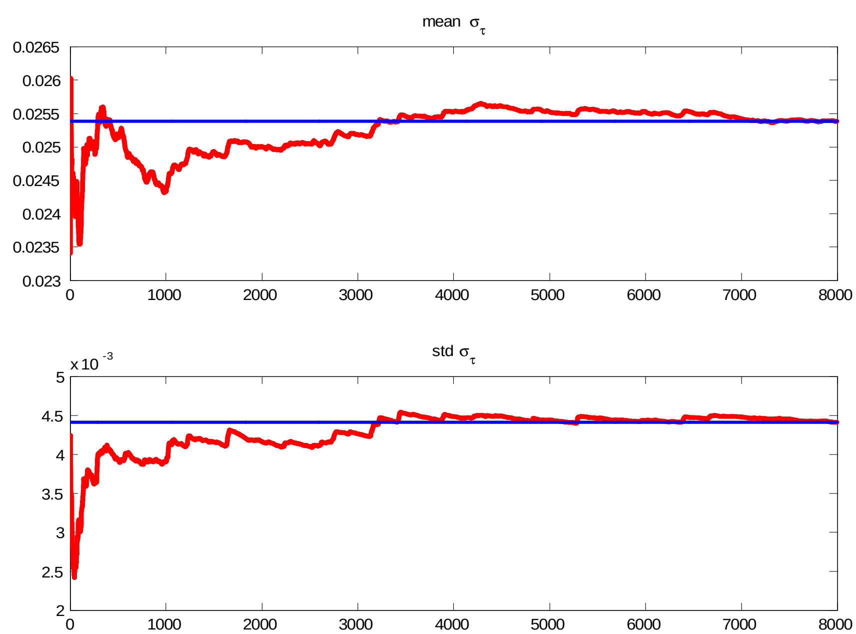
σ_j



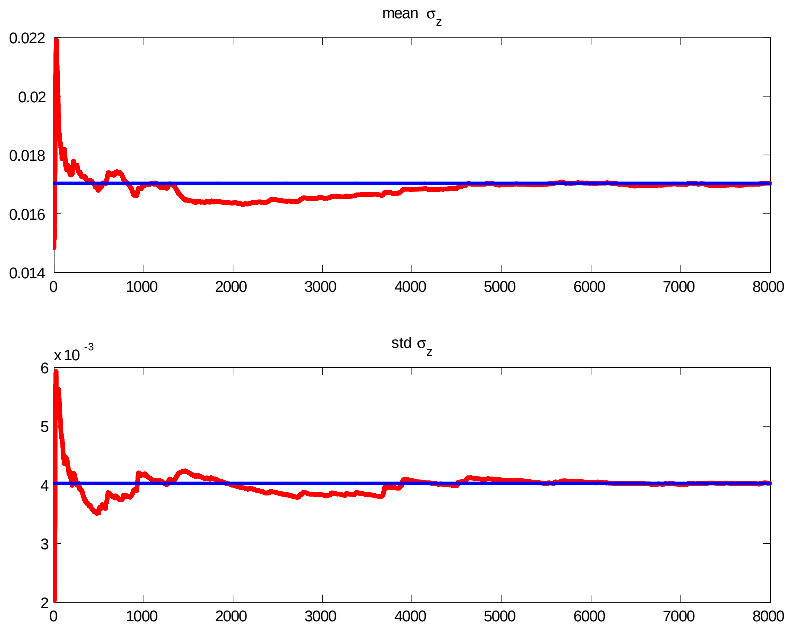
σ_p



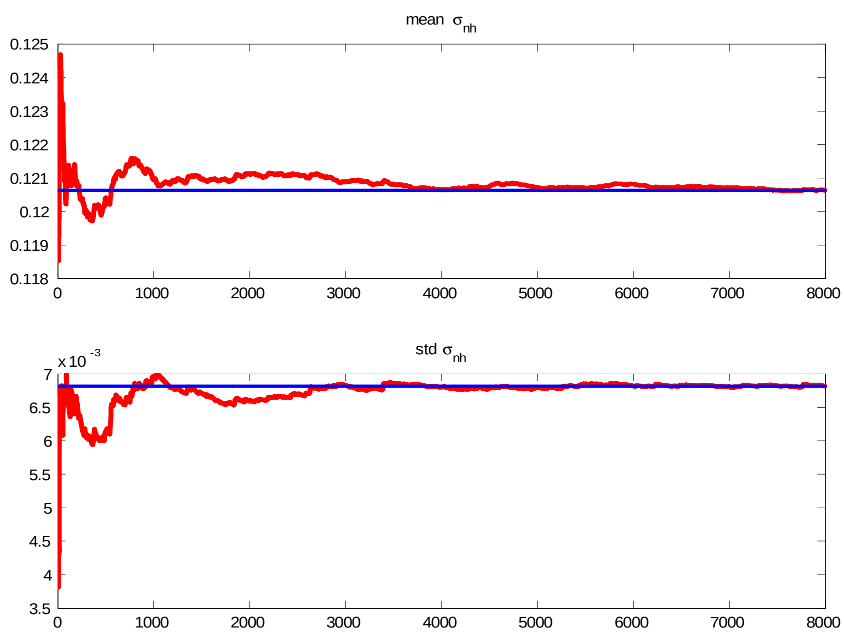
σ_s



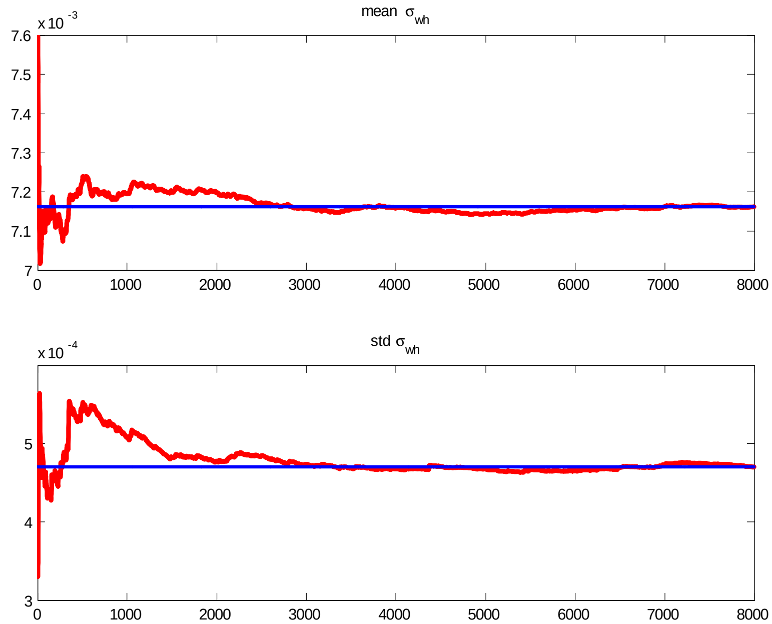
σ_τ



σ_z



$\sigma_{n,h}$

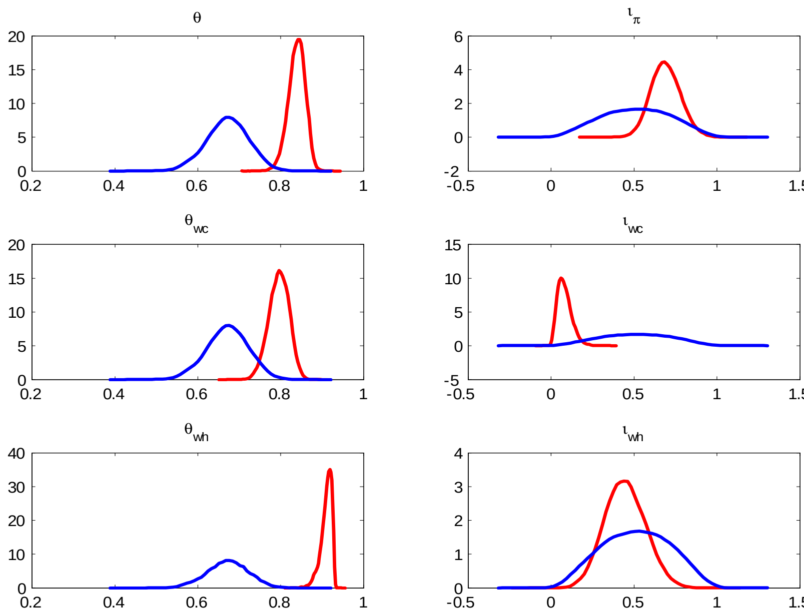


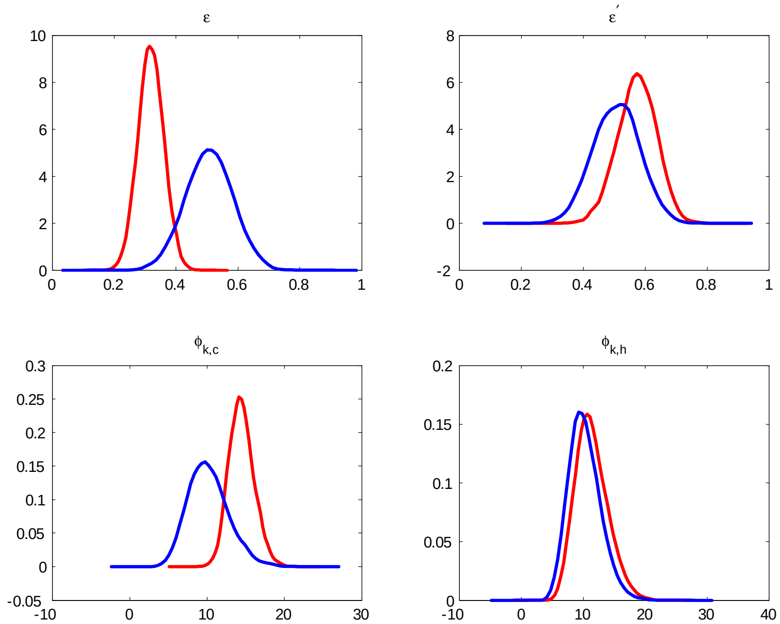
mean σwh
x 10 -3
7.6
7.5
7.4
7.3
7.2
7.1
7
0
1000
2000
3000
4000
5000
6000
7000
8000
std σwh
x 10 -4
5
4
3
0
1000
2000
3000
4000
5000
6000
7000
8000

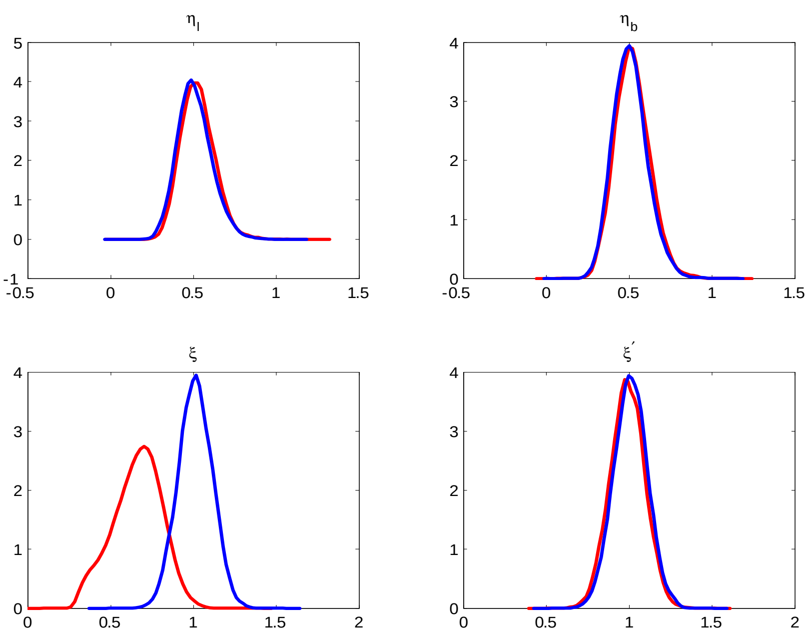
$\sigma_{n,h}$

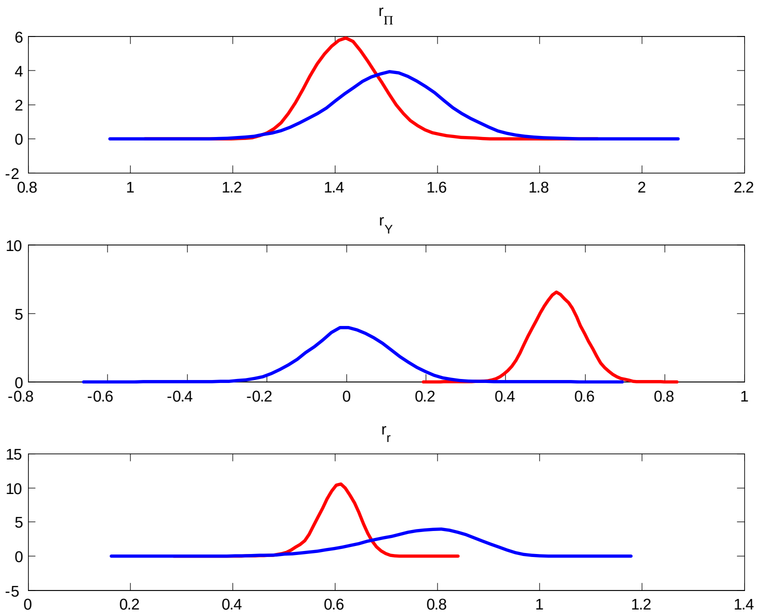
3.3 Prior and posterior densities

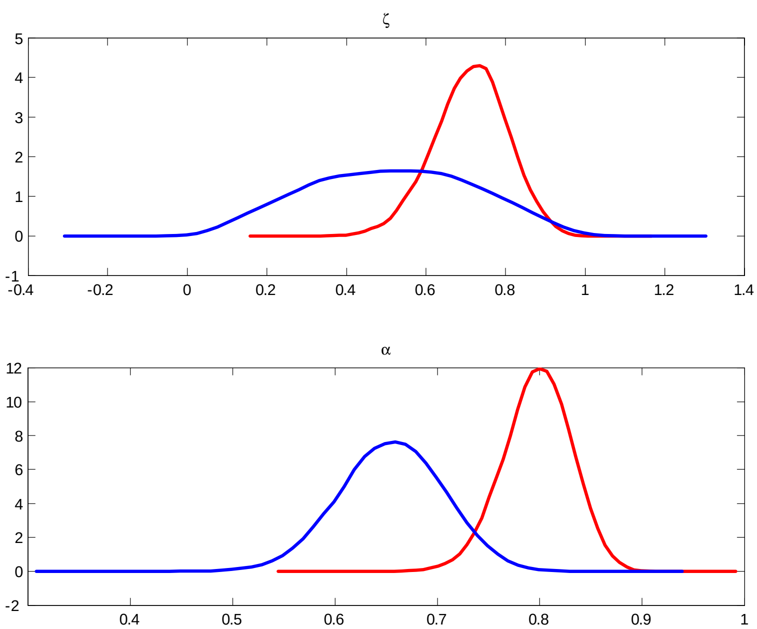
In the following graphs we report the prior and posterior densities of selected parameters. The posterior ones are based on 500,000 draws from the Metropolis algorithm and are estimated using a Gaussian kernel. Red lines denote the posterior density while the blue one the prior density.




ε
ε´
φk,c
φk,h


ηl
ηb
ξ
ξ´


rΠ
6
4
2
0
-2
0.8
1
1.2
1.4
1.6
1.8
2
2.2
rY
10
5
0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
rr
15
10
5
0
-5
0
0.2
0.4
0.6
0.8
1
1.2
1.4


ζ
5
4
3
2
1
0
-1
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
α
12
10
8
6
4
2
0
-2
0.4
0.5
0.6
0.7
0.8
0.9
1

Appendix D: Robustness Analysis for "Housing Market Spillovers: Evidence from an Estimated DSGE Model"

1 Overview

This Appendix reports the results of additional robustness exercises that are mentioned in the paper "Housing Market Spillovers: Evidence from an Estimated DSGE Model".

2 Robustness Analysis

2.1 The Role of Shocks and Frictions.

The ability of the various shocks and frictions to match certain moments of the data has been assessed by reestimating the model shutting one or more frictions or shocks off each time. Table D.1a reports the simulated (mean) volatilities of some of the observables used in estimation.[1] Table D.1b reports correlations among selected observables. Table D.2 reports the mode of the posterior distribution of the structural parameters.

Tables D.1a and D.1b report selected standard deviations and correlations for the data and the baseline model in columns (a) and (b). Column (c) reports statistics for the model without capital adjustment costs: this model generates excessive volatilility in business investment and a correlation between consumption and GDP that is far lower than in the data. The model with perfect labor mobility across sectors (column d) underestimates the positive correlation between housing prices and housing investment. The model with fixed capacity utilization (column e) generates excessive volatility of residential investment and house prices, and fails on the correlation between house prices and housing investment. The model with flexible wages and prices (column f) fails to account for the empirical volatilities and the correlations between the real variables (consumption, business and residential investment). Similar considerations apply to the model with flexible wage only (column g) and flexible prices only (column h). Finally, the model without borrowing constrained households (column i) is similar to the baseline model in terms of unconditional moments properties: however, as we already emphasize in the text, it generates a negative comovement between house prices and consumption conditional on a housing demand shock.

Moving to parameter estimates, Table D.2 reports the posterior distribution of the estimated parameters for the alternative model specifications in which real and nominal frictions are shut off one at a time.

The main differences concern the degree of substitutability between hours in the two sectors, the share of unconstrained agents and the parameters measuring the nominal rigidities.

