Linearization model

We define: $\frac{P_t^M}{P_t} = p_t^M$; $\frac{P_t^Z}{P_t} = p_t^Z$; $\frac{W_t}{P_t} = w_t$; $\frac{B_t}{P_t} = b_t$; $\frac{TAX_t}{P_t} = tax_t$.

FOC consumo:

$$\vartheta_t = \beta^t\, C_t^{-\sigma} \quad \rightarrow \quad \widetilde{\vartheta_t} = -\sigma \widetilde{c_t}$$

FOC labor:

$$\beta^t\, N_t^{\eta} = \vartheta_t \frac{W_t}{P_t} \quad \rightarrow \quad \widetilde{w_t} = \eta\, \widetilde{n_t} - \widetilde{\vartheta_t}$$

FOC bonds:

$$\frac{\vartheta_t}{P_t} = (1 + i_t)\, E_t \left(\frac{\vartheta_{t+1}}{P_{t+1}}\right)$$

ss. $\frac{\bar{\vartheta}}{\bar{P}} = (1 + \bar{\imath}) \frac{\bar{\vartheta}}{\bar{P}} = \bar{r} \frac{\bar{\vartheta}}{\bar{P}}$

$$\frac{\bar{\vartheta}}{\bar{P}} \left(\widetilde{\vartheta_t} - \widetilde{p_t}\right) = \bar{r} \frac{\bar{\vartheta}}{\bar{P}} \left(\widetilde{r_t} + \widetilde{\vartheta_{t+1}} - \widetilde{p_{t+1}}\right)$$

$$\widetilde{\vartheta_t} = \widetilde{r_t} + \widetilde{\vartheta_{t+1}} - \widetilde{p_{t+1}} + \widetilde{p_t}$$

$$\widetilde{\vartheta_t} = \widetilde{r_t} + \widetilde{\vartheta_{t+1}} - \ln p_{t+1} + \ln \bar{P} + \ln p_t - \ln \bar{P}$$

$$\widetilde{\vartheta_t} = \widetilde{r_t} + \widetilde{\vartheta_{t+1}} - (\ln p_{t+1} - \ln p_t) = \widetilde{\vartheta_t} = \widetilde{r_t} + \widetilde{\vartheta_{t+1}} - \pi_{t+1}$$

Energy price:

$$p_t^M = C_t^{\sigma}\, [(1 - er_t)\, \mu]^{1+e}\, M_t{}^{e}$$

ss. $\overline{p^M} = \bar{C}^{\sigma}\, \bar{M}^e \mu^{1+e} - \bar{C}^{\sigma}\, \bar{M}^e (\overline{er}\mu)^{1+e}$

$$\widetilde{p_t^M}\, \overline{p^M} = \sigma \widetilde{c_t}\, \bar{C}^{\sigma}\, \bar{M}^e \mu^{1+e} + e\, \widetilde{m_t}\, \bar{C}^{\sigma}\, \bar{M}^e \mu^{1+e} - \bar{C}^{\sigma}\, \bar{M}^e (\overline{er}\mu)^{1+e} \sigma \widetilde{c_t} - e\, \widetilde{m_t}\, \bar{C}^{\sigma}\, \bar{M}^e (\overline{er}\mu)^{1+e} - \bar{C}^{\sigma}\, \bar{M}^e (\overline{er}\mu)^{1+e} (1 + e)\, \widetilde{er_t}$$

$$\widetilde{p_t^M}\, \overline{p^M} = \bar{C}^{\sigma}\, \bar{M}^e \mu^{1+e} (1 - \overline{er}^{1+e})\, \sigma\, \widetilde{c_t} + \bar{C}^{\sigma}\, \bar{M}^e \mu^{1+e} (1 - \overline{er}^{\,1+e})\, e\, \widetilde{m_t} - \bar{C}^{\sigma}\, \bar{M}^e (\overline{er}\, \mu)^{1+e} (1 + e) \widetilde{er_t}$$

$$\widetilde{p_t^M} = \frac{\bar{C}^{\sigma}\, \bar{M}^e \mu^{1+e}}{\overline{p^M}} [(1 - \overline{er}^{1+e})(\sigma \widetilde{c_t} + e\, \widetilde{m_t}) - (1 + e)\, \widetilde{er_t}\, \overline{er}^{1+e}]$$

