1. Households

$$L = E_t \left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{{N_t}^{1+\eta}}{1+\eta} - \frac{[(1 - er_t)\mu\, M_t]^{1+e}}{1+e} \right] \right\}$$
$$+ \vartheta_t \left[\frac{W_t}{P_t} N_t + (1 + i_{t-1}) \frac{B_{t-1}}{P_t} + \pi_t + \frac{P_t^M}{P_t}\, M_t - C_t - \frac{B_t}{P_t} \right]$$

$\frac{\partial L}{\partial C_t}$: $\beta^t\, C_t^{-\sigma} - \vartheta_t = 0$

$\frac{\partial L}{\partial N_t}$: $-\beta^t\, N_t^{\eta} + \vartheta_t\, \frac{W_t}{P_t} = 0$

$\frac{\partial L}{\partial B_t}$: $-\frac{\vartheta_t}{P_t} + (1 + i_t)\, E_t\, \left(\frac{\vartheta_{t+1}}{P_{t+1}}\right) = 0$

$\frac{\partial L}{\partial M_t}$: $-\beta^t\, [(1 - er_t)\, \mu]^{1+e}\, {M_t}^{e} + \vartheta_t \frac{P_t^M}{P_t} = 0$

From these conditions we obtain respectively the labor supply (1), the Euler equation in consumption (2) and the price of energy (3):

$$\frac{N_t^{\eta}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \qquad (1)$$

$$1 = \beta\, (1 + i_t)\, E_t\, \left(\frac{1}{\pi_{t+1}} \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \right) \qquad (2)$$

$$\frac{P_t^M}{P_t} = C_t^{\sigma}\, [(1 - er_t)\mu]^{1+e}\, M_t^{e} \qquad (3)$$

2. Firms

Final goods producers

$$\max P_t \left[\int_0^1 Y_t(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

FOC: $Y_t P_t^{\theta} = Y_t(j) P_t(j)^{\theta}$

Intermediate goods producers

Emissions : $Z_t(j) = (1 - er_t(j))\, \mu\, M_t(j)$

Production function: $Y_t(j) = A_t\, {N_t}^{\alpha}(j)\, {M_t}^{1-\alpha}(j)$

The problem is:

$$\max\ \pi = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) - \frac{P_t^M}{P_t} M_t(j) - \frac{P_t^Z}{P_t}\left(1 - er_t(j)\right)\mu\, M_t(j) - CE_t(j)$$

$$\text{s.t.}\quad Y_t(j) = A_t\, N_t^{\ \alpha}(j)\, M_t^{\ 1-\alpha}(j)$$

where $CE_t(j) = -\delta\mu M_t(j)\left[\ln\left(1 - er_t(j)\right)\left(1 - er_t(j)\right) + er_t(j)\right]$ are total costs of emissions.

$$\frac{\partial\varsigma}{\partial N_t(j)}: -\frac{W_t}{P_t} - \lambda_t A_t \alpha\, N_t^{\alpha-1}(j) M_t^{1-\alpha}(j) = 0$$

$$\frac{\partial\varsigma}{\partial M_t(j)}: \frac{P_t^M}{P_t} + \frac{P_t^Z}{P_t}\left(1 - er_t(j)\right)\mu - \delta\mu[\ln(1 - er_t(j))\,(1 - er_t(j)) + er_t(j)]$$

$$= -\lambda_t\, A_t (1-\alpha) N_t^{\alpha}(j) M_t^{-\alpha}(j)$$

$$\frac{\partial\varsigma}{\partial er_t(j)}: \frac{P_t^Z}{P_t} = \delta \ln\left(1 - er_t(j)\right)$$

Solving the system, we obtain:

- $MC_t = -\lambda_t = \dfrac{\left({W_t}/{P_t}\right)^{\alpha}}{A_t\, (\alpha)^{\alpha}\, (1-\alpha)^{1-\alpha} \left[\frac{P_t^M}{P_t} - \delta\mu\, er_t\right]^{\alpha-1}}$
- $\frac{W_t}{P_t} = -\lambda_t\, \alpha\, A_t\, N_t^{\alpha-1}(j)\, M_t^{1-\alpha}(j) \quad where\ \ MPL_t = \alpha A_t N_t^{\alpha-1}(j) M_t^{1-\alpha}(j)$ is the marginal productivity of labor
- $M_t^{-\alpha}(j) = \frac{1}{MC_t\, A_t (1-\alpha) N_t^{\alpha}(j)} \left(\frac{P_t^M}{P_t} + \frac{P_t^Z}{P_t}\left(1 - er_t(j)\right)\mu - \delta\mu\left[\ln\left(1 - er_t(j)\right)\left(1 - er_t(j)\right) + er_t(j)\right]\right)$ from the second FOC

Potential output: $\tilde{y}_t^{\ f} = \frac{(1+\eta)}{1+\eta+\alpha(\sigma-1)} \tilde{a}_t$

Output gap : $x_t = Y_t - Y_t^f$

Government constraint: $P_t\, G_t + (1 + i_{t-1})\, B_{t-1} = TAX_t + B_t$

Government taxation: $TAX_t = W_t N_t + P_t^M M_t + P_t^Z\, Z_t$

Aggregate resource constraint: $Y_t = C_t + G_t + CE_t$