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A check for finite order VAR representations of DSGE models

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HIGHLIGHTS

- Economic shocks of DSGE models cannot always be recovered from VARs.
- We provide a way to check when this is the case.
- The check consists in verifying that a certain matrix defined in Fernández-Villaverde et al. (2007) is nilpotent.
- Our condition is equivalent to the one in Ravenna (2007).

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ABSTRACT

The present paper shows that a DSGE model can be represented by a finite order VAR if and only if the eigenvalues of the matrix defined in Fernández-Villaverde et al. (2007) are all equal to zero. Further it shows that this condition is equivalent to the unimodularity condition presented in Ravenna (2007).

1. Introduction

The analysis of how an economy reacts to shocks plays a central role in macroeconomics. Since the seminal work of Sims (1980), the mechanisms of propagation of economic shocks have been analyzed empirically using vector autoregressive (VAR) models. During the last thirty years, a vast literature has provided evidence on the effects of monetary policy, fiscal policy and other economically relevant shocks via the analysis of impulse response functions (IRFs) derived from VARs, see e.g. Blanchard and Quah (1989), Blanchard and Perotti (2002), Uhlig (2005), Mountford and Uhlig (2009) and Perotti (2008). On the theoretical side, dynamic stochastic general equilibrium (DSGE) models have recently gained a central role in formalizing these propagation mechanisms in a coherent framework; in many cases, one uses the 'stylized facts' derived from a VAR as a guidance for isolating properties that a theoretical model would need to possess.

A growing number of papers, see Chari et al. (2005), Christiano et al. (2006), Kapetanios et al. (2007), Fernández-Villaverde et al. (2007), Ravenna (2007), remarks that this requires that the data-generating process consistent with a DSGE admits a finite order VAR representation, and poses the following basic question: is it always possible to capture the economic shocks of a DSGE via the residuals of a VAR? That is, does a reduced form VAR always contain the economic shocks of a DSGE among its structural interpretations? This difficulty is related to the problem of non-invertibility (or non-fundamentalness) of economic models, see Hansen and Sargent (1980), Hansen and Sargent (1991), Lippi and Reichlin (1993), Lippi and Reichlin (1994) for early treatments of the issue.

Conditions to answer the questions above are provided in two recent papers: in Fernández-Villaverde et al. (2007) it is shown that if a certain matrix is stable, then an infinite order

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VAR representation of the DSGE exists.[1] The condition of stability of that matrix is called the 'poor man's invertibility condition'. Furthermore, Ravenna (2007) shows that if a certain matrix polynomial is unimodular, then the model admits a finite order VAR representation.[2] In the following, we refer to this condition as the 'unimodularity condition'.

The present paper shows that a finite order VAR representation of a DSGE model exists if and only if the matrix defined in Fernández-Villaverde et al. (2007) is nilpotent, i.e. its eigenvalues are all equal to zero. Therefore, one can check the existence of finite or infinite order VAR representations of a DSGE model within the same computation. Further we show that the unimodularity condition in Ravenna (2007) is equivalent to our nilpotency condition; through this equivalence, the relationship between the results in Ravenna (2007) and those in Fernández-Villaverde et al. (2007) is clarified.

The rest of the paper is organized as follows. Section 2 introduces the square case, Section 3 presents our results and Section 4 concludes. All proofs are collected in the Appendix.

2. The ABCD setup

Let an equilibrium of an economic model have the state space representation (see e.g. Uhlig (1999) for an exposition of how to obtain it):

$$\begin{aligned} x_t &= Ax_{t-1} + Bw_t \\ y_t &= Cx_{t-1} + Dw_t \\ w_t &= Hw_{t-1} + \varepsilon_t \end{aligned} \tag{1}$$

where x_t is an $n_x \times 1$ vector of possibly unobserved variables, y_t is an $n_y \times 1$ vector of observed variables, w_t is an $n_w \times 1$ autoregressive process, and ε_t is an $n_\varepsilon \times 1$ vector white noise of economic shocks, i.e. $E(\varepsilon_t) = 0$ and $E(\varepsilon_t \varepsilon_t')$ is a positive definite diagonal matrix.