Emissions:

$$Z_t = (1 - er_t) \mu M_t = \mu M_t - \mu\, er_t M_t$$

ss. $\bar{Z} = \mu \bar{M} - \mu \, \overline{er} \, \bar{M}$

$\widetilde{z_t} \bar{Z} = \mu \, \widetilde{m_t} \, \bar{M} - \mu \, \widetilde{er_t} \overline{er} \bar{M} - \mu \, \overline{er} \bar{M} \, \widetilde{m_t}$

$\widetilde{z_t} \bar{Z} = \mu \bar{M} (\widetilde{m_t} - \widetilde{er_t} \overline{er} - \overline{er} \widetilde{m_t})$

$$\widetilde{z_t} = \frac{\mu \bar{M}}{\bar{Z}} (\widetilde{m_t} - \widetilde{er_t} \overline{er} - \overline{er} \widetilde{m_t})$$

Emissions price:

$p_t^Z = \delta \ln(1 - er_t)$

ss. $\overline{p^Z} = \delta \ln(1 - \overline{er})$

$$\widetilde{p_t^Z} \, \overline{p^Z} = -\frac{\delta \, \widetilde{er_t} \, \overline{er}}{1 - \overline{er}}$$

$$\widetilde{p_t^Z} = -\frac{\delta \, \widetilde{er_t} \, \overline{er}}{\overline{p^Z}(1 - \overline{er})}$$

Total cost of emissions:

$CE_t(j) = -\delta \mu M_t \left[\ln(1 - er_t)(1 - er_t) + er_t \right]$

ss. $\overline{CE} = -\delta \mu \bar{M} \ln(1 - \overline{er}) + \delta \mu \bar{M} \, \overline{er} \ln(1 - \overline{er}) - \delta \mu \bar{M} \, \overline{er}$

$$\widetilde{ce_t} \overline{CE} = -\delta \mu \bar{M} \ln(1 - \overline{er}) \left(\widetilde{m_t} - \frac{\overline{er} \, \widetilde{er_t}}{1 - \overline{er}} \right) + \delta \mu \bar{M} \overline{er} \ln(1 - \overline{er}) \left(\widetilde{m_t} + \widetilde{er_t} - \frac{\overline{er} \, \widetilde{er_t}}{1 - \overline{er}} \right) - \delta \mu \bar{M} \widetilde{m_t} (\widetilde{m_t} + \widetilde{er_t})$$

$$\widetilde{ce_t} = \frac{1}{\overline{CE}} \left\{ -\delta \mu \bar{M} \ln(1 - \overline{er}) \left(\widetilde{m_t} - \frac{\overline{er} \, \widetilde{er_t}}{1 - \overline{er}} \right) + \delta \mu \bar{M} \overline{er} \ln(1 - \overline{er}) \left(\widetilde{m_t} + \widetilde{er_t} - \frac{\overline{er} \, \widetilde{er_t}}{1 - \overline{er}} \right) - \delta \mu \bar{M} \, \overline{er} \, (\widetilde{m_t} + \widetilde{er_t}) \right\}$$

Production function:

$Y_t = A_t N_t^{\alpha} M_t^{1-\alpha} \quad \rightarrow \quad \widetilde{y_t} = \widetilde{a_t} + \alpha \, \widetilde{n_t} + (1 - \alpha) \widetilde{m_t}$

Marginal productivity of labor:

$MPL_t = A_t \, \alpha \, N_t^{\alpha - 1} M_t^{1-\alpha} \quad \rightarrow \quad \widetilde{mpl_t} = \widetilde{a_t} + (\alpha - 1) \, \widetilde{n_t} + (1 - \alpha) \, \widetilde{m_t}$