Figures D.1 to D.4 show the estimated impulse responses for the alternative versions of the model. As we argue in the main text, wage rigidity is the most important friction to account for the differential responses of residential and business investment to monetary shocks. In the case of a housing preference shock borrowing constraints, nominal wage rigidity, variable capacity utilisation and imperfect labor mobility are all important elements in enhancing the model's ability to generate an increase in consumption following a shock that increases real house prics.

[1] The statistics are computed using the mode of the posterior distribution of the parameters and drawing 500 time series for each variable.

2.2 The Model with Technology Shocks only

The model has been estimated with only technology shocks. All the variables except residential investment, consumption and business investment have an AR(1) measurement error attached to the corresponding observation equation. Tables D.3 and D.4 report simulated volatilities and correlations of this model.

This model also generates a lower volatility of all the variables. It also produces a very low correlation between real house prices and real residential investment. It also generates a lower correlation between real residential investment and output, and misses the empirical correlation between consumption and housing prices (in the data it is 0.48, in the model it is 0.95).

2.3 The Role of House Price Data

In order to understand the implications of the choice of the data for house prices, we have estimated the model using the OFHEO price index in place of the Census one.[2] We have also estimated the model using both time series under the assumption that each series measures house prices up to some measurement error. In this case we have assumed the following measurement equations:

$$\hat{q}_t^{Census} = \widehat{q}_t + v_t^{Census}$$

$$\hat{q}_t^{OFHEO} = \widehat{q}_t + v_t^{OFHEO}$$

where the two measurement errors v_t^{Census} and v_t^{OFHEO} are assumed to be two mutually independent, serially correlated processes:

$$v_t^{Census} = \rho_C v_{t-1}^{Census} + \epsilon_t^{Census}$$

$$v_t^{OFHEO} = \rho_O v_{t-1}^{OFHEO} + \epsilon_t^{OFHEO}$$

and $\widehat{q}_t$ denotes the model counterpart of log real house prices, in deviation from the linear trend.

Figures D.5 and D.6 compare the impulse responses to housing preference (Figure D.5) and monetary policy (Figure D.6) shocks computed using the mode of the posterior distribution of the parameters obtained using the Census, the OFHEO and both indices. The implications for the results in the paper are evaluated in terms of parameters estimates, impulse responses and historical decompositions of real house prices. Figures D.5 and D.6 show that the three sets of impulse responses are virtually identical.

A comparison of the contribution of monetary policy to the historical decomposition of real house prices shows that the effects of changes in the nominal interest rate have had similar consequences on house prices, either measured by the Census or the OFHEO indices. Table D.5 shows that the mode of the parameters estimated using the two house price data, both separately and at the same time do not differ substantially.

2.4 The Role of Heterogeneous Household Preferences

We have estimated a version of our model where we constrain the preference parameters to be the same across the two types of agents. The results are reported in Table D.6 and are quantitatively similar to those reported in the paper.

[2] The Census series starts in 1965. The OFHEO series is only available from 1970. To ensure compatibility across the two sets of estimates, we extrapolate the OFHEO series backwards for the years 1965-1969 using the growth rate of the Census series during the same period.

Concerning the effects of a positive housing preference shock, the model in which constrained and unconstrained agents have identical preferences (except for the discount factor) suggests a larger elasticity of real consumption to real house prices. The response on impact is twice the one in the model with heterogeneous households' preferences.

With respect to the effects of an expansionary monetary policy shock, the model with identical preferences suggests a larger response of residential investment.

3 Some Additional Checks

3.1 What is the Role of Measurement Error?

In the baseline model, we allow for (iid) measurement error only in wages and hours of the construction sector. We have estimated two alternative versions of the model with different assumption regarding measurement error: (1) a version of our model where we allow for measurement error also in wages and hours in the non-housing sector; (2) a version of our model where we allow for measurement error in all time series. We found no major differences across models for the estimates of the key model parameters. The main discrepancy between our benchmark model and the versions with measurement error arises when we compare monetary shocks between the model with measurement error only for the housing labor market with the model with measurement error in all variables. Allowing for measurement error in all variables reduces the contribution of monetary policy shocks to business fluctuations. In practice, the model assigns a good deal of the fluctuations in interest rates to "noise", rather than random changes in the monetary policy rule. For this reason, the standard deviation of the monetary shocks is found to be smaller (it falls from around 0.3 percent to 0.1 percent), so that monetary shocks play a smaller role in the historical decomposition. Figure D.10 plots the impulse responses to a monetary shock in the three models. The response in the model with measurement error for all variables are a scaled-down version of those of the benchmark model.

3.2 What do Credit Shocks do?

As a further check, we have estimated our model by allowing m to change over time. We have treated m_t as an observable random variable that follows an AR(1) process of the form $\log m_t = (1-\rho_m)\log \overline{m} + \rho_m \log m_{t-1} + \varepsilon_{mt}$, where ε_{mt} is an i.i.d disturbance with standard deviation σ_m. We have then constructed a time series for m_t[3] using as a proxy household leverage, constructed as the ratio of outstanding home mortgages over holdings of residential real estate.[4] The implied series is plotted in Figure D.11 together with the real house price series used in estimation.[5] As the figure shows, the run-up in house prices since the late 1990s is roughly concomitant with an increase in leverage of the household sector. However, household leverage fell in the 1970s (when house prices also rose dramatically) and did not change much between 1997 and 2001 (at the beginning of the housing boom). Our estimates (including the results from the historical decomposition) for this

[3] We have normalized $\overline{m} = 0.85$ as in our benchmark model.

[4] The series are from the Flows of Funds. Home mortgages are in line 32 of Table B.100 - series FL153165105.Q -. Residential real estate holdings are in line 4 of Table B.100 - series FL155035015.Q -.

[5] We set the average loan-to-value ratio $\overline{m}$ to 0.85, and feed into the model the demeaned series for leverage plotted in Figure 2. Therefore, we do not use information on average leverage as an input in estimation. The reason why we do so is because in the data many households have a mortgage but behave as unconstrained households, smoothing consumption through other means (for instance, they might own equity and a mortgage at the same time, or they borrow less than the maximum amount).

version of the model are essentially unchanged from the baseline, with the only exception that we now estimate two additional parameters, ρ_m and σ_m. Their estimated values are respectively 0.994 and 0.0049.

A persistent shock to m_t leads to a protracted increase in debt, housing prices and investment, and consumption. However, the quantitative effects are small, and insufficient to generate large fluctuations in house prices. A one standard-deviation shock (impulse responses are plotted in Figure D.12) changes leverage by 50 basis points (from, say, 0.85 to 0.855) and, while it affects debt substantially (because it creates a large transfer of housing stock from lenders to borrowers), it produces a modest effect on house prices (house prices increase by less than 0.05 percent). Most of the effects of the leverage shock involve reallocation of the housing stock from one class of agents to another, but their effect on housing prices is limited.

To gain insight into why changes in m have little effect on prices, one can study the two key equations that determine the equilibrium behavior of housing demand (equations $A.2$ and $A.3$ in appendix B of the paper). After rearranging, these two equations can be combined to write the two relative housing demand equations as:

$$\underbrace{\frac{u_{h,t}}{u_{c,t}}}_{\text{MRS b/w housing and consumption}} = \underbrace{q_t - E_t\left(\frac{1}{RR_t}q_{t+1}\left(1-\delta_h\right)\right)}_{\text{user cost of housing for unconstrained agents}}$$

$$\underbrace{\frac{u_{h',t}}{u_{c',t}}}_{\text{MRS b/w housing and consumption}} = \underbrace{q_t - E_t\left(\frac{\beta' G_C u_{c',t+1}}{u_{c',t}}q_{t+1}\left(1-\delta_h\right) + \left(\frac{1}{RR_t} - \frac{\beta' G_C u_{c',t+1}}{u_{c',t}}\right)m_t q_{t+1}\right)}_{\text{user cost of housing for constrained agents}}$$

For changes in leverage to significantly affect housing demand and prices, it must be that they significantly alter the user cost (the right-hand side of the equations above). For unconstrained agents, changes in m do not affect their housing demand. For constrained agents, an increase in m lowers the user cost, with an effect that grows with gap between the stochastic discount factor (the term $1/RR_t - \beta' G_C u_{c',t+1}/u_{c',t}$) of the two groups. But, from a quantitative standpoint, this effect of changes in m is not large enough to generate large increases in asset prices.

4 Estimated Shocks and Newspaper Accounts

We have conducted a search of newspapers' articles for the period 1965-2006 trying to relate, from an informal standpoint, our estimated shocks to contemporary accounts of developments in the housing market at the time. Commentators' accounts in general agree with us that house prices are driven by a large variety of factors, such as inflation, technology, monetary policy. They also seem to refer from time to time to mysterious changes in housing demand that they could not immediately attribute to changes in fundamentals. Below, we report some examples:

For Housing preference shocks

- A positive housing shock in January 1970: "Privacy is another important factor for the buyer in today's market. With the increasing pressures of crowded living, more and more people are searching for solitude" [Anonymous (1970, January 18). "As Tastes Change, So Does Broker's Pitch: Brokers Attuned to Tastes," *New York Times.*]

- In 1975, a positive housing demand shock coming to an end: "People have been buying a lot more house than necessary... They've had empty living rooms with plastic covers on the furniture while they were using the family room..." [Lindsey, Robert (1975, December 7). "Less House for a Home; Less House, and More Money, for a Home," *New York Times.*]

And for the recent boom:

- In 1998, the beginning of the housing boom: "Another show of wealth is a trend for buyers to raze expensive houses to build newer, even more expensive houses more to their taste" [Rather, John (1998, January 11). "Luxury Houses: Strong Market, Low Inventory; A strong economy and Wall St. are just part of a boom," *New York Times.*]
- In 2001, signs of strong demand again, not based on clear fundamentals: "Housing strength also reflects surprisingly resilient consumer confidence. Memories have faded away of the early 1990s... Faith in real estate as an investment remains strong" [Norris, Floyd (2001, June 22). "Will This Slowdown Spare Housing, or Just Hit It Late?" *New York Times*]

For Housing supply shocks

The 1970's are also a period where supply-side conditions were often cited (at least, they were cited far more than in the recent housing cycle) by the press and commentators are one of the reasons for rising high prices in a period where the quantity could not keep up pace with demand. Examples include:

- In 1973: "Building executives complained of severe shortages of various types of lumber products" [Tomasson, Robert E. (1973, January 14). "Lumber Costs Smashing Control Barriers: So the Prices Of Homes Go Up and Up," *New York Times.*]
- In 1974: "As the cost of building materials have increased drastically, and the wages of construction workers have soared, the cost of new housing has brought up the selling price on existing homes" [Jensen, Micheal C. (1974, August 25). "Home Buyers All Over U.S. Feel the Economy's Crunch," *New York Times*]
- In 1978: "Unless some significant improvements occur on the supply side, [construction] prices will remain high... lumber mills have been shut or curtailed for lack of timber [...] The major question now is whether there will be a change in national forest policies" [Mullaney, Thomas E. (1978, April 28). "Trimming Cost of Home Building By Cutting Into National Forests," *New York Times.*]

Table D.1a. Volatilities of selected observables (percentages): the role of frictions

	data	baseline	no k. adj.	full lab. mob.	fixed cap.	flex. w and p.	flex. w.	flex. p.	no collat. constr.
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
	.	.	.	.	.	.	.	.	.
C	1.22	1.40	1.28	1.51	1.37	0.88	1.16	1.76	1.37
π	0.40	0.47	0.48	0.47	0.50	-	0.46	0.69	0.45
IH	9.97	8.01	9.05	8.37	12.59	5.96	6.16	9.87	7.81
q	1.87	2.08	1.80	2.12	4.83	1.90	2.04	2.73	2.06
R	0.32	0.32	0.47	0.31	0.36	-	0.33	0.39	0.31
IK	4.87	3.80	7.25	3.71	4.08	2.83	3.52	5.98	3.78
GDP	2.17	2.11	2.07	2.02	1.90	1.26	1.66	1.90	2.04

Notes: The volatilities are computed using 500 draws of the time series obtained by setting the parameters of the model at the mode of the posterior distribution. The business cycle component of each variable is obtained using the HP filter with smoothing parameter set at 1,600. C: real consumption; IH: real residential investment; q: real house prices; R: nominal interest rate; IK: real business investment; GDP: output.

Table D.1b. Selected correlations: the role of frictions

	data	baseline	no k. adj.	full lab. mob.	fixed cap.	flex. w and p.	flex. w.	flex. p.	no collat. constr.
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
$\rho\,(C, GDP)$	0.88	0.82	0.41	0.86	0.65	0.67	0.83	0.03	0.88
$\rho\,(IH, GDP)$	0.78	0.65	0.09	0.56	0.53	0.42	0.30	0.57	0.66
$\rho\,(IK, GDP)$	0.75	0.89	0.86	0.88	0.82	0.87	0.89	0.79	0.90
$\rho\,(q, GDP)$	0.58	0.65	0.12	0.60	0.20	0.36	0.51	0.45	0.62
$\rho\,(q, C)$	0.48	0.46	0.32	0.59	0.19	0.12	0.45	-0.25	0.58
$\rho\,(q, IH)$	0.41	0.45	0.29	0.25	-0.17	0.12	0.06	0.57	0.43

Notes: The correlations are computed using 500 draws of the time series obtained by setting the parameters of the model at the mode of the posterior distribution. The business cycle component of each variable is obtained using the HP filter with smoothing parameter set at 1,600. C: real consumption; IH: real residential investment; Q: real house prices; R: nominal interest rate; IK: real business investment; NC: hours worked in the goods sector; NH; hours worked in the residential sector; Y: output.

Table D.2. Posterior modes of alternative models: the role of frictions

par.	baseline (a)	no capital adj. cost (b)	full labor mobility (c)	fixed capital util. (d)	flex. wage and price (e)	flex. wage (f)	flex. price (g)	no collateral constraint (h)
	.	.	.	.	.	.	.	.
ε	0.3117	0.3774	0.3053	0.3883	0.2642	0.1656	0.4110	0.2802
ε'	0.5749	0.5471	0.5495	0.6608	0.3547	0.4914	0.5708	-
η	0.4789	0.4415	0.4520	0.5662	0.4553	0.4604	0.6640	0.5057
η'	0.4738	0.4731	0.4858	0.5577	0.4680	0.5175	0.4818	-
ξ	0.7523	0.7367	-	1.0703	1.2294	0.6810	0.7740	0.6862
ξ'	0.9790	0.9802	-	1.0319	1.1485	0.9938	0.9590	-
$\phi_{k,c}$	16.0126	-	16.4534	16.5198	15.6971	16.0854	15.2134	16.0838
$\phi_{k,h}$	10.0026	-	10.0170	9.9059	13.3400	8.5111	9.3556	10.0392
α	0.7970	0.7785	0.8170	0.8841	0.6086	0.8725	0.7632	-
r_R	0.6071	0.2607	0.6091	0.6539	-	0.5771	0.6699	0.6008
r_π	1.3743	1.5156	1.3902	1.3883	-	1.3895	1.7849	1.3404
r_Y	0.4938	0.7128	0.5104	0.4010	-	0.4284	0.3220	0.4785
θ_π	0.8393	0.7987	0.8453	0.7987	-	0.8157	-	0.8548
ι_π	0.6961	0.0452	0.6791	0.8223	-	0.7294	-	0.6373
$\theta_{w,c}$	0.7901	0.9118	0.7674	0.7460	-	-	0.7497	0.7598
$\iota_{w,c}$	0.0656	0.0783	0.0988	0.0354	-	-	0.1996	0.0656
$\theta_{w,h}$	0.9218	0.9169	0.7995	0.7202	-	-	0.9352	0.9170
$\iota_{w,h}$	0.4134	0.6371	0.4298	0.2787	-	-	0.4482	0.4101
ζ	0.7469	0.6171	0.7683	-	0.1747	0.6784	0.9866	0.7520
γ_{AC}	0.0032	0.0033	0.0032	0.0032	0.0031	0.0032	0.0055	0.0032
γ_{AH}	0.0008	0.0014	0.0010	0.0008	0.0007	0.0005	0.0103	0.0007
γ_{AK}	0.0027	0.0026	0.0027	0.0027	0.0029	0.0028	-0.0009	0.0027

Table D.3. Volatilities: the model with only technology shocks

	C	π	IH	q	R	IK	n_c	n_h	GDP
Data	1.22	0.40	10.00	1.87	0.32	4.85	1.43	4.07	2.17
Benchmark	1.40	0.47	8.01	2.08	0.32	3.80	2.40	8.74	2.11
Only tech. shocks	0.92	0.11	5.72	0.90	0.09	3.07	0.80	5.73	1.53

Notes: The volatilities are computed using 500 draws of the time series obtained by setting the parameters of the model at the mode of the posterior distribution. The business cycle component of each variable is obtained using the HP filter with smoothing parameter set at 1,600. C: real consumption; IH: real residential investment; Q: real house prices; R: nominal interest rate; IK: real business investment; NC: hours worked in the goods sector; NH; hours worked in the residential sector; Y: output.

Table D.4. Selected correlations: the model with only technology shocks

	$\rho\,(C, GDP)$	$\rho\,(IH, GDP)$	$\rho\,(IK, GDP)$	$\rho\,(q, GDP)$	$\rho\,(q, C)$	$\rho\,(q, IH)$
Data	0.88	0.78	0.75	0.58	0.48	0.41
Benchmark	0.82	0.65	0.89	0.65	0.46	0.45
Only tech. shocks	0.83	0.54	0.84	0.73	0.95	0.17

Notes: The correlations are computed using 500 draws of the time series obtained by setting the parameters of the model at the mode of the posterior distribution. The business cycle component of each variable is obtained using the HP filter with smoothing parameter set at 1,600. C: real consumption; IH: real residential investment; Q: real house prices; R: nominal interest rate; IK: real business investment; NC: hours worked in the goods sector; NH; hours worked in the residential sector; Y: output.

