——————————————-

Capital accumulation period by period:
$K_1 = \left[\phi\left(\frac{I_1}{K_0}\right) + 1 - \delta\right] K_0$
Setting $\chi(I_1) = \phi\left(\frac{I_1}{K_0}\right) + 1 - \delta$
$K_1 = \chi(I_1)K_0$
$K_2 = \chi(I_2)K_1 \Leftrightarrow K_2 = \chi(I_2)\chi(I_1)K_0$
$K_3 = \chi(I_3)K_2 \Leftrightarrow K_3 = \chi(I_3)\chi(I_2)\chi(I_1)K_0$
$K_4 = \chi(I_4)K_3 \Leftrightarrow K_4 = \chi(I_4)\chi(I_3)\chi(I_2)\chi(I_1)K_0$
$\vdots$

——————————————-

Budget constraints period by period:
$C_1 = A + r_1K_0 - I_1$
$C_2 = A + r_2K_1 - I_2$
$C_3 = A + r_3K_2 - I_3$
$C_4 = A + r_4K_3 - I_4$
$\vdots$

where $A$ is just an endowment as I'd like to keep labour income fixed for now as it's not relevant for the problem.

Also, the reason I'd like to have consumption on the left-hand side is because I'll plug it straight into the utility function and take the FOCs.
Rewritting the sequence of budget constraints yields:
$C_1 = A + r_1K_0 - I_1$
$C_2 = A + r_2\chi(I_1)K_0 - I_2$
$C_3 = A + r_3\chi(I_2)\chi(I_1)K_0 - I_3$
$C_4 = A + r_4\chi(I_4)\chi(I_3)\chi(I_2)\chi(I_1)K_0 - I_4$

The utility maximization problem is given by:
$\max U = u(C_1) + \beta u(C_2) + \beta^2 u(C_3) + \beta^3 u(C_4) + ...$
Substituting consumption everywhere with the RHS of the sequence of budget constraints:
$\max U = u\left[A + r_1K_0 - I_1\right] + \beta u\left[A + r_2\chi(I_1)K_0 - I_2\right] + \beta^2 u\left[A + r_3\chi(I_2)\chi(I_1)K_0 - I_3\right]$
$+\beta^3 u\left[A + r_4\chi(I_4)\chi(I_3)\chi(I_2)\chi(I_1)K_0 - I_4\right] + ...$

The only variable of choice is now $I_i$. The first-order condition with respect to $I_1$ is:
$\frac{\partial U}{\partial I_1} = 0 \Leftrightarrow -u'(C_1) + \beta u'(C_2)r_2K_0\frac{\partial\chi(I_1)}{\partial I_1} + \beta^2 u(C_3)r_3K_0\left[\chi(I_2)\chi(I_1)\right]' + ..... = 0$
and thereby

$$-u'(C_1) + \beta u'(C_2)r_2K_0\frac{\partial\chi(I_1)}{\partial I_1} + \beta^2 u'(C_3)r_3K_0\left[\frac{\partial\chi(I_2)}{\partial I_1}\chi(I_1) + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_2)\right] + \cdots = 0$$

Set $\beta^{i-1}u'(C_i) = \lambda_i$

$$\frac{\lambda_1}{K_0} = \lambda_2 r_2 \frac{\partial\chi(I_1)}{\partial I_1} + \left[\frac{\partial\chi(I_2)}{\partial I_1}\chi(I_1) + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_2)\right]\lambda_3 r_3 + \left[\frac{\partial\chi(I_3)}{\partial I_1}\chi(I_2)\chi(I_1) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_3)\chi(I_1) + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_3)\chi(I_2)\right]\lambda_4 r_4$$
$$+\left[\frac{\partial\chi(I_4)}{\partial I_1}\chi(I_3)\chi(I_2)\chi(I_1) + \frac{\partial\chi(I_3)}{\partial I_1}\chi(I_4)\chi(I_2)\chi(I_1) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_4)\chi(I_3)\chi(I_1) + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_4)\chi(I_3)\chi(I_2)\right]\lambda_5 r_5 + \dots$$