Marginal costs:

$$MC_t = \frac{(w_t)^{\alpha}}{A_t \, (\alpha)^{\alpha} \, (1-\alpha)^{1-\alpha} \left[p_t^M - \delta \mu \, er_t \right]^{\alpha - 1}}$$

$$MC_t = (w_t)^{\alpha}\ A_t^{-1}\ (\alpha)^{-\alpha}\ (1-\alpha)^{\alpha-1}\ [p_t^M - \delta\mu\, er_t]^{1-\alpha}$$

$$MC_t^{\frac{1}{1-\alpha}} = (w_t)^{\frac{\alpha}{1-\alpha}} A_t^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\ [p_t^M - \delta\mu\, er_t]$$

s.s. $\overline{MC}^{\frac{1}{1-\alpha}} = \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\left[\overline{p^M} - \delta\mu\, \overline{er}\right]$

$$\begin{aligned}\frac{1}{1-\alpha}\widetilde{MC_t}\ \overline{MC}^{\frac{1}{1-\alpha}} &\\ &= \frac{\alpha}{1-\alpha}\widetilde{w_t}\ \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\ \overline{p^M}\\ &- \frac{\alpha}{1-\alpha}\widetilde{w_t}\ \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\ \delta\mu\, \overline{er}\\ &- \frac{1}{1-\alpha}\widetilde{a_t}\ \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\ \overline{p^M}\\ &+ \frac{1}{1-\alpha}\widetilde{a_t}\ \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\delta\mu\, \overline{er}\\ &+ \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\ \overline{p^M}\ \widetilde{p_t^M}\\ &- \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\ \delta\mu\, \overline{er}\,\widetilde{er_t}\end{aligned}$$

$$\begin{aligned}\frac{1}{1-\alpha}\widetilde{MC_t}\ \overline{MC}^{\frac{1}{1-\alpha}} &\\ &= \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ (1-\alpha)^{-1}\left[\overline{p^M}\left(\frac{\alpha}{1-\alpha}\widetilde{w_t} - \frac{1}{1-\alpha}\widetilde{a_t} + \widetilde{p_t^M}\right)\right.\\ &\left.- \delta\mu\, \overline{er}\left(\frac{\alpha}{1-\alpha}\widetilde{w_t} - \frac{1}{1-\alpha}\widetilde{a_t} + \widetilde{er_t}\right)\right]\end{aligned}$$

$$\begin{aligned}\widetilde{MC_t} &= \overline{w}^{\frac{\alpha}{1-\alpha}} \bar{A}^{-\frac{1}{1-\alpha}}\ (\alpha)^{-\frac{\alpha}{1-\alpha}}\ \overline{MC}^{-\frac{1}{1-\alpha}}\left[\overline{p^M}\left(\frac{\alpha}{1-\alpha}\widetilde{w_t} - \frac{1}{1-\alpha}\widetilde{a_t} + \widetilde{p_t^M}\right)\right.\\ &\left.- \delta\mu\, \overline{er}\left(\frac{\alpha}{1-\alpha}\widetilde{w_t} - \frac{1}{1-\alpha}\widetilde{a_t} + \widetilde{er_t}\right)\right]\end{aligned}$$

FOC energy:

$$p_t^M + p_t^Z(1-er_t)\mu - \delta\mu\ln(1-er_t) + \delta\mu er_t\ln(1-er_t) - \delta\mu\, er_t = MC_t\, A_t(1-\alpha)N_t^{\alpha}M_t^{-\alpha}$$

$$M_t^{-\alpha} = \frac{1}{MC_t\, A_t(1-\alpha)N_t^{\alpha}}\left[p_t^M + p_t^Z(1-er_t)\mu - \delta\mu\ln(1-er_t) + \delta\mu er_t\ln(1-er_t) - \delta\mu\, er_t\right]$$

ss. $\bar{M}^{-\alpha} = \frac{1}{\overline{MC}\ \bar{A}\ (1-\alpha)\ \bar{N}^{\alpha}}\left[\overline{p^M} + \overline{p^Z}(1-\overline{er})\mu - \delta\mu\ln(1-\overline{er}) + \delta\mu\overline{er}\ln(1-\overline{er}) - \delta\mu\overline{er}\right]$

$$-\alpha\widetilde{m_t}\bar{M}^{-\alpha} = \frac{1}{\overline{MC}\ \bar{A}\ (1-\alpha)\bar{N}^{\alpha}}\Big[\overline{p^M}\Big(\widetilde{p_t^M} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\Big) + \overline{p^Z}\mu\left(\widetilde{p_t^Z} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\right)$$
$$- \overline{p^Z}\,\overline{er}\mu\left(\widetilde{p_t^Z} + \widetilde{er_t} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\right)$$
$$- \delta\mu\ln(1-\overline{er})\left(\frac{-\overline{er}\,\widetilde{er_t}}{(1-\overline{er})} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\right)$$
$$+ \delta\mu\,\overline{er}\ln(1-\overline{er})\left(\widetilde{er_t} - \frac{\overline{er}\,\widetilde{er_t}}{(1-\overline{er})} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\right)$$
$$- \delta\mu\overline{er}(\widetilde{er_t} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t})\Big]$$