Table D.5. Posterior modes of the model using alternative data for house prices

parameter	Census (a)	OFHEO (b)	Both (c)
ε	0.3117	0.2953	0.2925
ε'	0.5749	0.5671	0.5263
η	0.4789	0.5013	0.5091
η'	0.4738	0.4903	0.4902
ξ	0.7523	0.7908	0.7761
ξ'	0.9790	0.9931	0.9743
$\phi_{k,c}$	16.0126	15.0621	16.7160
$\phi_{k,h}$	10.0026	10.0242	10.6020
α	0.7970	0.7891	0.7741
r_R	0.6071	0.6066	0.5975
r_π	1.3743	1.3686	1.3847
r_Y	0.4938	0.4906	0.4959
θ_π	0.8393	0.8448	0.8523
ι_π	0.6961	0.6899	0.6515
$\theta_{w,c}$	0.7901	0.7774	0.7770
$\iota_{w,c}$	0.0656	0.0593	0.0627
$\theta_{w,h}$	0.9218	0.9276	0.9237
$\iota_{w,h}$	0.4134	0.3346	0.3776
ζ	0.7469	0.6919	0.6244
γ_{AC}	0.0032	0.0034	0.0032
γ_{AH}	0.0008	-0.0013	0.0005
γ_{AK}	0.0027	0.0025	0.0027

Table D.6. Posterior modes of alternative models: the role of household preferences

parameter	Benchmark (a)	Same preferences (b)
ε	0.3117	0.3929
ε'	0.5749	0.3929
η	0.4789	0.5115
η'	0.4738	0.5115
ξ	0.7523	0.8691
ξ'	0.9790	0.8691
$\phi_{k,c}$	16.0126	16.1596
$\phi_{k,h}$	10.0026	10.0581
α	0.7970	0.8062
r_R	0.6071	0.6358
r_π	1.3743	1.3913
r_Y	0.4938	0.5012
θ_π	0.8393	0.8526
ι_π	0.6961	0.6771
$\theta_{w,c}$	0.7901	0.7964
$\iota_{w,c}$	0.0656	0.0642
$\theta_{w,h}$	0.9218	0.9311
$\iota_{w,h}$	0.4134	0.4144
ζ	0.7469	0.5981
γ_{AC}	0.0032	0.0032
γ_{AH}	0.0008	0.0008
γ_{AK}	0.0027	0.0027

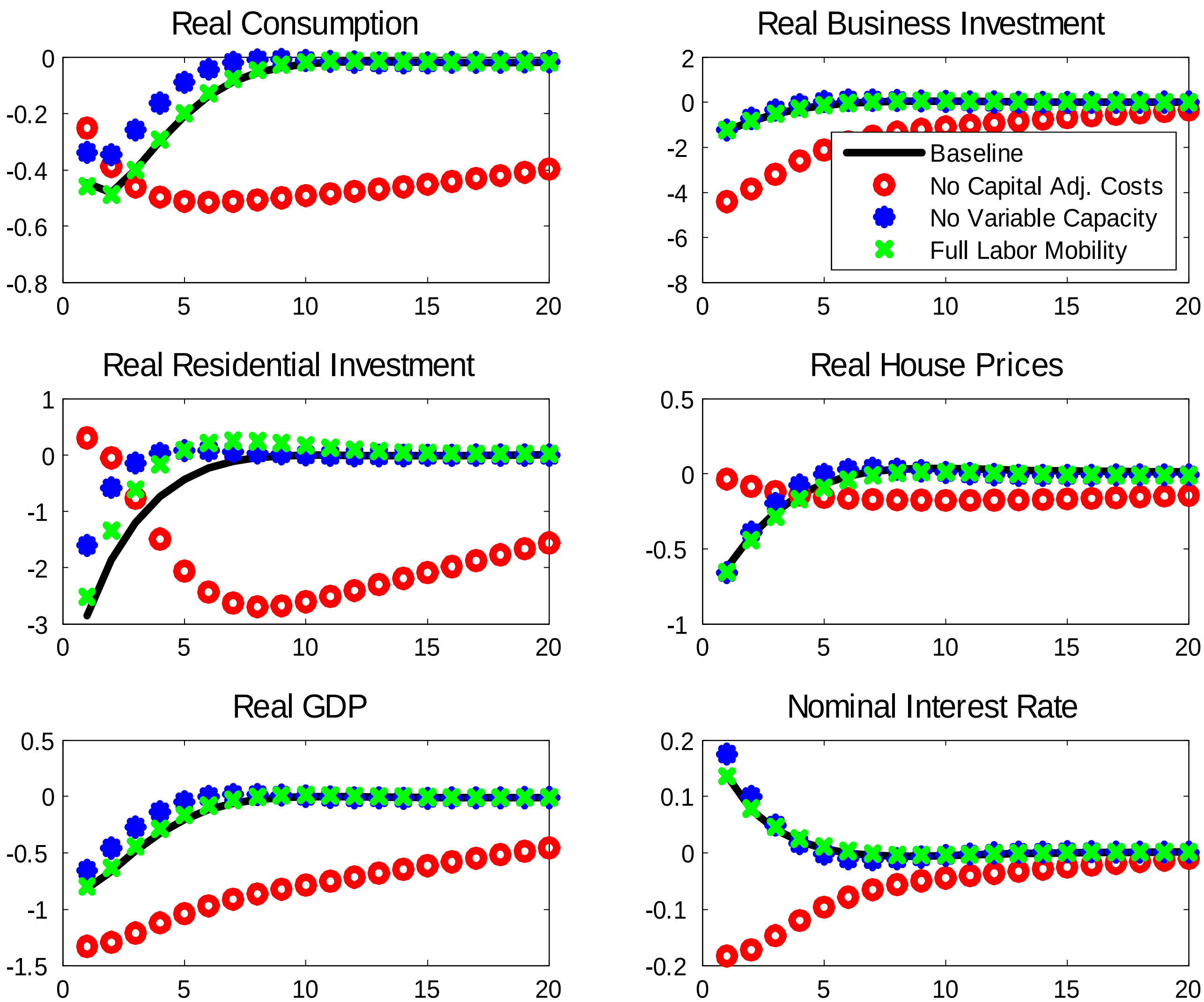


Figure D.1. Impulse responses to a monetary policy shock: the role of real rigidities

Note: horizontal axis: quarters from the shock; vertical axis: percentage deviation from the steady state.

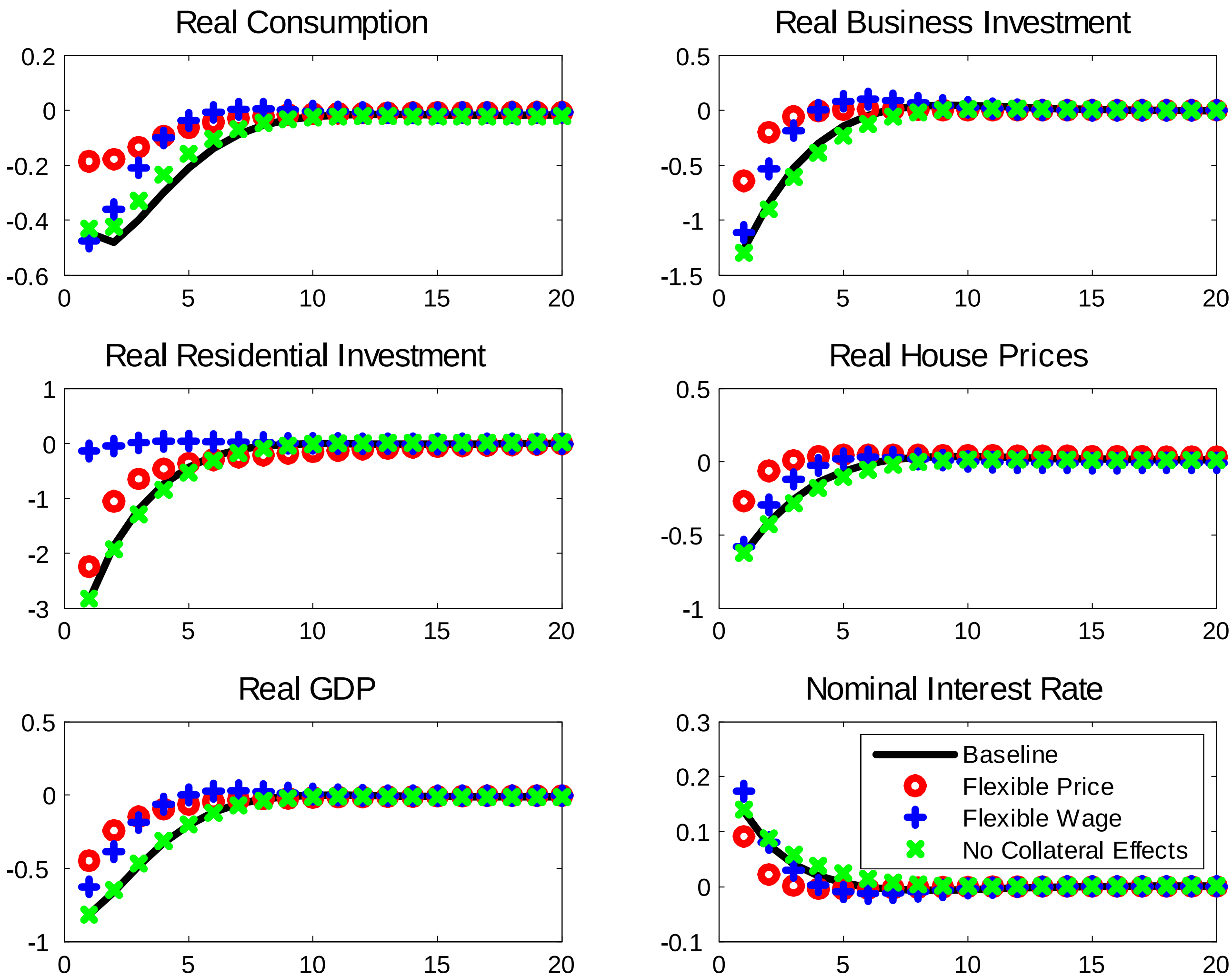


Figure D.2. Impulse responses to a monetary policy shock: the role of nominal rigidities and collateral constraints

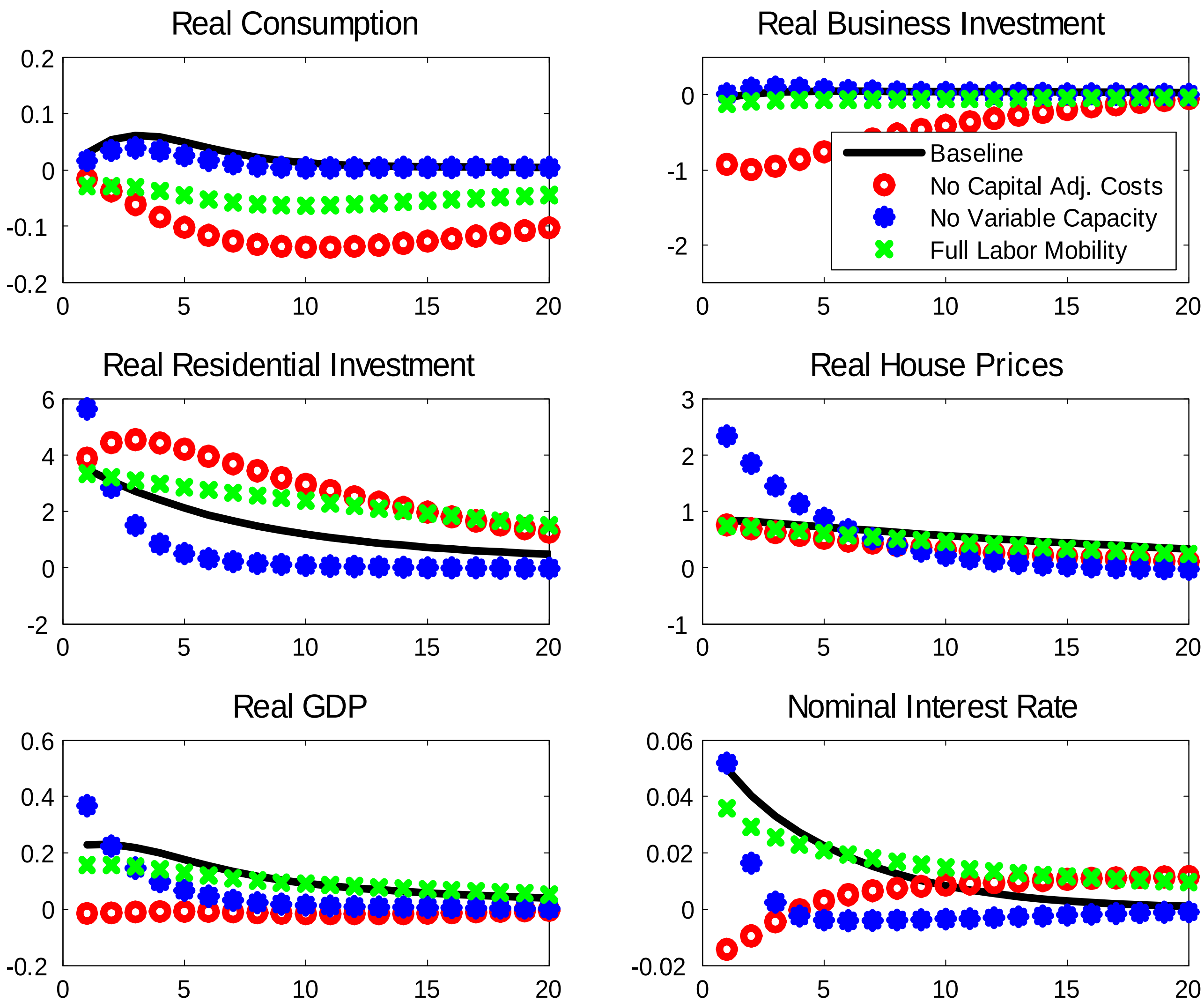


Figure D.3. Impulse responses to a housing preference shock: the role of real rigidities

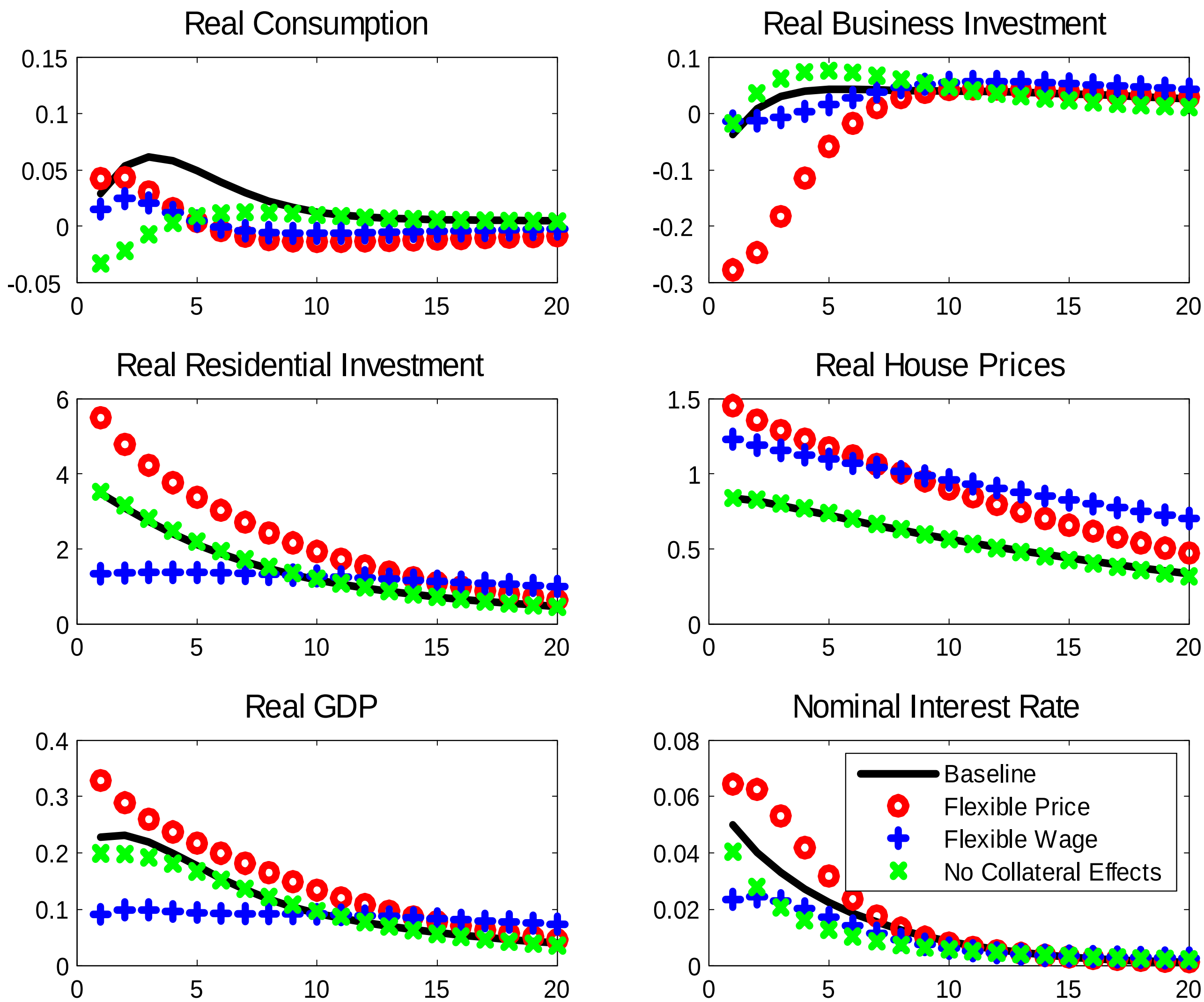


Figure D.4. Impulse responses to a housing preference shock: the role of nominal rigidities and collateral constraints