Assumption 2.1. Assume that (1) satisfies the following requirements:

(i) $n_\varepsilon = n_y$, i.e. the number of economic shocks is equal to the number of observables;
(ii) D is an invertible matrix;
(iii) the system is in minimal form.[3]

We are interested in characterizing situations in which the structural shocks of a DSGE match up with those of a finite order VAR on the observable y_t. That is, we wish to give a necessary and sufficient condition for (1) to admit a finite order VAR representation

$$y_t = \sum_{j=1}^{k} A_j y_{t-j} + u_t, \quad u_t = Q\varepsilon_t, \tag{2}$$

where the reduced form errors are a linear combination of the economic shocks of the DSGE via an invertible matrix Q.

It is well known that if all the eigenvalues of the matrix $A - BD^{-1}C$ are less than one in modulus, then (2) holds for $k = \infty$. This is the poor man's invertibility condition in Fernández-Villaverde et al. (2007). Since this condition only requires to compute the eigenvalues of $A - BD^{-1}C$, it is of straightforward numerical application. If the matrix polynomial[4] $|I - AL|I + C(I - AL)^{\mathrm{adj}}BD^{-1}L$, or equivalently $I - (A - BD^{-1}C)L$, is unimodular, then (2) holds for finite k. This is the unimodularity condition in Ravenna (2007).

We observe that the relationship between the poor man's invertibility condition and the unimodularity condition is not clear: the former ensures the DSGE to admit an infinite order VAR representation and the latter a finite order one, so the intuition is that the condition in Ravenna (2007) is stronger than the one in Fernández-Villaverde et al. (2007). That is, one expects the unimodularity condition to imply the poor man's invertibility condition (but not vice versa) and hence to be able to write the former as a restriction of the latter.

3. Main result

This section presents our results: in Proposition 3.1 we state that a necessary and sufficient condition for a DSGE to admit a finite order VAR representation is that the matrix $A - BD^{-1}C$ defined in Fernández-Villaverde et al. (2007) has all eigenvalues equal to zero, i.e. it is nilpotent. Therefore, one can check the existence of finite or infinite order VAR representations of a DSGE model within the same computation. Proposition 3.2 illustrates the relationship between our condition and the one in Ravenna (2007) and shows that the latter holds if and only if $A - BD^{-1}C$ is nilpotent. Finally, the relationship between the nilpotency and the poor man's invertibility condition is presented in Corollary 3.3. Combining these results, one sees why the condition in Ravenna (2007) implies the one in Fernández-Villaverde et al. (2007).

Proposition 3.1. *A finite order VAR representation for y_t exists if and only if the eigenvalues of $A - BD^{-1}C$ are all equal to zero, that is $A - BD^{-1}C$ is nilpotent.*

It is interesting to observe how this result is related to the one in Fernández-Villaverde et al. (2007), who show that if the eigenvalues of $A - BD^{-1}C$ are all less than one in modulus, then y_t admits an infinite order VAR representation. Our result shows that if all the eigenvalues of $A - BD^{-1}C$ are not only stable but also equal to zero, then the VAR is of finite order. Therefore this result has a practical implication: one can check the poor man's invertibility condition and the nilpotency condition by simply looking at the eigenvalues of the matrix $A - BD^{-1}C$. In minimal systems the converse also holds; that is, nilpotency of $A - BD^{-1}C$ characterizes the existence of a finite order VAR representation of a DSGE model.

Next we discuss the relationship between our condition and the one in Ravenna (2007).

Proposition 3.2. *The unimodularity condition in Ravenna (2007) holds if and only if $A - BD^{-1}C$ is nilpotent.*

Because the two conditions are equivalent, one may check one or the other. Nevertheless, this result implies that one can control the nilpotency condition, that is the existence of finite order VAR representations of square minimal ABCD forms, as one checks the poor man's invertibility condition. This only requires to compute the eigenvalues of $A - BD^{-1}C$.

The relationship between our condition and the one in Fernández-Villaverde et al. (2007) is discussed next.

Corollary 3.3. *The nilpotency condition implies the poor man's invertibility condition; the converse does not hold.*

The nilpotency condition is stronger than the poor man's invertibility condition and thus it is able to eliminate the infinitely many lags from the autoregressive representation. This is because a nilpotent matrix is stable, but a stable matrix may not be nilpotent. Taken together, Propositions 3.1 and 3.2 and Corollary 3.3 allow to

[1] A matrix is called stable when all its eigenvalues are strictly less than one in modulus.

[2] A matrix polynomial is called unimodular if its determinant is a non-zero constant, see e.g. Antsaklis and Michel (2006, p. 526).

