Casting a DSGE model into the corresponding VAR(1) representation

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To rewrite the canonical form of the DSGE

$$\Gamma_0 s_t = \Gamma_1 s_{t-1} + \Psi \varepsilon_t + \Pi \eta_t \tag{1}$$

as a VAR(1), like eq. (2.10) in Herbst & Schorfheide (2015),

$$s_t = \Phi_1(\theta) s_{t-1} + \Phi_\varepsilon(\theta) \varepsilon_t \tag{2}$$

we proceed as follows.

First, apply generalized Schur (e.g. $\Gamma_0 = Q'\Lambda Z'$ and $\Gamma_1 = Q'\Omega Z'$) which yields

$$Q'\Lambda Z' s_t = Q'\Omega Z' s_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{3}$$

Second, multiply by Q (using $QQ' = I$) and define $w_t = Z' s_t$.

$$\Lambda w_t = \Omega w_{t-1} + Q\left(\Psi \varepsilon_t + \Pi \eta_t\right). \tag{4}$$

Third, recall that Λ, Ω are upper triangular, reorder the eigenvalues such that the unstable eigenvalues are located in the lower subsystem and decompose $w_t = (w_{1,t}' \;\; w_{2,t}')'$ as well as Λ, Ω, Q.

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix} \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + \begin{pmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{pmatrix} \left(\Psi \varepsilon_t + \Pi \eta_t\right) \tag{5}$$

Due to the upper triangular structure of Λ and Ω the lower subsystem can be solved independently from the upper subsystem. Since the lower subsystem depends on the unstable eigenvalues a stable solution for the lower block can only be obtained by forward iteration.

The forward solution for the lower subsystem is

$$\begin{aligned} w_{2,t} &= \Omega_{22}^{-1}\Lambda_{22} w_{2,t+1} - \Omega_{22}^{-1} Q_{2\cdot}\left(\Psi \varepsilon_{t+1} + \Pi \eta_{t+1}\right) && (6) \\ &= M^2 w_{2,t+2} - M\Omega_{22}^{-1} Q_{2\cdot}\left(\Psi \varepsilon_{t+2} + \Pi \eta_{t+2}\right) - \Omega_{22}^{-1} Q_{2\cdot}\left(\Psi \varepsilon_{t+1} + \Pi \eta_{t+1}\right) && (7) \\ &= \ldots && (8) \\ &= -\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1} Q_{2\cdot}\left(\Psi \varepsilon_{t+s} + \Pi \eta_{t+s}\right) && (9) \end{aligned}$$

where we set $M = \Omega_{22}^{-1}\Lambda_{22}$ and use the condition $\lim_{s\to\infty} M^s w_{2,t+s} = 0$. Because $E_t \varepsilon_{t+s} = E_t \eta_{t+s} = 0$ if $s \geq 1$, it holds that

$$\begin{aligned} w_{2,t} &= E_t w_{2,t} = 0 && (10) \\ &= -\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1} Q_{2\cdot}\left(\Psi \varepsilon_{t+s} + \Pi \eta_{t+s}\right) = 0. && (11) \end{aligned}$$

This implies that

$$Q_{2\cdot}\Pi\eta_t = -Q_{2\cdot}\Psi\varepsilon_t \tag{12}$$

$$\eta_t = -(Q_{2\cdot}\Pi)^{-1}Q_{2\cdot}\Psi\varepsilon_t. \tag{13}$$

From eq. (10) we can infer that the stable solution for the lower subsystem is

$$w_{2,t} = 0 \quad \forall \; t. \tag{14}$$

Now, the problem arises that the upper subsystem still depends on η_t. Following Sims (2002, p. 11) a necessary and sufficient condition for a unique solution is that there exist some matrix Φ such that

$$Q_{1\cdot}\Pi = \Phi Q_{2\cdot}\Pi. \tag{15}$$

Multiplying the lower subsystem in (5) with Φ and subtracting it from the upper subsystem yields

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix} \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + \begin{pmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{pmatrix} \Psi\varepsilon_t \tag{16}$$

where we replace the lower subsystem by its solution in (14).
Now, recall that $Z's_t = w_t$ and define

$$H = Z \cdot \begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}^{-1} = Z \cdot \begin{pmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12} - \Phi\Lambda_{22}) \\ 0 & I \end{pmatrix}. \tag{17}$$

Multiplying with H from the left (recall $ZZ' = I$) yields

$$s_t = Z \begin{pmatrix} \Lambda_{11}^{-1}\Omega_{11} & \Lambda_{11}^{-1}(\Omega_{12} - \Phi\Omega_{22}) \\ 0 & 0 \end{pmatrix} Z's_{t-1} + H \begin{pmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{pmatrix} \Psi\varepsilon_t \tag{18}$$

$$= Z_{\cdot 1} \begin{pmatrix} \Lambda_{11}^{-1}\Omega_{11} & \Lambda_{11}^{-1}(\Omega_{12} - \Phi\Omega_{22}) \end{pmatrix} Z's_{t-1} + H \begin{pmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{pmatrix} \Psi\varepsilon_t \tag{19}$$

where we partition $Z = [Z_{\cdot 1} Z_{\cdot 2}]$.

Finally, we obtain the VAR(1) representation of our DSGE model

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\varepsilon(\theta)\varepsilon_t \tag{20}$$

by defining

$$\Phi_1(\theta) = Z_{\cdot 1}\Lambda_{11}^{-1} \begin{pmatrix} \Omega_{11} & (\Omega_{12} - \Phi\Omega_{22}) \end{pmatrix} Z' \tag{21}$$

$$\Phi_\varepsilon(\theta) = H \begin{pmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{pmatrix} \Psi \tag{22}$$

where Φ is obtained as

$$\Phi = ? \tag{23}$$