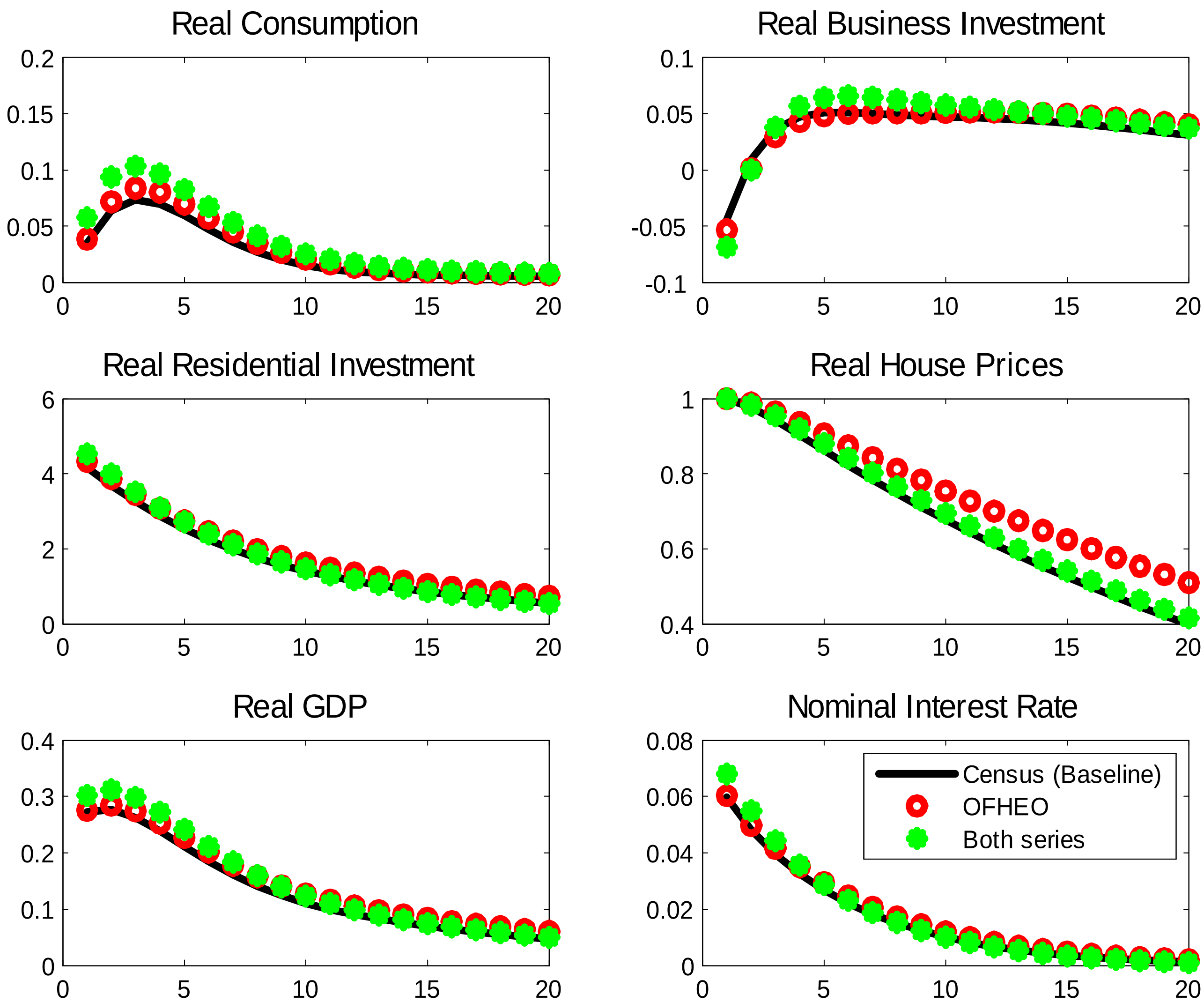


Figure D.5. Impulse responses to a housing preference shock: parameters estimated using the Census, the OFHEO and both indices at the same time

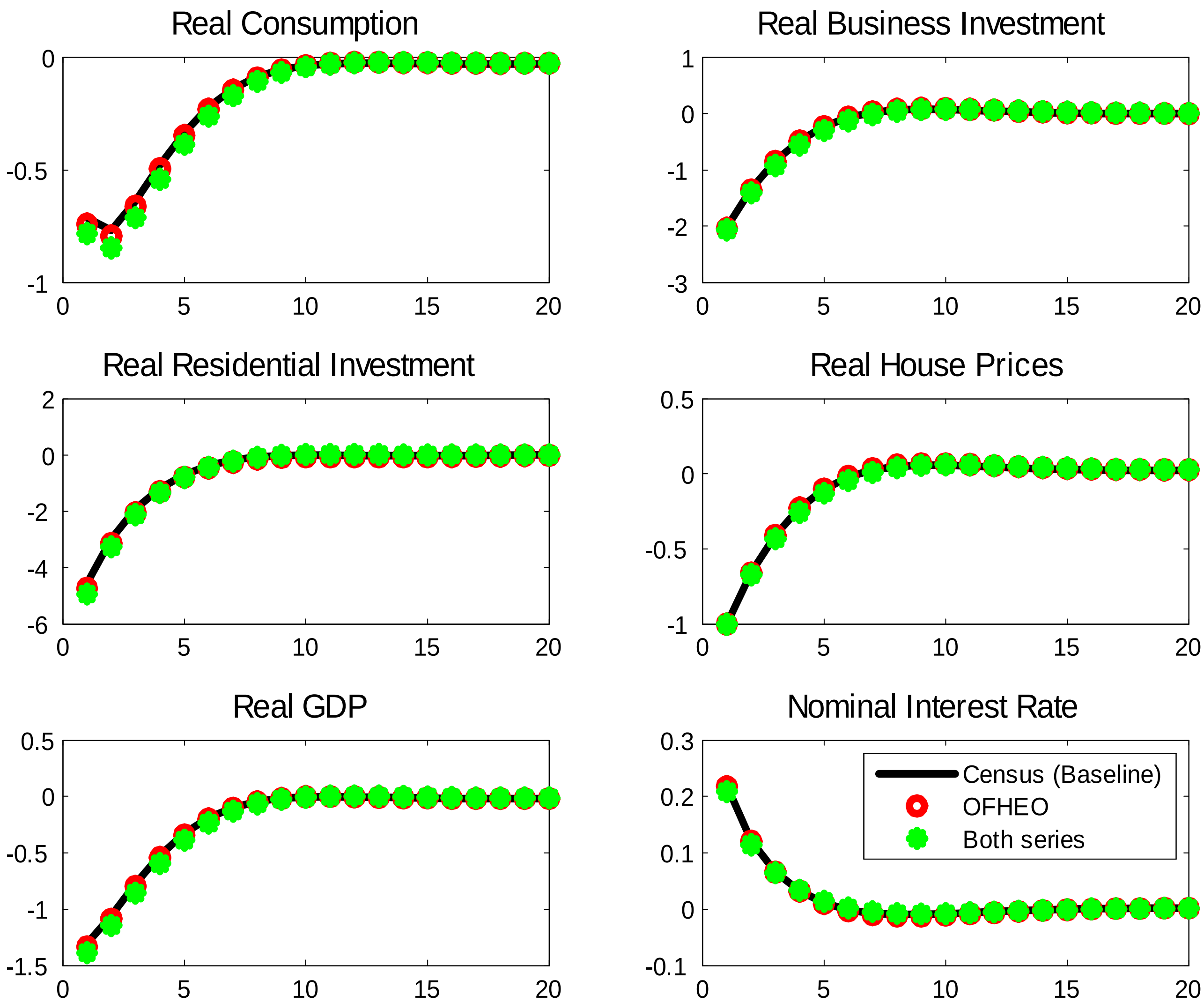


Figure D.6. Impulse responses to a monetary policy shock: parameters estimated using the Census, the OFHEO and both indices at the same time

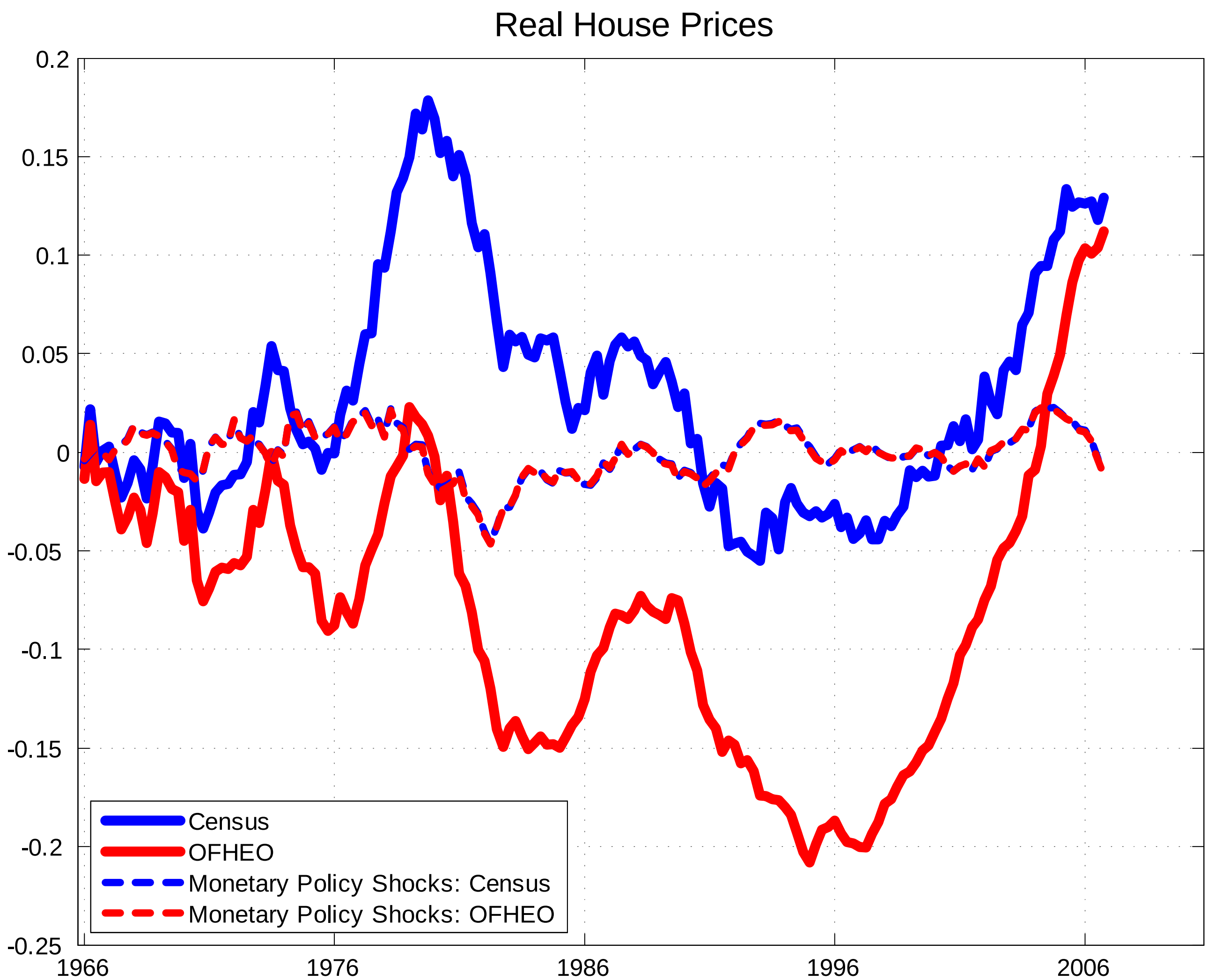


Figure D.7. Historical decomposition of real house prices: contribution of monetary policy (Census vs. OFHEO)

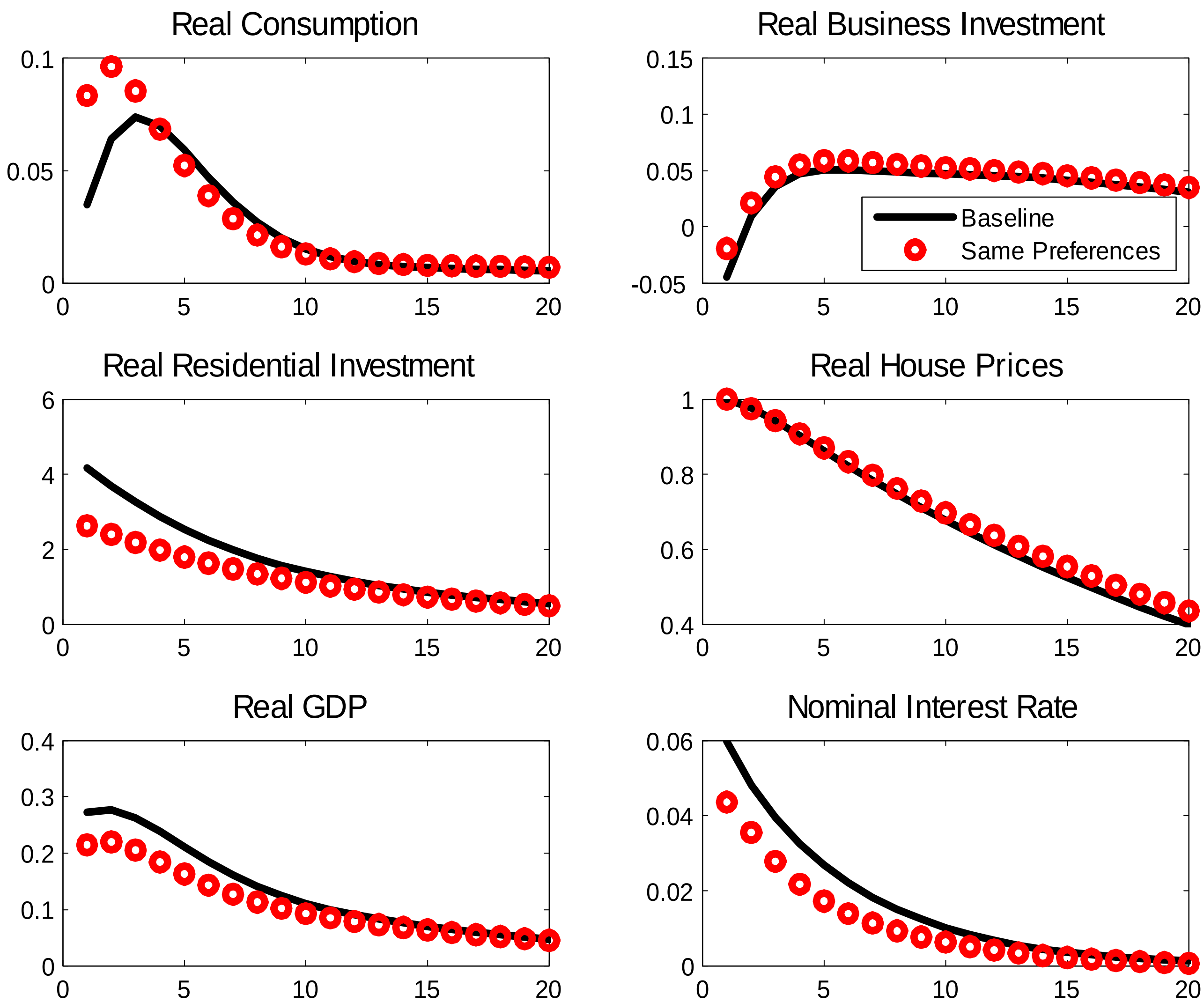


Figure D.8. Impulse responses to a housing preference shock: the role of heterogeneous preferences