[3] One can easily check minimality of (1) via the condition $\mathrm{rank}(B, AB, \ldots, A^{n_x-1}B) = \mathrm{rank}(C', A'C', \ldots, (A^{n_x-1})'C') = n_x$, see e.g. Antsaklis and Michel (2006, p. 395).

[4] $|I - AL|$ and $(I - AL)^{\mathrm{adj}}$ indicate determinant and adjoint of $I - AL$, where L is the lag operator.

link the results in Ravenna (2007) to those in Fernández-Villaverde et al. (2007). In particular, one sees that the unimodularity condition implies the poor man's invertibility condition and the converse does not hold.

4. Conclusion

In the present paper we have shown that a DSGE model can be represented by a finite order VAR if and only if the eigenvalues of the matrix $A - BD^{-1}C$ defined in Fernández-Villaverde et al. (2007) are all equal to zero. This nilpotency condition is shown to be equivalent to the unimodularity condition in Ravenna (2007). Hence our results show that one can use the same computation of the poor man's invertibility condition to check the existence of a finite order VAR representation of the model, because the eigenvalues of $A - BD^{-1}C$ contain all the necessary information.

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Appendix. Proofs

In the proofs we replace the lag operator L with a scalar $z \in \mathbb{C}$, see e.g. Hamilton (1994, p. 30), and define $F = A - BD^{-1}C$ and $F(z) = I - Fz$.

The proof of Proposition 3.1 is based on the following lemma.

Lemma A.1. *Let $\lambda_0 \neq 0$ be an eigenvalue of F and define $z_0 = \lambda_0^{-1}$; then*

$$(I - Fz)^{-1} = \frac{G(z)}{(z - z_0)^m g(z)}, \quad m \geq 1, \quad g(z_0) \neq 0, \qquad G(z_0) = u\varphi v' \neq 0, \tag{A.1}$$

where u, v are bases of the right and left eigenspaces[5] of F corresponding to λ_0. Moreover, if (1) is minimal, then

$$\operatorname{rank}(v'B) = \operatorname{rank}(Cu) = n_x - r, \tag{A.2}$$

where $r = \operatorname{rank}(F - \lambda_0 I)$.

Proof. Let $\lambda_0 \neq 0$ be an eigenvalue of F and define $z_0 = \lambda_0^{-1}$. One can write

$$|I - Fz| = (z - z_0)^a g(z), \quad a \geq 1, \quad g(z_0) \neq 0,$$
$$(I - Fz)^{\text{adj}} = (z - z_0)^b G(z), \quad 0 \leq b < a, \quad G(z_0) \neq 0;$$

canceling common factors from determinant and adjoint, one then has

$$(I - Fz)^{-1} = \frac{G(z)}{(z - z_0)^m g(z)}, \quad m = a - b.$$

The identity $(I - Fz)(I - Fz)^{\text{adj}} = (I - Fz)^{\text{adj}}(I - Fz) = |I - Fz|I$ delivers

$$(I - Fz)G(z) = G(z)(I - Fz) = (z - z_0)^m g(z)I,$$

which evaluated for $z = z_0$ gives

$$(F - \lambda_0 I)G(z_0) = G(z_0)(F - \lambda_0 I) = 0,$$

because $I - Fz_0 = (-z_0)(F - \lambda_0 I)$. The last equation implies that the non-zero columns (rows) of $G(z_0)$ are right (left) eigenvectors of F corresponding to λ_0; hence $G(z_0) = u\varphi v'$, where $\varphi \neq 0$ and u, v are bases of the right and left eigenspaces of F corresponding to λ_0. With this notation, one can write the rank factorization $(F - \lambda_0 I) = v_\perp u'_\perp$, where $x_\perp$ indicates a basis of the orthogonal complement of the space generated by x. This completes the proof of the first statement.