$$\widetilde{m_t} = \frac{-\bar{M}^{\alpha}}{\overline{MC}\ \bar{A}\ \alpha(1-\alpha)\bar{N}^{\alpha}}\Big[\overline{p^M}\Big(\widetilde{p_t^M} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\Big) + \overline{p^Z}\mu\Big(\widetilde{p_t^Z} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\Big)$$
$$- \overline{p^Z}\,\overline{er}\mu\Big(\widetilde{p_t^Z} + \widetilde{er_t} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\Big)$$
$$- \delta\mu\ln(1-\overline{er})\left(\frac{-\overline{er}\,\widetilde{er_t}}{(1-\overline{er})} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\right)$$
$$+ \delta\mu\,\overline{er}\ln(1-\overline{er})\left(\widetilde{er_t} - \frac{\overline{er}\,\widetilde{er_t}}{(1-\overline{er})} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t}\right)$$
$$- \delta\mu\overline{er}(\widetilde{er_t} - \widetilde{mc_t} - \widetilde{a_t} - \alpha\widetilde{n_t})\Big]$$

Government constraint:

$$G_t = tax_t + b_t - (1+i_{t-1})\,b_{t-1} = tax_t + b_t - r_{t-1}\frac{P_{t-1}}{P_t}\,b_{t-1}$$

ss. $\bar{G} = \overline{tax} + \bar{b} - \bar{r}\,\frac{\bar{P}}{\bar{P}}\bar{b}$

$$\widetilde{g_t}\bar{G} = \widetilde{tax_t}\,\overline{tax} + \bar{b}\,\widetilde{b_t} - \widetilde{r_{t-1}}\,\bar{r}\,\bar{b} - \widetilde{p_{t-1}}\,\bar{r}\,\bar{b} + \widetilde{p_t}\bar{r}\bar{b} - \widetilde{b_{t-1}}\,\bar{r}\,\bar{b}$$

$$\widetilde{g_t}\bar{G} = \widetilde{tax_t}\,\overline{tax} + \bar{b}\,\widetilde{b_t} - \bar{r}\,\bar{b}\left(\widetilde{r_{t-1}} + \widetilde{p_{t-1}} - \widetilde{p_t} + \widetilde{b_{t-1}}\right)$$

$$\widetilde{g_t} = \frac{1}{\bar{G}}\left\{\widetilde{tax_t}\,\overline{tax} + \bar{b}\,\widetilde{b_t} - \bar{r}\,\bar{b}\left(\widetilde{r_{t-1}} - \pi_t + \widetilde{b_{t-1}}\right)\right\}$$

Taxation:

$$tax_t = w_t N_t + p_t^M M_t + p_t^Z Z_t \quad \rightarrow$$

$$\widetilde{tax_t} = \frac{1}{\overline{tax}}\left[\bar{w}\bar{N}\,(\widetilde{w_t} + \widetilde{n_t}) + \overline{p^M}\bar{M}\left(\widetilde{p_t^M} + \widetilde{m_t}\right) + \overline{p^Z}\bar{Z}\left(\widetilde{p_t^Z} + \widetilde{z_t}\right)\right]$$

Aggregate resource constraint:

$$Y_t = C_t + G_t + CE_t \quad \rightarrow \quad \widetilde{y_t} = \frac{1}{\bar{Y}}(\widetilde{c_t}\,\bar{C} + \widetilde{g_t}\bar{G} + \widetilde{ce_t}\,\overline{CE})$$

<u>Flexible price Output</u>: $\widetilde{y_t}^f = \frac{(1+\eta)}{1+\eta+\alpha(\sigma-1)} \widetilde{a_t}$

<u>NK Phillips Curve</u>: $\pi_t = \tilde{k}\, MC_t + \beta\, E_t(\pi_{t+1})$