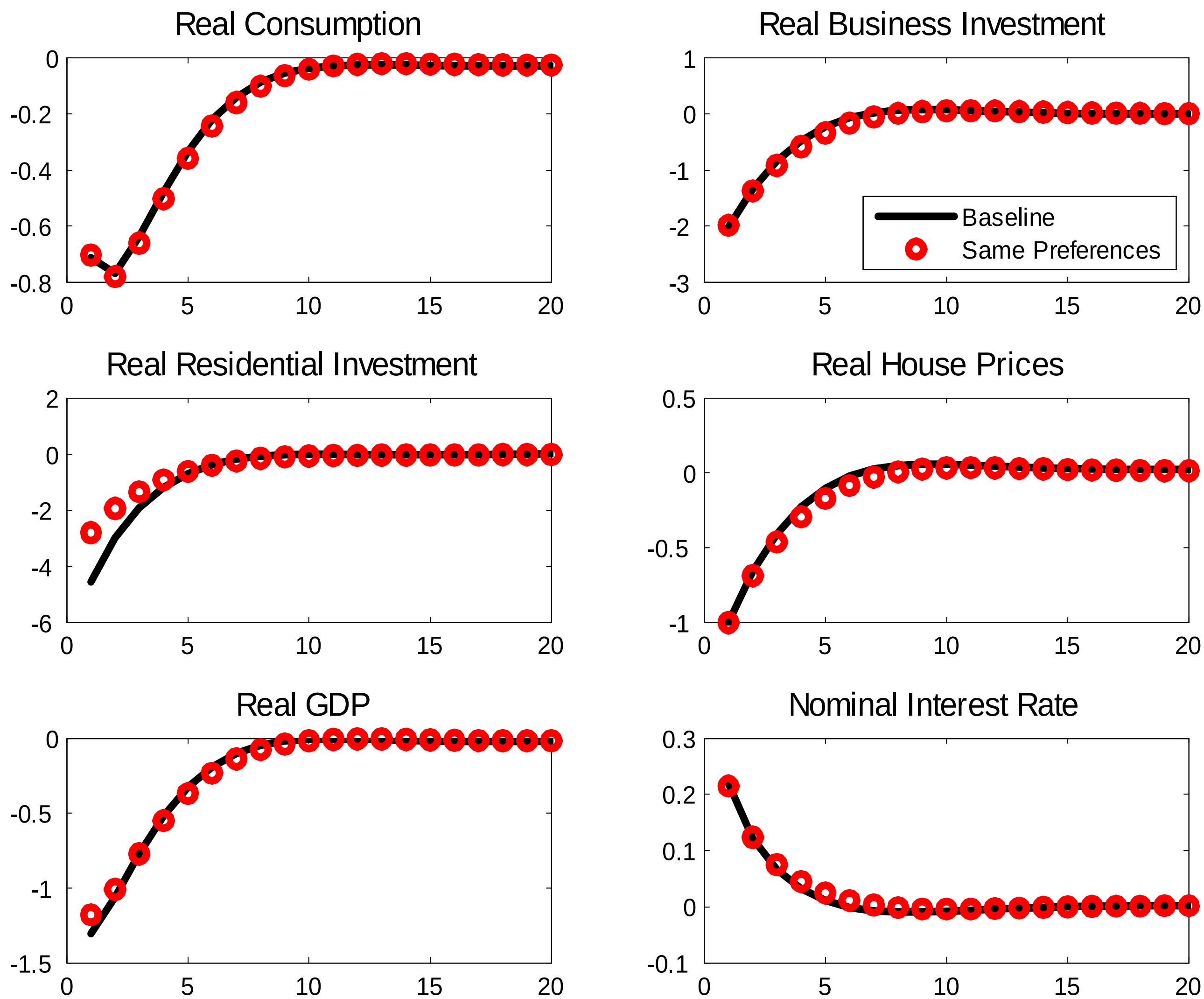


Figure D.9. Impulse responses to a monetary policy shock: the role of heterogeneous preferences

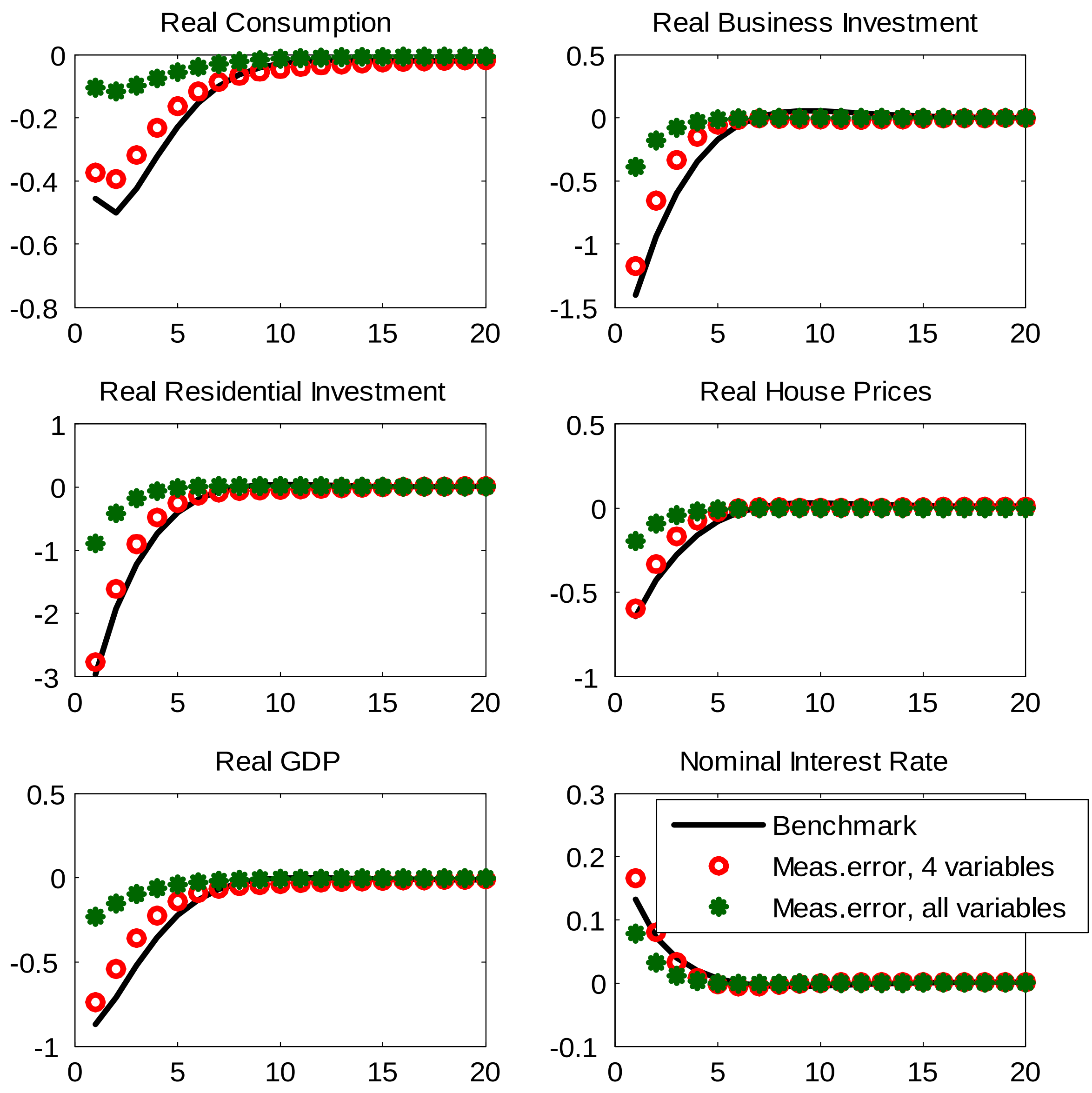


Figure D.10: Impulse Responses to a Monetary Shock. The Role of Measurement Error.

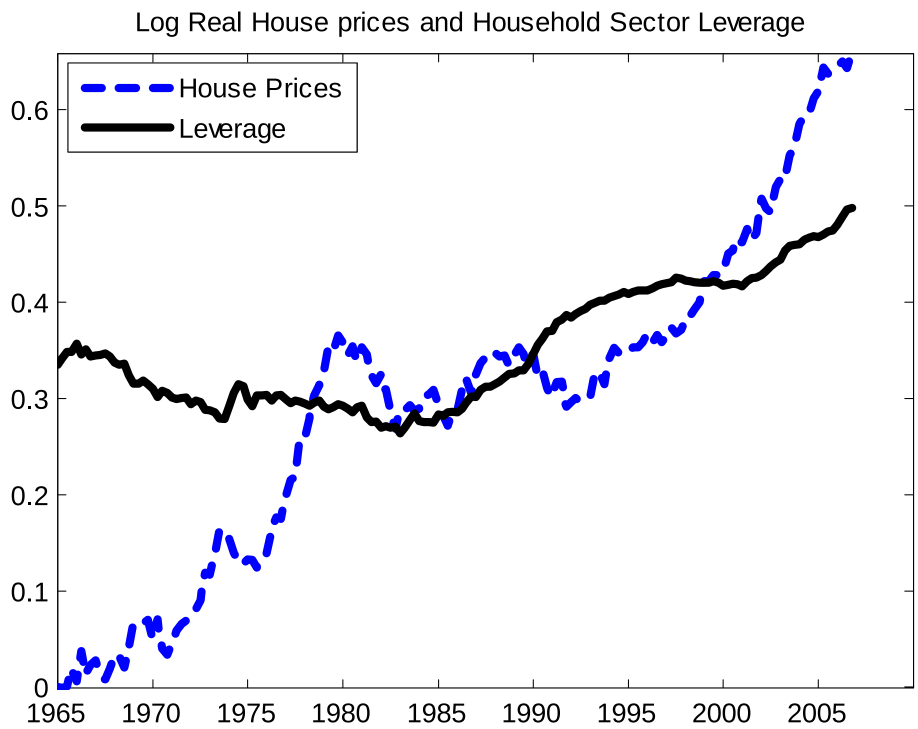


Figure D.11: House Prices and Households' Mortgage Debt to Housing Wealth Ratio

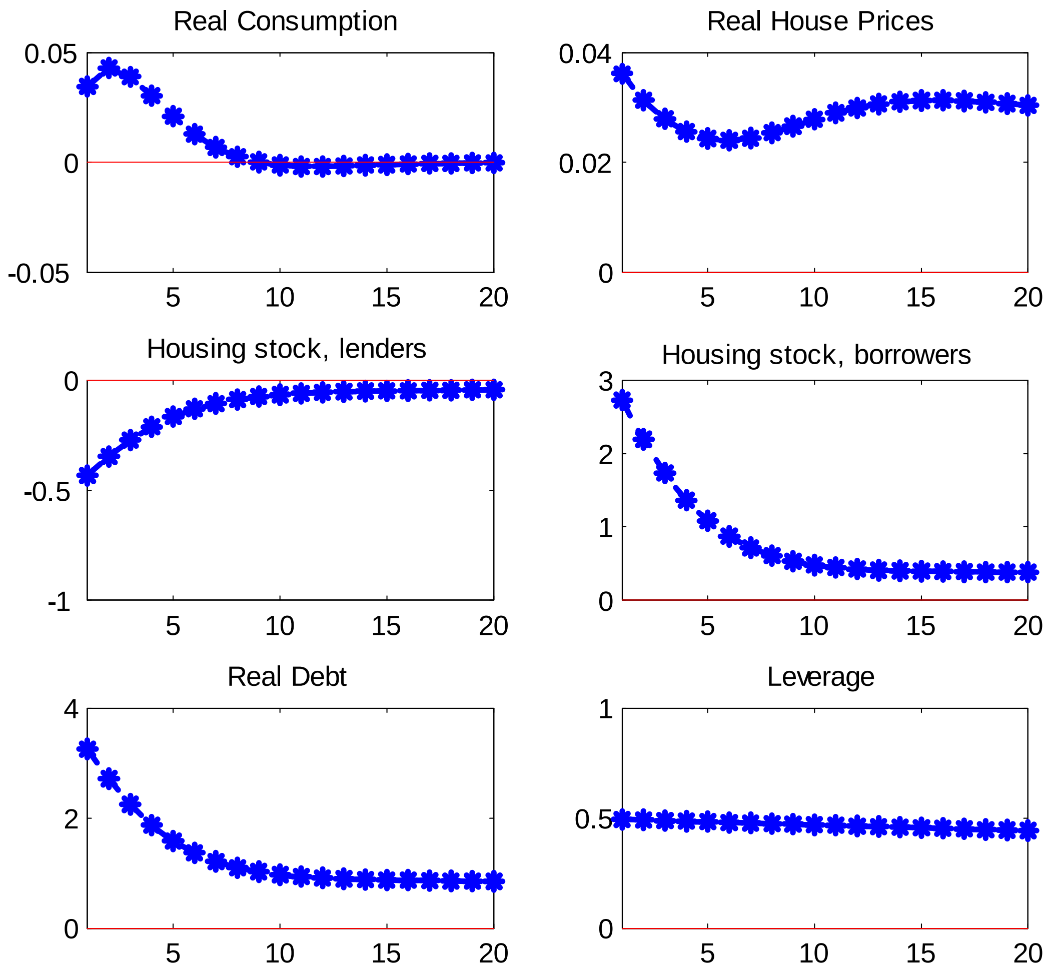


Figure D.12: Impulse responses to an estimated innovation in the loan-to-value ratio

Appendix E: Mathematical Derivations for the Equations of "Housing Market Spillovers: Evidence from an Estimated DSGE Model"

1 The model

1.1 Patient households

Lifetime utility is given by:

$$V_t = E_0 \sum_{t=0}^{\infty} (\beta G_C)^t \, \mathsf{z}_t \left[\frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \log (c_t - \varepsilon c_{t-1}) + \mathsf{j}_t \log h_t - \frac{\tau_t}{1+\eta} \left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{1+\eta}{1+\xi}} \right]$$

where the term in square brackets represents period utility. With this formulation, the marginal utility of consumption is given by:

$$u_{ct} = \mathsf{z}_t \left(\frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \right) \left(\frac{1}{c_t - \varepsilon c_{t-1}} - \frac{\beta G_C \varepsilon}{c_{t+1} - \varepsilon c_t} \right)$$

the marginal utility of housing is:

$$u_{ht} = \frac{\mathsf{z}_t \mathsf{j}_t}{h_t}$$

and the marginal disutility of working in the goods and housing sector:

$$u_{nct} = \mathsf{z}_t \mathsf{j}_t (1+\eta) \, n_{ct}^{\xi} \left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta-\xi}{1+\xi}}$$

$$u_{nht} = \mathsf{z}_t \mathsf{j}_t (1+\eta) \, n_{ht}^{\xi} \left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi} \right)^{\frac{\eta-\xi}{1+\xi}}$$

Since along the balance growth path (BGP) consumption grows at the rate G_C every quarter, the marginal utility of consumption falls at this rate. Hence the transformed marginal utility $\widetilde{u}_{ct} = u_{ct} G_C^t$ is stationary around the steady state and equal to:

$$\begin{aligned}
\widetilde{u}_{ct} &= G_C^t u_{ct} = \frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \left(\frac{G_C^t}{c_t - \varepsilon c_{t-1}} - \frac{\beta G_C^{t+1} \varepsilon}{c_{t+1} - \varepsilon c_t} \right) \\
&= \frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \left(\frac{1}{\frac{c_t}{G_C^t} - \frac{\varepsilon}{G_C} \frac{c_{t-1}}{G_C^{t-1}}} - \frac{\beta\varepsilon}{\frac{c_{t+1}}{G_C^{t+1}} - \frac{\varepsilon}{G_C} \frac{c_t}{G_C^t}} \right) \\
&= \frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \left(\frac{1}{\widetilde{c}_t - \frac{\varepsilon}{G_C} \widetilde{c}_{t-1}} - \frac{\beta\varepsilon}{\widetilde{c}_{t+1} - \frac{\varepsilon}{G_C} \widetilde{c}_t} \right) \\
&= \frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \left(\frac{1}{\widetilde{c}_t - \frac{\varepsilon}{G_C} \widetilde{c}_{t-1}} - \frac{\beta\varepsilon}{\widetilde{c}_{t+1} - \frac{\varepsilon}{G_C} \widetilde{c}_t} \right)
\end{aligned}$$

Tranformed consumption, $\widehat{c}_t = c_t / G^t$, and the scaled marginal utility of consumption $\widetilde{u}_{ct}$:

$$\begin{aligned}
\widetilde{u}_c &= \frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \left(\frac{1}{1 - \frac{\varepsilon}{G_C}} - \frac{\beta\varepsilon}{1 - \frac{\varepsilon}{G_C}} \right) \frac{1}{\widetilde{c}} = \\
&= \frac{G_C - \varepsilon}{G_C - \beta\varepsilon G_C} \left(\frac{G_C (1 - \beta\varepsilon)}{G_C - \varepsilon} \right) \frac{1}{\widetilde{c}} = \frac{1}{\widetilde{c}}
\end{aligned}$$

are both constant in steady state.
The marginal utility of housing $u_{ht} = \frac{\mathsf{j}_t \mathsf{z}_t}{h_t}$ declines at the rate G_H. Therefore the transformed marginal utility $\widetilde{u}_{ht} = u_{ht} G_H^t$ is stationary around the steady state and equal to:

$$\widetilde{u}_{ht} = \frac{\mathsf{j}_t \mathsf{z}_t}{\tilde{h}_t}$$

In steady state it is equal to $\overline{\widetilde{u}_h} = \frac{1}{\tilde{h}}$ since both j_t and z_t are equal to one.

Due to the assumptions on preferences and technology hours worked in the two sector are stationary already in the level economy.