Next assume (1) is minimal; Lancaster and Rodman (1995, Theorems 4.3.3 and 6.1.5) show that this means

$$\operatorname{rank}(A - \lambda I \quad B) = \operatorname{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n_x, \quad \forall \lambda \in \mathbb{C}.$$

We observe that this is equivalent to

$$\operatorname{rank}(F - \lambda I \quad B) = \operatorname{rank}\begin{pmatrix} F - \lambda I \\ C \end{pmatrix} = n_x, \quad \forall \lambda \in \mathbb{C};$$

in fact $(A - \lambda I, B)$ and $(F - \lambda I, B)$ are connected by the invertible transformation

$$(A - \lambda I \quad B)\begin{pmatrix} I_{n_x} & 0 \\ -D^{-1}C & I_{n_y} \end{pmatrix} = (F - \lambda I \quad B).$$

Similarly one shows that $\operatorname{rank}(A' - \lambda I, C') = \operatorname{rank}(F' - \lambda I, C') = n_x$. Next use the projection identity $I = P_{v_\perp} + P_v = v_\perp \bar{v}'_\perp + \bar{v}v'$, where $P_x = x(x'x)^{-1}x'$ is the projection on the space generated by x and $\bar{x} = x(x'x)^{-1}$, to write

$$\begin{aligned}(F - \lambda_0 I \quad B) &= (v_\perp u'_\perp \quad v_\perp \bar{v}'_\perp B + \bar{v}v'B) \\ &= (v_\perp \quad \bar{v})\begin{pmatrix} u'_\perp & \bar{v}'_\perp B \\ 0 & v'B \end{pmatrix}.\end{aligned}$$

Because $\operatorname{rank}(F - \lambda_0 I, B) = n_x$ and $(v_\perp, \bar{v})$ is invertible, this shows that $\operatorname{rank}(v'B) = n_x - r$, where $r = \operatorname{rank}(F - \lambda_0 I)$. Similarly, because

$$\begin{pmatrix} F - \lambda_0 I \\ C \end{pmatrix} = \begin{pmatrix} v_\perp & 0 \\ C\bar{u}_\perp & Cu \end{pmatrix}\begin{pmatrix} u'_\perp \\ \bar{u}' \end{pmatrix},$$

one has that $\operatorname{rank}(Cu) = n_x - r$. □

Proof of Proposition 3.1. Let $z_t = Dw_t$ and rewrite (1) as

$$x_t = Fx_{t-1} + BD^{-1}y_t, \quad F = A - BD^{-1}C,$$
$$y_t = Cx_{t-1} + z_t$$
$$z_t = Mz_{t-1} + D\varepsilon_t, \quad M = DHD^{-1};$$

combining the first two equations one finds $(I - C(I - FL)^{-1}BD^{-1}L)y_t = z_t$ and thus using the third equation

$$(I - ML)T(L)y_t = D\varepsilon_t, \qquad T(z) = I - C(I - Fz)^{-1}BD^{-1}z, \quad z \in \mathbb{C}. \tag{A.3}$$

Note that (A.3) is a finite order VAR if and only if $T(z)$ is a polynomial matrix.

Suff. We first prove that if F is nilpotent then a finite order VAR representation for y_t exists. If F is nilpotent then $F - \lambda I$ has full rank for all $\lambda \neq 0$; because $F - \lambda I = -\lambda(I - Fz)$, where $z = \lambda^{-1}$, this implies $|I - Fz| \neq 0$ for all $z \in \mathbb{C}$. Then $(I - Fz)^{-1}$ is a matrix polynomial and hence the same holds for $T(z)$.

Nec. We now show that if a finite order VAR representation for y_t exists then F must be nilpotent. First we observe that $T(z)$ is a polynomial matrix if and only if $C(I - Fz)^{-1}B$ is a polynomial matrix. Next we show that if $C(I - Fz)^{-1}B$ is a polynomial matrix then F must be nilpotent. Suppose that this is not the case, namely assume that there exists $\lambda_0 \neq 0$ eigenvalue of F. Write $G(z) = \sum_{j=0}^{k} G_j(z - z_0)^j$, so that $G_0 = G(z_0)$; then, see (A.1) in Lemma A.1,

[5] We say that $x \neq 0$ is a right (left) eigenvector corresponding to the eigenvalue λ_0 if $(F - \lambda_0 I)x = 0$ $(x'(F - \lambda_0 I) = 0)$. By right (left) eigenspace of F corresponding to λ_0 we mean the space generated by its right (left) eigenvectors.

$$C(I - Fz)^{-1}B = \frac{CG(z)B}{(z-z_0)^m g(z)} =$$

$$\frac{1}{g(z)}\left(\frac{CG_0B}{(z-z_0)^m} + \frac{CG_1B}{(z-z_0)^{m-1}} + \cdots + \frac{CG_kB}{(z-z_0)^{m-k}}\right),$$

$$G_0 = u\varphi v' \neq 0.$$

Because $T(z)$ is a polynomial matrix it must be that $CG_0B = 0$, i.e. that $Cu\varphi v'B = 0$. However, in minimal systems Cu and $B'v$ have full column rank, see (A.2) in Lemma A.1, and this implies $\varphi = 0$, which is a contradiction because $G_0 \neq 0$. Hence it must be that F is nilpotent. □