The patient household's budget constraint:

$$\begin{aligned} c_t & + \frac{k_{ct}}{\mathsf{A}_{kt}} + k_{ht} + k_{bt} + q_t \left[h_t - (1-\delta_h) h_{t-1} \right] + p_{lt} l_t = \frac{w_{ct}}{X_{wct}} n_{ct} + \frac{w_{ht}}{X_{wht}} n_{ht} \\ + & \; Div_t - \phi_t + \left(R_{ct} z_{ct} + \frac{1-\delta_k}{\mathsf{A}_{kt}} \right) k_{ct-1} + (R_{ht} z_{ht} + 1 - \delta_k) k_{ht-1} + p_{bt} k_{bt} \\ & + b_t - \frac{R_{t-1} b_{t-1}}{\pi_t} + (R_{lt} + p_{lt}) l_{t-1} - \frac{a(z_{ct})}{\mathsf{A}_{kt}} k_{ct-1} - a(z_{ht}) k_{ht-1} \end{aligned}$$

where the adjustment costs on capital are

$$\phi_t = \frac{\phi_{kc}}{2} \left(\frac{k_{ct}}{k_{ct-1}} - G_{KC} \right)^2 \frac{k_{ct-1}}{\Gamma_{Ak}^t} + \frac{\phi_{kh}}{2} \left(\frac{k_{ht}}{k_{ht-1}} - G_C \right)^2 k_{ht-1}$$

where Γ_{Ak} is the gross growth rate of the investment specific technology process in the goods sector and G_{KC} is the BGP gross growth rate of capital in the goods sector. Adjustment costs on capacity utilisation are:

$$\begin{aligned} a(z_{ct}) &= R_c \left(\frac{1}{2} \varpi z_{ct}^2 + (1-\varpi) z_{ct} + \left(\frac{\varpi}{2} - 1 \right) \right) \\ a(z_{ht}) &= R_h \left(\frac{1}{2} \varpi z_{ht}^2 + (1-\varpi) z_{ht} + \left(\frac{\varpi}{2} - 1 \right) \right) \end{aligned}$$

where R_c and R_h are the steady state levels of the rental rate of capital in, respectively, the goods and the housing sector.
The budget constraint can be transformed as follows:

$$\begin{aligned} & \frac{c_t}{G_C^t} + \frac{k_{ct}}{\mathsf{A}_{kt} G_C^t} + \frac{k_{ht}}{G_C^t} + \frac{k_{bt}}{G_C^t} + \frac{q_t}{G_C^t} \left(\frac{h_t}{G_C^t} - (1-\delta_h) \frac{h_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^{t-1}} \right) + \frac{p_{lt}}{G_C^t} l_t = \frac{w_{ct}}{X_{wct} G_C^t} n_{ct} + \frac{w_{ht}}{X_{wht} G_C^t} n_{ht} \\ & + \frac{Div_t}{G_C^t} - \frac{\phi_t}{G_C^t} + \frac{\Gamma_{AK}^t}{\Gamma_{AK}^{t-1} \Gamma_{AK}} R_{ct} z_{ct} k_{ct-1} \frac{1}{G_C^{t-1} G_C} + (1-\delta_k) \frac{k_{ct-1}}{\mathsf{A}_{kt}} \frac{\mathsf{A}_{kt-1}}{\mathsf{A}_{kt-1}} \frac{1}{G_C^t} \\ & + (R_{ht} z_{ht} + 1 - \delta_k) \frac{k_{ht-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t} + p_{bt} \frac{k_{bt}}{G_C^t} + \frac{b_t}{G_C^t} - \frac{R_{t-1}}{\pi_t} \frac{b_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t} + \frac{R_{lt} + p_{lt}}{G_C^t} l_{t-1} \\ & - \frac{a(z_{ct})}{\mathsf{A}_{kt}} \frac{k_{ct-1}}{\mathsf{A}_{kt-1}} \frac{\mathsf{A}_{kt-1}}{G_C^t} - a(z_{ht}) \frac{k_{ht-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t} \end{aligned}$$

$$
\begin{aligned}
&\frac{c_t}{G_C^t}+\frac{k_{ct}}{\mathsf{A}_{kt}G_C^t}+\frac{k_{ht}}{G_C^t}+\frac{k_{bt}}{G_C^t}+\frac{q_t}{G_C^t}\left(\frac{h_t}{G_C^t}-(1-\delta_h)\frac{h_{t-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^{t-1}}\right)+\frac{p_{lt}}{G_C^t}l_t=\frac{w_{ct}}{X_{wct}G_C^t}n_{ct}+\frac{w_{ht}}{X_{wht}G_C^t}n_{ht}\\
&+\frac{Div_t}{G_C^t}-\frac{\phi_t}{G_C^t}+\frac{R_{ct}\Gamma_{AK}^t}{\Gamma_{AK}G_C}z_{ct}\frac{k_{ct-1}}{\Gamma_{AK}^{t-1}G_C^{t-1}}+(1-\delta_k)\frac{k_{ct-1}}{\mathsf{A}_{kt}}\frac{\mathsf{A}_{kt-1}}{\mathsf{A}_{kt-1}}\frac{1}{G_C^t}\\
&+(R_{ht}z_{ht}+1-\delta_k)\frac{k_{ht-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}+p_{bt}\frac{k_{bt}}{G_C^t}+\frac{b_t}{G_C^t}-\frac{R_{t-1}}{\pi_t}\frac{b_{t-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}+\frac{R_{lt}+p_{lt}}{G_C^t}l_{t-1}\\
&-\frac{a(z_{ct})}{\mathsf{A}_{kt}}\frac{k_{ct-1}}{\mathsf{A}_{kt-1}}\frac{\mathsf{A}_{kt-1}}{G_C^t}-a(z_{ht})\frac{k_{ht-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{c}_t+\frac{\tilde{k}_{ct}}{\mathsf{a}_{kt}}+\tilde{k}_{ht}+\tilde{k}_{bt}+\tilde{q}_t\tilde{h}_t-(1-\delta_h)\tilde{q}_t\frac{\tilde{h}_{t-1}}{G_H}+\tilde{p}_{lt}l_t=\frac{\tilde{w}_{ct}}{X_{wct}}n_{ct}+\frac{\tilde{w}_{ht}}{X_{wht}}n_{ht}+\tilde{Div}_t-\tilde{\phi}_t\\
&+\tilde{R}_{ct}z_{ct}\frac{\tilde{k}_{ct-1}}{G_{KC}}+\frac{(1-\delta_k)}{G_{KC}}\frac{\tilde{k}_{ct-1}}{a_{kt}}+(R_{ht}z_{ht}+1-\delta_k)\frac{\tilde{k}_{ht-1}}{G_C}+p_{bt}\tilde{k}_{bt}\\
&-\frac{a(z_{ct})}{G_{KC}}\frac{\tilde{k}_{ct-1}}{\mathsf{a}_{kt}}-a(z_{ht})\frac{\tilde{k}_{ht-1}}{G_C}+\tilde{b}_t-\frac{R_{t-1}}{\pi_t}\frac{\tilde{b}_{t-1}}{G_C}+\left(\tilde{R}_{lt}+\tilde{p}_{lt}\right)l_{t-1}
\end{aligned}
$$

where the following result, which will be derived later, has been used:

$$G_{KC}^t=\Gamma_{AK}^tG_C^t$$

Adjustment costs for capital can be transformed as follows:

$$\frac{\phi_t}{G_C^t}=\frac{\phi_{kc}}{2}\left(\frac{k_{ct}}{k_{ct-1}}-G_{KC}\right)^2\frac{k_{ct-1}}{G_C^{t-1}G_C\Gamma_{Ak}^{t-1}\Gamma_{Ak}}+\frac{\phi_{kh}}{2}\left(\frac{k_{ht}}{k_{ht-1}}-G_C\right)^2\frac{k_{ht-1}}{G_C^{t-1}G_C}$$

$$\tilde{\phi}_t=\frac{\phi_{kc}}{2G_{KC}}\left(G_{KC}\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}}-G_{KC}\right)^2\tilde{k}_{ct-1}+\frac{\phi_{kh}}{2G_C}\left(G_C\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-G_C\right)^2\tilde{k}_{ht-1}$$

Using the definition of dividends:

$$DIV_t=\left(1-\frac{1}{X_{wct}}\right)w_{ct}n_{ct}+\left(1-\frac{1}{X_{wht}}\right)w_{ht}n_{ht}+\left(1-\frac{1}{X_t}\right)Y_t$$

the terms $\frac{1}{X_{wct}}w_{ct}n_{ct}$ and $\frac{1}{X_{wht}}w_{ht}n_{ht}$ cancel out in the budget constraint so that dividends to the patient households are given by:

$$DIV_t=\left(1-\frac{1}{X_t}\right)Y_t$$

The final expression for the budget constraint is:

$$\tilde{c}_t + \frac{\tilde{k}_{ct}}{\mathsf{a}_{kt}} + \tilde{k}_{ht} + \tilde{k}_{bt} + \tilde{q}_t \tilde{h}_t - (1-\delta_h)\,\tilde{q}_t \frac{\tilde{h}_{t-1}}{G_H} + \tilde{p}_{lt} l_t = \tilde{w}_{ct} n_{ct} + \tilde{w}_{ht} n_{ht} + \left(1 - \frac{1}{X_t}\right)\tilde{Y}_t$$
$$+ \left(\tilde{R}_{ct} z_{ct} + \frac{(1-\delta_k)}{a_{kt}}\right)\frac{\tilde{k}_{ct-1}}{G_{KC}} + (R_{ht} z_{ht} + 1 - \delta_k)\,\frac{\tilde{k}_{ht-1}}{G_C} + p_{bt}\tilde{k}_{bt}$$
$$-\tilde{b}_t + \frac{R_{t-1}}{\pi_t}\frac{\tilde{b}_{t-1}}{G_C} + \left(\tilde{R}_{lt} + \tilde{p}_{lt}\right) l_{t-1} - \frac{a(z_{ct})}{G_{KC}}\frac{\tilde{k}_{ct-1}}{\mathsf{a}_{kt}} - a(z_{ht})\,\frac{\tilde{k}_{ht-1}}{G_C} - \frac{\phi_{kc}}{2G_{KC}}\left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - G_{KC}\right)^2 \tilde{k}_{ct-1}$$
$$+\frac{\phi_{kh}}{2G_C}\left(\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}} - G_C\right)^2 \tilde{k}_{ht-1}$$

The choice variables for the patient household are the following: c_t, h_t, k_{ct}, k_{ht}, b_t, n_{ct}, n_{ht}, k_{bt}, z_{ct} and z_{ht}. The first-order conditions of the patient household's maximisation problem are:

$$\begin{aligned}
u_{ct} q_t &= u_{ht} + \beta G_C E_t \left[u_{ct+1} q_{t+1}(1-\delta_h)\right] \\
u_{ct} &= \beta G_C E_t \left(u_{ct+1} R_t / \pi_{t+1}\right) \\
\frac{u_{ct}}{\mathsf{A}_{kt}}\left(1 + \frac{\partial \phi_{ct}}{\partial k_{ct}}\right) &= \beta G_C E_t \left[u_{ct+1}\left(R_{ct+1} z_{ct+1} - \frac{a(z_{ct})}{\mathsf{A}_{kt}} + \frac{1-\delta_k}{\mathsf{A}_{kt+1}} - \frac{\partial \phi_{ct+1}}{\partial k_{ct}}\right)\right] \\
u_{ct}\left(1 + \frac{\partial \phi_{ht}}{\partial k_{ht}}\right) &= \beta G_C E_t \left[u_{ct+1}\left(R_{ht+1} z_{ht+1} - a(z_{ht}) + 1 - \delta_k - \frac{\partial \phi_{ht+1}}{\partial k_{ht}}\right)\right] \\
u_{nct} &= u_{ct}\frac{w_{ct}}{X_{wct}} \\
u_{nht} &= u_{ct}\frac{w_{ht}}{X_{wht}} \\
u_{ct}(p_{bt} - 1) &= 0 \\
R_{ct} &= \frac{a'(z_{ct})}{\mathsf{A}_{kt}} \\
R_{ht} &= a'(z_{ht}) \\
u_{ct} p_{lt} &= \beta G_C E_t \left[u_{ct+1}(p_{lt+1} + R_{lt+1})\right]
\end{aligned}$$

where we have substitute away the Lagrange multiplier on the budget constraint. These optimality condition must be transformed to take into account the fact the some of the variables are growing over time.

The first order condition with respect to h_t is transformed in the following way:

$$\begin{aligned}
u_{ct} q_t &= u_{ht} + \beta G_C E_t \left(u_{ct+1} q_{t+1}(1-\delta_h)\right) \\
u_{ct} G_C^t \frac{q_t}{G_Q^t} &= \left(u_{ht} G_H^t\right) + \beta G_C E_t \left[\left(u_{ct+1} G_C^t\right)\left(\frac{q_{t+1}}{G_Q^t}\right)(1-\delta_h)\right] \\
u_{ct} G_C^t \frac{q_t}{G_Q^t} &= \left(u_{ht}\frac{G_C^t}{G_Q^t}\right) + \beta G_C E_t \left[\left(u_{ct+1} G_C^{t+1}\frac{1}{G_C}\right)\left(\frac{q_{t+1}}{G_Q^{t+1}} G_Q\right)(1-\delta_h)\right] \\
\widetilde{u}_{ct}\widetilde{q}_t &= \widetilde{u}_{ht} + \beta G_C E_t \left[\widetilde{u}_{ct+1}\widetilde{q}_{t+1}(1-\delta_h)\right]\frac{G_Q}{G_C}
\end{aligned}$$

where G_Q is the BGP growth rate of real house prices whose expression will be derived later.

The transformation that must be applied to the first order condition with respect to lending, b_t, is:

$$\begin{aligned}
u_{ct} &= \beta G_C E_t \left(\frac{u_{ct+1} R_t}{\pi_{t+1}} \right) \\
u_{ct} G_C^t &= \beta G_C E_t \left(u_{ct+1} \frac{G_C^{t+1}}{G_C} \frac{R_t}{\pi_{t+1}} \right) \\
\widetilde{u}_{ct} &= \beta \widetilde{u}_{ct+1} \frac{R_t}{\pi_{t+1}}
\end{aligned}$$

The first order condition with respect to k_{ct} is:

$$\begin{aligned}
u_{ct} \left[\frac{1}{\mathsf{A}_{kt}} + \frac{\phi_{kc}}{G_{KC}} \left(\frac{k_{ct}}{k_{ct-1}} - G_{KC} \right) \right] &= \beta G_C E_t \left[u_{ct+1} \left(R_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{A}_{kt+1}} + \frac{1-\delta_k}{\mathsf{A}_{kt+1}} - \right. \right. \\
&+ \left. \left. \frac{\phi_{kc}}{2G_{KC}} \left(\frac{k_{ct+1}^2}{k_{ct}^2} - G_{KC}^2 \right) \frac{1}{\Gamma_{AK}^{t+1}} \right) \right.
\end{aligned}$$

It can be transformed in the following way:

$$\begin{aligned}
G_C^t u_{ct} \left[\frac{1}{\mathsf{A}_{kt}} + \frac{\phi_{kc}}{G_{KC}} \left(G_{KC} \frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - G_{KC} \right) \right] &= \beta G_C E_t \left[G_C^t u_{ct+1} \frac{G_C^{t+1}}{G_C^{t+1}} \left(\frac{\Gamma_{AK}^{t+1}}{\Gamma_{AK}^{t+1}} R_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{A}_{kt+1}} + \right. \right. \\
&+ \left. \left. \frac{1-\delta_k}{\mathsf{A}_{kt+1}} + \frac{\phi_{kc}}{2G_{KC}} \left(G_{KC}^2 \frac{\tilde{k}_{ct+1}^2}{\tilde{k}_{ct}^2} - G_{KC}^2 \right) \frac{1}{\Gamma_{AK}^{t+1}} \right) \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{u}_{ct} \left[\frac{1}{\mathsf{a}_{kt}} + \phi_{kc} \left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - 1 \right) \right] &= \beta G_C E_t \left[\frac{\tilde{u}_{ct+1} \Gamma_{AK}^t}{G_C \Gamma_{AK}^{t+1}} \left(\tilde{R}_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{a}_{kt+1}} + \right. \right. \\
&+ \left. \left. \frac{1-\delta_k}{\mathsf{a}_{kt+1}} + \frac{G_{KC}\phi_{kc}}{2} \left(\frac{\tilde{k}_{ct+1}^2}{\tilde{k}_{ct}^2} - 1 \right) \right) \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{u}_{ct} \left[\frac{1}{\mathsf{a}_{kt}} + \phi_{kc} \left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}} - 1 \right) \right] &= \beta G_C E_t \left[\frac{\tilde{u}_{ct+1}}{G_{KC}} \left(\tilde{R}_{ct+1} z_{ct+1} - \frac{a\left(z_{ct+1}\right)}{\mathsf{a}_{kt+1}} + \right. \right. \\
&+ \left. \left. \frac{1-\delta_k}{\mathsf{a}_{kt+1}} + \frac{G_{KC}\phi_{kc}}{2} \left(\frac{\tilde{k}_{ct+1}^2}{\tilde{k}_{ct}^2} - 1 \right) \right) \right]
\end{aligned}$$

The first order condition with respect to k_{ht} is:

$$\begin{aligned}
u_{ct} \left[1 + \frac{\phi_{kh}}{G_C} \left(\frac{k_{ht}}{k_{ht-1}} - G_C \right) \right] &= \beta G_C E_t \left[u_{ct+1} \left(R_{ht+1} z_{ht+1} - a\left(z_{ht+1}\right) + 1 - \delta_k - \right. \right. \\
&- \left. \left. \frac{\phi_{kh}}{2G_C} \left(\frac{k_{ht+1}^2}{k_{ht}^2} - G_C^2 \right) \right) \right]
\end{aligned}$$

This first order condition can be transformed into:

$$\begin{aligned}
G_C^t u_{ct}\left[1+\frac{\phi_{kh}}{G_C}\left(G_C\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-G_C\right)\right] &= \beta G_C E_t\left[G_C^t u_{ct+1}\frac{G_C^{t+1}}{G_C^{t+1}}\left(R_{ht+1}z_{ht+1}-a\left(z_{ht+1}\right)+\right.\right. \\
&+ \left.\left. 1-\delta_k-\frac{\phi_{kh}}{2G_C}\left(G_C^2\frac{\tilde{k}_{ht+1}^2}{\tilde{k}_{ht}^2}-G_C^2\right)\right)\right]
\end{aligned}$$

$$\begin{aligned}
\tilde{u}_{ct}\left[1+\phi_{kh}\left(\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-1\right)\right] &= \beta G_C E_t\left[\frac{\tilde{u}_{ct+1}}{G_C}\left(R_{ht+1}z_{ht+1}-a\left(z_{ct}\right)+\right.\right. \\
&+ \left.\left. 1-\delta_k-\frac{\phi_{kh}G_C}{2}\left(\frac{\tilde{k}_{ht+1}^2}{\tilde{k}_{ht}^2}-1\right)\right)\right]
\end{aligned}$$

The first order conditions with respect to u_{nct} and u_{nht} are:

$$\begin{aligned}
u_{nct} &= u_{ct}\frac{w_{ct}}{X_{wct}} \\
u_{nht} &= u_{ct}\frac{w_{ht}}{X_{wht}}
\end{aligned}$$

which can be transformed as follow:

$$\begin{aligned}
u_{nct} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ct}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = u_{ct}G_C^t\frac{w_{ct}}{X_{wct}G_C^t} \\
u_{nht} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ht}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = u_{ct}G_C^t\frac{w_{ht}}{X_{wht}G_C^t} \\
u_{nct} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ct}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = \tilde{u}_{ct}\frac{\tilde{w}_{ct}}{X_{wct}} \\
u_{nht} &= \mathsf{z}_t\mathsf{j}_t\left(1+\eta\right)n_{ht}^{\xi}\left(n_{ct}^{1+\xi}+n_{ht}^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}} = \tilde{u}_{ct}\frac{\tilde{w}_{ht}}{X_{wht}}
\end{aligned}$$

The first order condition with respect to intermediate inputs k_{bt} is:

$$u_{ct}\left(p_{bt}-1\right)=0$$

which implies that their price is always equal to 1.
The first order conditions with respect to capacity utilisation are:

$$\begin{aligned}
R_{ct} &= \frac{a'\left(z_{ct}\right)}{\mathsf{A}_{kt}} \\
R_{ht} &= a'\left(z_{ht}\right)
\end{aligned}$$

which are transformed as:

$$\begin{aligned}
R_{ct}\mathsf{A}_{kt} &= a'\left(z_{ct}\right) \\
R_{ht} &= a'\left(z_{ht}\right)
\end{aligned}$$

$$
\begin{aligned}
\tilde{R}_{ct} &= \frac{a'(z_{ct})}{\mathsf{a}_{kt}} \\
R_{ht} &= a'(z_{ht})
\end{aligned}
$$

where we have taken into account the definition $\tilde{R}_{ct} = R_{ct}\Gamma_k^t$.

The first order condition with respect to land l_t

$$
u_{ct}p_{lt} = \beta G_C E_t\left(u_{ct+1}\left(p_{lt+1} + R_{lt+1}\right)\right)
$$

becomes after transformation:

$$
\begin{aligned}
u_{ct}G_C^t\frac{p_{lt}}{G_C^t} &= \beta G_C E_t\left[u_{ct+1}G_C^{t+1}\left(\frac{p_{lt+1}}{G_C^{t+1}} + \frac{R_{lt+1}}{G_C^{t+1}}\right)\right] \\
\widetilde{u}_{ct}\widetilde{p}_{lt} &= \beta G_C E_t\left[\widetilde{u}_{ct+1}\left(\widetilde{p}_{lt+1} + \widetilde{R}_{lt+1}\right)\right].
\end{aligned}
$$

1.2 Impatient households

Lifetime utility is given by:

$$
V_t = E_0\sum_{t=0}^{\infty}\left(\beta' G_C\right)^t \mathsf{z}_t\left[\frac{G_C - \varepsilon'}{G_C - \beta'\varepsilon' G_C}\log\left(c_t' - \varepsilon' c_{t-1}'\right) + \mathsf{j}_t \log h_t' - \frac{\tau_t}{1+\eta'}\left(\left(n_{ct}'\right)^{1+\xi'} + \left(n_{ht}'\right)^{1+\xi'}\right)^{\frac{1+\eta'}{1+\xi'}}\right]
$$

With this formulation, the marginal utility of consumption is given by:

$$
u_{ct}' = \frac{G_C - \varepsilon'}{G_C - \beta'\varepsilon' G_C}\left(\frac{1}{c_t' - \varepsilon' c_{t-1}'} - \frac{\beta' G_C \varepsilon'}{c_{t+1}' - \varepsilon' c_t'}\right)
$$

which can be made stationary using the same transformation employed for the patient households $\widetilde{u}_{ct}' = u_{ct}' G_C^t$. The marginal utility of housing and the marginal disutilities of working are similar to those of the patient households with the exception that the household-specific variables and parameters are denoted with a prime.

The optimality conditions must be transformed to take into account the fact the some of the variables are growing over time. The first order condition with respect to housing, h_t:

$$
\begin{aligned}
u_{c't}q_t &= u_{h't} + \beta' G_C E_t\left(u_{c't+1}\left(q_{t+1}\left(1-\delta_h\right)\right)\right) + E_t\left(\lambda_t\frac{mq_{t+1}\pi_{t+1}}{R_t}\right) \\
u_{c't}G_C^t\frac{q_t}{G_Q^t} &= u_{h't}\frac{G_C^t}{G_Q^t} + \beta G_C E_t\left(\left(u_{c't+1}G_C^{t+1}\frac{1}{G_C}\right)\left(\frac{q_{t+1}}{G_Q^{t+1}}G_Q\right)\left(1-\delta_h\right)\right) \\
&+ E_t\left(\lambda_t G_C^t\frac{mq_{t+1}\pi_{t+1}}{R_t}\frac{1}{G_Q^t}\frac{G_Q^{t+1}}{G_Q^{t+1}}\right) \\
\widetilde{u}_{c't}\widetilde{q}_t &= \widetilde{u}_{h't} + \beta G_C E_t\left(\widetilde{u}_{c't+1}\widetilde{q}_{t+1}\left(1-\delta_h\right)\right)\frac{G_Q}{G_C} + E_t\left(\tilde{\lambda}_t\frac{m\tilde{q}_{t+1}\pi_{t+1}}{R_t}G_Q\right)
\end{aligned}
$$

where G_Q is the BGP growth rate of real house prices whose expression will be derived later.

The first order condition with respect to lending, b_t:

$$\begin{aligned}
u_{c't} &= \beta G_C E_t \left(\frac{u_{c't+1} R_t}{\pi_{t+1}} \right) + \lambda_t \\
u_{c't} G_C^t &= \beta G_C E_t \left(u_{c't+1} \frac{G_C^{t+1}}{G_C} \frac{R_t}{\pi_{t+1}} \right) + \lambda_t G_C^t \\
\widetilde{u}_{c't} &= \beta \widetilde{u}_{c't+1} \frac{R_t}{\pi_{t+1}} + \tilde{\lambda}_t
\end{aligned}$$

The budget constraint:

$$c'_t + q_t \left(h'_t - (1-\delta_h) h'_{t-1} \right) = \frac{w'_{ct}}{X_{wct}} n'_{ct} + \frac{w'_{ht}}{X_{wht}} n'_{ht} + Div'_t + b'_t - \frac{R_{t-1} b'_{t-1}}{\pi_t}$$

$$\frac{c'_t}{G_C^t} + \frac{q_t}{G_C^t} \left(\frac{h'_t}{G_C^t} - (1-\delta_h) \frac{h'_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^{t-1}} \right) = \frac{w'_{ct}}{X_{wct} G_C^t} n'_{ct} + \frac{w'_{ht}}{X_{wht} G_C^t} n'_{ht} + \frac{Div'_t}{G_C^t} + \frac{b'_t}{G_C^t} - \frac{R_{t-1}}{\pi_t} \frac{b'_{t-1}}{G_C^{t-1}} \frac{G_C^{t-1}}{G_C^t}$$

By substituting the expression for the dividends from the unions the transformed budget constraint becomes:

$$\tilde{c}'_t + \tilde{q}_t \tilde{h}'_t - (1-\delta_h) \tilde{q}_t \frac{\tilde{h}'_{t-1}}{G_H} = \tilde{w}'_{ct} n'_{ct} + \tilde{w}'_{ht} n'_{ht} + \tilde{b}'_t - \frac{R_{t-1}}{\pi_t} \frac{\tilde{b}'_{t-1}}{G_C}$$

The borrowing constraint:

$$b'_t = m E_t \left(\frac{q_{t+1} h'_t \pi_{t+1}}{R_t} \right)$$

can be transformed as follows:

$$\frac{b'_t}{G_C^t} = m E_t \left(\frac{q_{t+1} h'_t \pi_{t+1}}{G_C^t R_t} \right)$$

$$\tilde{b}_t = m E_t \left(\frac{q_{t+1} h'_t \pi_{t+1}}{G_H^t G_Q^t R_t} \right)$$

$$\tilde{b}_t = m E_t \left(\frac{G_Q q_{t+1} h'_t \pi_{t+1}}{G_Q^{t+1} G_H^t R_t} \right)$$

$$\tilde{b}_t = m E_t \left(\frac{G_Q \tilde{q}_{t+1} \tilde{h}'_t \pi_{t+1}}{R_t} \right)$$

1.3 Intermediate goods firms

Wholesale firms solve the following maximization problem:

$$\max \frac{Y_t}{X_t} + q_t IH_t - \left(\sum w_{it} n_{it} + R_{ct} z_{ct} k_{ct-1} + R_{ht} z_{ht} k_{ht-1} + R_{lt} l_{t-1} + p_{bt} k_{bt}\right)$$

The two production technologies are:

$$\begin{aligned} Y_t &= \left[\mathsf{A}_{ct}\left(n_{ct}^{\alpha} n_{ct}'^{1-\alpha}\right)\right]^{1-\mu_c} \left(z_{ct} k_{ct-1}\right)^{\mu_c} \\ IH_t &= \left[\mathsf{A}_{ht}\left(n_{ht}^{\alpha} n_{ht}'^{1-\alpha}\right)\right]^{1-\mu_h-\mu_b-\mu_l} \left(z_{ht} k_{ht-1}\right)^{\mu_h} k_{bt}^{\mu_b} l_{t-1}^{\mu_l} \end{aligned}$$

The first order condition with respect to n_{ct} is:

$$\left(1-\mu_c\right)\alpha \frac{Y_t}{X_t n_{ct}} = w_{ct}$$

which after taking into account that both Y_t and w_{ct} growth at rate G_C along the BGP becomes:

$$\begin{aligned} \left(1-\mu_c\right)\alpha \frac{Y_t}{G_C^t X_t n_{ct}} &= \frac{w_{ct}}{G_C^t} \\ \left(1-\mu_c\right)\alpha \frac{\tilde{Y}_t}{X_t n_{ct}} &= \tilde{w}_{ct} \end{aligned}$$

Similarly for n'_{ct}:

$$\begin{aligned} \left(1-\mu_c\right)\left(1-\alpha\right) \frac{Y_t}{X_t n'_{ct}} &= w'_{ct} \\ \left(1-\mu_c\right)\left(1-\alpha\right) \frac{\tilde{Y}_t}{X_t n'_{ct}} &= \tilde{w}'_{ct} \end{aligned}$$

and for n_{ht}:

$$\begin{aligned} \left(1-\mu_h-\mu_l\right)\alpha \frac{q_t IH_t}{n_{ht}} &= w_{ht} \\ \left(1-\mu_h-\mu_l\right)\alpha \frac{q_t}{G_Q^t} \frac{IH_t}{G_H^t} n_{ht} &= \frac{w_{ht}}{G_C^t} \\ \left(1-\mu_h-\mu_l\right)\alpha \frac{\tilde{q}_t \tilde{IH}_t}{n_{ht}} &= \tilde{w}_{ht} \end{aligned}$$

and for n'_{ht}:

$$\begin{aligned} \left(1-\mu_h-\mu_l\right)\left(1-\alpha\right) \frac{q_t IH_t}{n'_{ht}} &= w'_{ht} \\ \left(1-\mu_h-\mu_l\right)\left(1-\alpha\right) \frac{q_t}{G_Q^t} \frac{IH_t}{G_H^t} n'_{ht} &= \frac{w'_{ht}}{G_C^t} \\ \left(1-\mu_h-\mu_l\right)\left(1-\alpha\right) \frac{\tilde{q}_t \tilde{IH}_t}{n'_{ht}} &= \tilde{w}'_{ht} \end{aligned}$$

The first-order condition with respect to k_{ct-1} is:

$$\mu_c \frac{Y_t}{X_t k_{ct-1}} = R_{ct} z_{ct}$$

which is transformed into:

$$\frac{\mu_c}{X_t}\frac{\frac{Y_t}{G_C^t}}{\frac{k_{ct-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}\frac{1}{\Gamma_{AK}^t}} = R_{ct}\Gamma_{AK}^t z_{ct}$$

$$\frac{\mu_c}{X_t}\frac{\tilde{Y}_t}{\frac{\tilde{k}_{ct-1}}{G_{KC}}} = \tilde{R}_{ct} z_{ct}.$$

Similarly with respect to k_{ht-1} is

$$\begin{aligned}
\mu_h \frac{q_t IH_t}{k_{ht-1}} &= R_{ht} z_{ht} \\
\mu_h \frac{\frac{q_t}{G_Q^t}\frac{IH_t}{G_H^t}}{\frac{k_{ht-1}}{G_C^{t-1}}\frac{G_C^{t-1}}{G_C^t}} &= R_{ht} z_{ht} \\
\mu_h \frac{\tilde{q}_t \tilde{IH}_t}{\frac{\tilde{k}_{ht-1}}{G_C}} &= R_{ht} z_{ht}.
\end{aligned}$$

The first order condition with respect to l_t, after setting $l_t = 1$, is:

$$\begin{aligned}
\mu_l q_t IH_t &= R_{lt} \\
\mu_l \frac{q_t}{G_Q^t}\frac{IH_t}{G_H^t} &= \frac{R_{lt}}{G_C^t} \\
\mu_l \tilde{q}_t \tilde{IH}_t &= \tilde{R}_{lt}
\end{aligned}$$

and with respect to k_{bt}:

$$\begin{aligned}
\mu_b \frac{q_t IH_t}{k_{bt}} &= p_{bt} \\
\mu_b \frac{\frac{q_t}{G_Q^t}\frac{IH_t}{G_H^t}}{\frac{k_{bt}}{G_C^t}} &= p_{bt} \\
\mu_b \frac{\tilde{q}_t \tilde{IH}_t}{\tilde{k}_{bt}} &= p_{bt}
\end{aligned}$$

1.4 Wage stickiness

Patient and impatient households supply their homogeneous labor services to labor unions. There are four unions, two for each sector, each one acting in the interest of either patient or impatient households. The unions differentiate labor services, set nominal wages subject to a Calvo scheme and offer labor services to intermediate labor packers who assemble the differentiated labor services into the homogeneous labor composites n_c, n_h, n'_c and n'_h. The probability of unions being allowed to change nominal wages in each sector is common to both households. Wholesale firms hire labor services from the labor packers. Under partial indexation of nominal wages to past inflation, the wage-setting rules set by the union imply four wage Phillips curves that are isomorphic to the one in the goods sector:[1]

$$
\begin{aligned}
\ln \omega_{c,t} - \iota_{wc} \ln \pi_{t-1} &= \beta G_C \left(E_t \ln \omega_{c,t+1} - \iota_{wc} \ln \pi_t \right) - \varepsilon_{wc} \ln \left(X_{wc,t}/X_{wc} \right) \\
\ln \omega'_{c,t} - \iota_{wc} \ln \pi_{t-1} &= \beta' G_C \left(E_t \ln \omega'_{c,t+1} - \iota_{wc} \ln \pi_t \right) - \varepsilon'_{wc} \ln \left(X_{wc,t}/X_{wc} \right) \\
\ln \omega_{h,t} - \iota_{wh} \ln \pi_{t-1} &= \beta G_C \left(E_t \ln \omega_{h,t+1} - \iota_{wh} \ln \pi_t \right) - \varepsilon_{wh} \ln \left(X_{wh,t}/X_{wh} \right) \\
\ln \omega'_{h,t} - \iota_{wh} \ln \pi_{t-1} &= \beta' G_C \left(E_t \ln \omega'_{h,t+1} - \iota_{wh} \ln \pi_t \right) - \varepsilon'_{wh} \ln \left(X_{wh,t}/X_{wh} \right)
\end{aligned}
$$

with $\omega_{i,t}$ nominal wage inflation, that is, $\omega_{i,t} = \frac{w_{i,t}\pi_t}{w_{i,t-1}}$ for each sector/household pair, and

$$
\begin{aligned}
\varepsilon_{wc} &= \left(1-\theta_{wc}\right)\left(1-\beta G_C \theta_{wc}\right)/\theta_{wc} \\
\varepsilon'_{wc} &= \left(1-\theta_{wc}\right)\left(1-\beta' G_C \theta_{wc}\right)/\theta_{wc} \\
\varepsilon_{wh} &= \left(1-\theta_{wh}\right)\left(1-\beta G_C \theta_{wh}\right)/\theta_{wh} \\
\varepsilon'_{wh} &= \left(1-\theta_{wh}\right)\left(1-\beta' G_C \theta_{wh}\right)/\theta_{wh}
\end{aligned}
$$

define the slope of the wage equations.

1.5 Price stickiness

Price stickiness in the consumption-business investment sector is introduced by assuming monopolistic competition at the retail level, implicit costs of adjusting nominal prices following Calvo-style contracts and partial indexation to lagged inflation of those prices that can not be reoptimized. The resulting inflation equation is:

$$
\log \pi_t - \iota_\pi \log \pi_{t-1} = \beta \left(E_t \log \pi_{t+1} - \iota_\pi \log \pi_t \right) - \varepsilon_\pi \log \left(\frac{X_t}{X} \right) + \log \mathsf{u}_{p,t}
$$

where the parameter ε_π is equal to $\varepsilon_\pi = \frac{(1-\theta_\pi)(1-\beta G_C \theta_\pi)}{\theta_\pi}$.