Proof of Proposition 3.2. The proof consists in showing that $I - Fz$ is unimodular if and only if F is nilpotent. Observe that $\lambda_0 \neq 0$ is an eigenvalue of F if and only if $z_0 = \lambda_0^{-1}$ is a root of $|I - Fz| = 0$. We next show that if the eigenvalues of F are all equal to zero, then $F(z)$ is unimodular. Suppose that this is not the case, namely that there exists $z_0 \neq 0$ such that $|F(z_0)| = 0$. Because $I - Fz_0 = (-z_0)(F - z_0^{-1}I)$, one has $|F - z_0^{-1}I| = 0$, i.e. $\lambda_0 = z_0^{-1} \neq 0$ is an eigenvalue of F. This contradicts the hypothesis and hence $F(z)$ must be unimodular. Similarly one proves that if $F(z)$ is unimodular, then the eigenvalues of F are all equal to zero. □

Proof of Corollary 3.3. In the text below the statement. □

References

Antsaklis, P., Michel, A., 2006. Linear Systems. Birkhäuser, Basel, Boston, Berlin.

Blanchard, O., Perotti, R., 2002. An empirical characterization of the dynamic effects of changes in government spending and taxes on output. The Quarterly Journal of Economics 117, 1329–1368.

Blanchard, O., Quah, D., 1989. The dynamic effects of aggregate demand and supply disturbances. American Economic Review 79, 655–673.

Chari, V., Kehoe, P., McGrattan, E., 2005. A critique of structural VARs using real business cycle theory. Working Papers 631, Federal Reserve Bank of Minneapolis.

Christiano, L.J., Eichenbaum, M., Vigfusson, R., 2006. Assessing structural VARs. In: Acemoglu, D., Rogoff, K., Woodford, M. (Eds.), NBER Macroeconomics Annual 2006, Vol. 21. The MIT Press, pp. 1–106.

Fernández-Villaverde, J., Rubio-Ramírez, J., Sargent, T., Watson, M., 2007. ABCs (and Ds) of understanding VARs. American Economic Review 97, 1021–1026.

Hamilton, J., 1994. Time Series Analysis. Princeton University Press.

Hansen, L., Sargent, T., 1980. Formulating and estimating dynamic linear rational expectations models. Journal of Economic Dynamics and Control 2, 7–46.

Hansen, L., Sargent, T., 1991. Two difficulties in interpreting vector autoregressions. In: Hansen, L., Sargent, T. (Eds.), Rational Expectations Econometrics. Westview Press, pp. 77–119.

Kapetanios, G., Pagan, A., Scott, A., 2007. Making a match: combining theory and evidence in policy-oriented macroeconomic modeling. Journal of Econometrics 136, 565–594.

Lancaster, P., Rodman, L., 1995. Algebraic Riccati Equations. Clarendon Press.

Lippi, M., Reichlin, L., 1993. The dynamic effects of aggregate demand and supply: comment. American Economic Review 83, 644–652.

Lippi, M., Reichlin, L., 1994. Var analysis, non-fundamental representations, Blaschke matrices. Journal of Econometrics 63, 307–325.

Mountford, A., Uhlig, H., 2009. What are the effects of fiscal policy shocks? Journal of Applied Econometrics 24, 960–992.

Perotti, R., 2008. In search of the transmission mechanism of fiscal policy. In: NBER Macroeconomics Annual 2007, Vol. 22. pp. 169–226.

Ravenna, F., 2007. Vector autoregressions and reduced form representations of DSGE models. Journal of Monetary Economics 54, 2048–2064.

Sims, C., 1980. Macroeconomics and reality. Econometrica 48, 1–48.

Uhlig, H., 1999. A toolkit for analyzing nonlinear dynamic stochastic models easily. In: Marimon, R., Scott, A. (Eds.), Computational Methods for the Study of Dynamic Economies. Oxford University Press, pp. 30–61.

Uhlig, H., 2005. What are the effects of monetary policy on output? Results from an agnostic identification procedure. Journal of Monetary Economics 52, 381–419.