1.6 Monetary policy

$$
R_t = \left(R_{t-1}\right)^{r_R} \left[\pi_t^{r_\pi} \left(\frac{GDP_t}{G_C GDP_{t-1}} \right)^{r_Y} \right]^{1-r_R} \overline{rr}^{1-r_R} \frac{\mathsf{e}_{Rt}}{\mathsf{A}_{St}}
$$

where GDP_t is the sum of the value added of the two sectors, that is $GDP_t = Y_t + \overline{q} IH_t + IK_t$

1.7 Market clearing

The market clearing conditions are:

$$
\begin{aligned}
C_t + IK_{ct}/\mathsf{A}_{kt} + IK_{ht} + k_{bt} &= Y_t - \frac{\phi_{kc}}{2} \left(\frac{k_{ct}}{k_{ct-1}} - G_{KC} \right)^2 \frac{k_{ct-1}}{\Gamma_{AK}^t} - \frac{\phi_{kh}}{2} \left(\frac{k_{ht}}{k_{ht-1}} - G_C \right)^2 k_{ht-1} \\
h_t + h'_t - \left(1-\delta_h\right)\left(h_{t-1} + h'_{t-1}\right) &= IH_t \\
b_t + b'_t &= 0
\end{aligned}
$$

[1] Here we make use of the result that the price-setter stochastic discount factor for nominal payoffs (the ratio between future and current marginal utility of consumption) cancels out in the linearization of the Phillips curve itself, so that the effective discount factor is simply βG_C, rather than $\beta G_C E_t u_{c,t+1}/u_{c,t}$.

which are transformed as follows:

$$\begin{aligned}
\frac{C_t}{G_C^t}+\frac{IK_{ct}/\mathsf{A}_{kt}}{G_C^t}+\frac{IK_{ht}}{G_C^t}+\frac{k_{bt}}{G_C^t} &= \frac{Y_t}{G_C^t}-\frac{\phi_{kc}}{2}\left(\frac{k_{ct}}{k_{ct-1}}\frac{G_{KC}^t}{G_{KC}^t}-G_{KC}\right)^2\frac{k_{ct-1}}{G_C^t\Gamma_{AK}^t}- \\
-\frac{\phi_{kh}}{2}\left(\frac{k_{ht}}{k_{ht-1}}\frac{G_C^t}{G_C^t}-G_C\right)^2\frac{k_{ht-1}}{G_C^t} \\
\frac{h_t}{G_H^t}+\frac{h'_t}{G_H G_H^{t-1}}-(1-\delta_h)\left(\frac{h_{t-1}}{G_H G_H^{t-1}}+\frac{h'_{t-1}}{G_H G_H^{t-1}}\right) &= \frac{IH_t}{G_H^t} \\
\frac{b_t}{G_C^t}+\frac{b'_t}{G_C^t} &= 0
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_t+\frac{I\tilde{K}_{ct}}{\mathsf{a}_{kt}}+I\tilde{K}_{ht}+\tilde{k}_{bt} &= \tilde{Y}_t-\frac{\phi_{kc}}{2G_{KC}}\left(\frac{\tilde{k}_{ct}}{\tilde{k}_{ct-1}}-G_{KC}\right)^2\tilde{k}_{ct-1}- \\
-\frac{\phi_{kh}}{2G_C}\left(\frac{\tilde{k}_{ht}}{\tilde{k}_{ht-1}}-G_C\right)^2\tilde{k}_{ht-1} \\
\tilde{h}_t+\tilde{h}'_t-(1-\delta_h)\left(\frac{\tilde{h}_{t-1}}{G_H}+\frac{\tilde{h}'_{t-1}}{G_H}\right) &= I\tilde{H}_t \\
\tilde{b}_t+\tilde{b}'_t &= 0
\end{aligned}$$

2 Linear deterministic trends

Suppose there are linear deterministic trends in the technologies A_c, A_h and A_k. Let the corresponding gross growth rates be respectively:

$$\gamma_C,\gamma_H,\gamma_K$$

Because of these trends, the variables:

$$Y,c,c',\frac{k_c}{A_k},k_h,k_b,qI$$

all grow at a common rate along the balanced growth path. This result stems from the form of the utility function and the assumption of constant returns to scale in the production functions, which implies common expenditure shares. To compute the net growth rate (x) of Y, we observe from the production function that $x_Y=(1-\mu_c)\,\gamma_C+\mu_c x_{KC}$. We also know that $x_Y=x_{KC}-\gamma_K$. It then follows that

$$x_Y=\gamma_C+\frac{\mu_c}{1-\mu_c}\gamma_K$$

$$x_{KC}=\gamma_C+\frac{1}{1-\mu_c}\gamma_K$$

$$x_{KH}=\gamma_C+\frac{\mu_c}{1-\mu_c}\gamma_K$$

In order to disentangle q and I separately, we use the formula for I to obtain the steady state growth rate of I as $x_I = (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H + \mu_h x_{KH} + \mu_b x_{KB}$.
Hence the steady state growth rate of I is:

$$x_I = (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H + \mu_h \left(\gamma_C + \frac{\mu_c}{1-\mu_c}\gamma_K\right) + \mu_b \left(\gamma_C + \frac{\mu_c}{1-\mu_c}\gamma_K\right)$$

$$x_I = x_H = (\mu_h + \mu_b)\,\gamma_C + \frac{(\mu_h + \mu_b)\,\mu_c}{1-\mu_c}\gamma_K + (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H$$

and the growth rate of q is

$$x_Q = (1 - \mu_h - \mu_b)\,\gamma_C + \frac{(1 - \mu_h - \mu_b)\,\mu_c}{1-\mu_c}\gamma_K - (1 - \mu_h - \mu_l - \mu_b)\,\gamma_H$$

$$x_Q = x_Y - x_I$$

3 Steady state of the model

We are interested in finding the steady state of the transformed model. In the transformed model, each variable is scaled by its long-run growth rate, e.g.

$$\begin{aligned} \widetilde{c}_t &= \frac{c_t}{G_C^t} \\ \widetilde{c}_{t-1} &= \frac{c_{t-1}}{G_C^{t-1}} \end{aligned}$$

hence in each equation we perform the necessary replacements such as the following:

$$\begin{aligned} c_t &= \widetilde{c}_t G_C^t \\ c_{t-1} &= \widetilde{c}_{t-1} G_C^{t-1} \\ q_t &= \widetilde{q}_t G_Q^t \\ u_{ct} &= \widetilde{u}_{ct} G^{-t} \end{aligned}$$

3.1 Calculations

Marginal utility of consumption and housing are equal, respectively, to $1/c$ and j/h in steady state. From the transformed consumption Euler equation:

$$u_{ct} = \beta G_C u_{ct+1} \frac{R_t}{\pi_{t+1}}$$

the G_C term disappears and we can derive the steady state level of the real interest rate once we have imposed $\overline{\pi} = 1$:

$$R = \frac{1}{\beta}$$

From the Euler equations for the two capital stocks we can derive the steady state values for the rental rates:

$$\begin{aligned} R_{kc} &= \frac{\Gamma_K}{\beta} - (1 - \delta_k) \\ R_{kh} &= \frac{1}{\beta} - (1 - \delta_k) \\ r &\equiv \frac{R}{G_C} - 1 \end{aligned}$$

Combining the Euler equation for k_c and the expression for R_{kc} (from the optimal demand for capital by firms in the good sector) the following ratio is obtained:

$$\zeta_0 = \frac{k_c}{Y} = \left(\frac{\beta G_{KC} \mu_c}{\Gamma_K - \beta (1 - \delta_{kc})} \right) \frac{1}{X}$$

Combining the Euler equation for k_h and the expression for R_{kh} from the optimal demand for capital by firms in the good sector the following ratio is obtained:

$$\zeta_1 = \frac{k_h}{qI} = \frac{\beta G_C \mu_h}{1 - \beta (1 - \delta_{kh})}$$

From the Euler equation for h:

$$\zeta_2 = \frac{qh}{c} = \frac{j}{1 - \beta G_Q (1 - \delta_h)}$$

while from the Euler equation for h' and b':

$$\begin{aligned}
\zeta_3 &= \frac{j}{1 - \beta' G_Q (1 - \delta_h) - G_Q \left(\beta - \beta'\right) m} \\
\lambda &= \frac{1 - \beta'/\beta}{c'} \\
f &= \frac{X - 1}{X} Y
\end{aligned}$$

For land, let $l = 1$, so that

$$R_l = \mu_l q I$$

The following equations describe the steady state (using $b + b' = 0$, where $b = mG_Q qh'/R$ and steady state repayment is $\left(\frac{R}{G_C} - 1\right) b$, so that repayment equals $\left(\frac{R}{G_C} - 1\right) \frac{mG_Q}{R} qh' = \zeta_4 qh'$):
Define the adjusted depreciation rates:

$$\begin{aligned}
\delta'_h &= 1 - \frac{1 - \delta_h}{G_H} \\
\delta'_k &= 1 - \frac{1 - \delta_k}{G_{KC}}
\end{aligned}$$

From the above ratios and using the budget constraints of the two types of households, we have:

$$\begin{aligned}
k_c &= \zeta_0 Y \\
k_h &= \zeta_1 q I \\
qh &= \zeta_2 c \\
qh' &= \zeta_3 c' \\
\delta'_h \left(qh + qh'\right) &= qI \\
c + c' + \delta'_k (k_c + k_h) &= Y \\
c + \delta'_h qh &= f + r k_c + r k_h + \mu_l q I + \sum wn + \zeta_4 qh' + \text{div} \\
c' + \delta'_h qh' &= \sum wn - \zeta_4 qh' + \text{div}
\end{aligned}$$

Simple algebra yields:

$$\begin{aligned}
\delta'_h \left(\zeta_2 c + \zeta_3 c'\right) &= qI \\
c + c' + \delta'_k (\zeta_0 Y + \zeta_1 q I) &= Y \\
c + \delta'_h \zeta_2 c &= f + r\zeta_0 Y + r\zeta_1 qI + \mu_l qI + \sum wn + \zeta_4 \zeta_3 c' + div \\
c' + \delta'_h \zeta_3 c' &= \sum wn - \zeta_4 \zeta_3 c' + \text{div}
\end{aligned}$$

The equations in labor market satisfy from the demand side:

$$\begin{aligned}
(1-\mu_c)\,\alpha\frac{Y}{Xn_c} &= w_c \\
(1-\mu_c)\,(1-\alpha)\,\frac{Y}{Xn'_c} &= w'_c \\
(1-\mu_h-\mu_b-\mu_l)\,\alpha\frac{qI}{n_h} &= w_h \\
(1-\mu_h-\mu_b-\mu_l)\,(1-\alpha)\,\frac{qI}{n'_h} &= w'_h
\end{aligned}$$

To compute the steady state, we simply need to know the total wage bill plus union dividends earned by each group, which equals

$$\begin{aligned}
w_cn_c+w_hn_h &= \alpha\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right) \\
w'_cn'_c+w'_hn'_h &= (1-\alpha)\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right)
\end{aligned}$$

Using $\phi=(X-1)/X$, we have

$$\delta'_h\left(\zeta_2c+\zeta_3c'\right)=qI$$

$$c+c'+\delta'_k\left(\zeta_0Y+\zeta_1qI\right)=Y$$

$$c+\delta'_h\zeta_2c=\phi Y+r\zeta_0Y+r\zeta_1qI+\mu_lqI+\alpha\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right)+\zeta_4\zeta_3c'$$

$$c'+\delta'_h\zeta_3c'=(1-\alpha)\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,qI\right)-\zeta_4\zeta_3c'$$

Eliminating one redundant equation (for example the second) and using the formula for qI

$$\begin{aligned}
c+\delta_h\zeta_2c &= (\phi+r\zeta_0)\,Y+r\zeta_1\delta_h\left(\zeta_2c+\zeta_3c'\right)+\alpha\left(\frac{(1-\mu_c)\,Y}{X}+(1-\mu_b-\mu_h-\mu_l)\,\delta_h\left(\zeta_2c+\zeta_3c'\right)\right)+\zeta_4\zeta_3c' \\
c'+\delta_h\zeta_3c' &= (1-\alpha)\left((1-\mu_c)\,\frac{Y}{X}+(1-\mu_h-\mu_b-\mu_l)\,\delta'_h\left(\zeta_2c+\zeta_3c'\right)\right)-\zeta_4\zeta_3c'
\end{aligned}$$

Hence the consumption-output ratios c/Y and c'/Y solve:

$$\begin{aligned}
& \left(1+\delta'_h\zeta_2\left(1-r\zeta_1-\mu_l-\alpha\left(1-\mu_b-\mu_h-\mu_l\right)\right)\right)c-\left(\left(r\zeta_1+\mu_l+\alpha\left(1-\mu_h-\mu_b-\mu_l\right)\right)\delta'_h\zeta_3+\zeta_4\zeta_3\right)c' \\
= & \left(\frac{X-1}{X}+r\zeta_0X+\alpha\frac{(1-\mu_c)}{X}\right)Y \\
& \left(1+\delta'_h\zeta_3-(1-\alpha)\,(1-\mu_h-\mu_b-\mu_l)\,\delta'_h\zeta_3+\zeta_4\zeta_3\right)c'-(1-\alpha)\,(1-\mu_h-\mu_b-\mu_l)\,\delta'_h\zeta_2c \\
= & (1-\alpha)\,(1-\mu_c)\,\frac{1}{X}Y
\end{aligned}$$

Solve for c/Y, c'/Y and qI/Y. Defining the following variables:

$$\begin{aligned}
\chi_1 &= 1+\delta_h'\zeta_2\left(1-r\zeta_1-\mu_l-\alpha\left(1-\mu_h-\mu_b-\mu_l\right)\right)\\
\chi_2 &= \left(r\zeta_1+\mu_l+\alpha\left(1-\mu_h-\mu_b-\mu_l\right)\right)\delta_h'\zeta_3+\zeta_4\zeta_3\\
\chi_3 &= \frac{X-1}{X}+r\zeta_0X+\alpha\frac{(1-\mu_c)}{X}\\
\chi_4 &= 1+\delta_h'\zeta_3-(1-\alpha)\left(1-\mu_h-\mu_b-\mu_l\right)\delta_h'\zeta_3+\zeta_4\zeta_3\\
\chi_5 &= (1-\alpha)\left(1-\mu_h-\mu_b-\mu_l\right)\delta_h'\zeta_2\\
\chi_6 &= (1-\alpha)\left(1-\mu_c\right)\frac{1}{X}
\end{aligned}$$

delivers the following solution:

$$\begin{aligned}
\frac{c}{Y} &= \frac{\chi_3\chi_4+\chi_2\chi_6}{\chi_1\chi_4-\chi_2\chi_5}\\
\frac{c'}{Y} &= \frac{\chi_1\chi_6+\chi_3\chi_5}{\chi_1\chi_4-\chi_2\chi_5}\\
\frac{qI}{Y} &= \delta_h'\left(\zeta_2c+\zeta_3c'\right)
\end{aligned}$$

3.2 Levels

In order to compute the levels of the variables in steady state we need to find first the value of hours worked. It is useful to normalize τ to 1. The labor market equilibrium is of the kind:

$$\begin{aligned}
(1-\mu_c)\,\alpha\frac{Y}{X} &= X_wc\left(n_c^{1+\xi}+n_h^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}}n_c^{1+\xi}\\
(1-\mu_h-\mu_b-\mu_l)\,\alpha qI &= X_wc\left(n_c^{1+\xi}+n_h^{1+\xi}\right)^{\frac{\eta-\xi}{1+\xi}}n_h^{1+\xi}
\end{aligned}$$

so that the ratio of hours worked is:

$$\frac{n_h}{n_c}=\left(\frac{(1-\mu_h-\mu_b-\mu_l)\,qIX}{(1-\mu_c)\,Y}\right)^{\frac{1}{1+\xi}}$$

plug back to get

$$(1-\mu_c)\,\alpha\frac{Y}{Xc}=X_w\left(1+\frac{(1-\mu_h-\mu_b-\mu_l)\,qIX}{(1-\mu_c)\,Y}\right)^{\frac{\eta-\xi}{1+\xi}}n_c^{1+\eta}$$

knowing $\frac{Y}{c}$ and $\frac{qI}{Y}$, this can be solved for n_c, and consequently for all the variables of the model:

$$n_c=\left(\frac{(1-\mu_c)\,\alpha\frac{Y}{X_wXc}}{\left(1+\frac{(1-\mu_h-\mu_b-\mu_l)qIX}{(1-\mu_c)Y}\right)^{\frac{\eta-\xi}{1+\xi}}}\right)^{\frac{1}{1+\eta}}$$

Similar formulas apply to n_h, n_c' and n_h'. Once we know the levels of hours worked by the two households in the two sectors, we can compute Y, c, c', k_c, k_h and the product qI. To find q and I separately we use:

$$k_b=\mu_bqI$$

and

$$I = \left(A_h n_h^{\alpha} n_h'^{1-\alpha}\right) \left(\zeta_1 q\right)^{\frac{\mu_h}{1-\mu_h-\mu_b-\mu_l}} \left(\mu_b q\right)^{\frac{\mu_h}{1-\mu_h-\mu_b-\mu_l}}$$

Let qI be equal to θ, then:

$$\begin{aligned} qI &= \theta \\ I &= \left(A_h n_h^{\alpha} n_h'^{1-\alpha}\right)^{1-\mu_h-\mu_l} \left(\zeta_1 \theta\right)^{\mu_h} \left(\mu_b \theta\right)^{\mu_b} \end{aligned}$$

Given the values of hours worked, we can use the production function in the goods sector to compute output Y:

$$Y =$$

The levels of capital stock in the two sectors are respectively:

$$\begin{aligned} k_c &= \zeta_0 Y \\ k_c &= \zeta_1 Q I \end{aligned}$$

the levels of consumption of the two agents:

$$\begin{aligned} c &= \left(\frac{\chi_3\chi_4 + \chi_2\chi_6}{\chi_1\chi_4 - \chi_2\chi_5}\right) Y \\ c' &= \left(\frac{\chi_1\chi_6 + \chi_3\chi_5}{\chi_1\chi_4 - \chi_2\chi_5}\right) Y \end{aligned}$$

and their stock of housing:

$$\begin{aligned} h &= \zeta_2 \frac{c}{q} \\ h' &= \zeta_3 \frac{c'}{q} \end{aligned}$$

Finally, the level of loans is:

$$b = m\, q\, G_Q\, \frac{h'}{r}$$