Then, in period 2:

$$\frac{\lambda_2}{K_1} = \lambda_3 r_3 \frac{\partial\chi(I_2)}{\partial I_2} + \left[\frac{\partial\chi(I_3)}{\partial I_2}\chi(I_2) + \frac{\partial\chi(I_2)}{\partial I_2}\chi(I_3)\right]\lambda_4 r_4 + \left[\frac{\partial\chi(I_4)}{\partial I_2}\chi(I_3)\chi(I_2) + \frac{\partial\chi(I_3)}{\partial I_2}\chi(I_4)\chi(I_2) + \frac{\partial\chi(I_2)}{\partial I_2}\chi(I_4)\chi(I_3)\right]\lambda_5 r_5$$
$$+\left[\frac{\partial\chi(I_5)}{\partial I_2}\chi(I_4)\chi(I_3)\chi(I_2) + \frac{\partial\chi(I_4)}{\partial I_2}\chi(I_5)\chi(I_3)\chi(I_2) + \frac{\partial\chi(I_3)}{\partial I_2}\chi(I_5)\chi(I_4)\chi(I_2) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_5)\chi(I_4)\chi(I_3)\right]\lambda_6 r_6 + \dots$$

At that point I'd like to find a way how to plug $\frac{\lambda_2}{K_1}$ into the period 1 equation, that is, $\frac{\lambda_1}{K_0}$. For this we can re-arrange some terms in the period 1 equation. For example, all terms that have $\frac{\partial\chi(I_1)}{\partial I_1}$ I'd like to isolate.

$$\frac{\lambda_1}{K_0} = \frac{\partial\chi(I_1)}{\partial I_1}\lambda_2 r_2 + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_2)\lambda_3 r_3 + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_3)\chi(I_2)\lambda_4 r_4 + \frac{\partial\chi(I_1)}{\partial I_1}\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5 + \dots$$
$$+\frac{\partial\chi(I_2)}{\partial I_1}\chi(I_1)\lambda_3 r_3 + \left[\frac{\partial\chi(I_3)}{\partial I_1}\chi(I_2)\chi(I_1) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_3)\chi(I_1)\right]\lambda_4 r_4$$
$$+\left[\frac{\partial\chi(I_4)}{\partial I_1}\chi(I_3)\chi(I_2)\chi(I_1) + \frac{\partial\chi(I_3)}{\partial I_1}\chi(I_4)\chi(I_2)\chi(I_1) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_4)\chi(I_3)\chi(I_1)\right]\lambda_5 r_5 + \dots$$

Rearranging yields:

$$\frac{\lambda_1}{K_0} = \frac{\partial\chi(I_1)}{\partial I_1}\left\{\lambda_2 r_2 + \chi(I_2)\lambda_3 r_3 + \chi(I_3)\chi(I_2)\lambda_4 r_4 + \chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5 + \dots\right\}$$
$$+\chi(I_1)\left\{\frac{\partial\chi(I_2)}{\partial I_1}\lambda_3 r_3 + \left[\frac{\partial\chi(I_3)}{\partial I_1}\chi(I_2) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_3)\right]\lambda_4 r_4 + \right.$$
$$\left.\left[\frac{\partial\chi(I_4)}{\partial I_1}\chi(I_3)\chi(I_2) + \frac{\partial\chi(I_3)}{\partial I_1}\chi(I_4)\chi(I_2) + \frac{\partial\chi(I_2)}{\partial I_1}\chi(I_4)\chi(I_3)\right]\lambda_5 r_5 + \dots\right\}$$

Let's do something here which will help us plug $\frac{\lambda_2}{K_1}$ into $\frac{\lambda_1}{K_0}$.

$$\frac{\partial\chi(I_2)}{\partial I_1} = \frac{\partial\chi(I_2)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}$$

$$\frac{\partial\chi(I_3)}{\partial I_1} = \frac{\partial\chi(I_3)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}$$

$$\frac{\partial\chi(I_4)}{\partial I_1} = \frac{\partial\chi(I_4)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}$$

$$\frac{\partial\chi(I_5)}{\partial I_1} = \frac{\partial\chi(I_5)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}$$

We need to do that since the first FOC is w.r.t $I_1$ and the second FOC is w.r.t. $I_2$.

Then, let's continue:

$$\frac{\lambda_1}{K_0} = \frac{\partial\chi(I_1)}{\partial I_1}\left\{\lambda_2 r_2 + \chi(I_2)\lambda_3 r_3 + \chi(I_3)\chi(I_2)\lambda_4 r_4 + \chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5 + \dots\right\}$$

$$+\chi(I_1)\Bigg\{\frac{\partial\chi(I_2)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\lambda_3 r_3+\left[\frac{\partial\chi(I_3)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\chi(I_2)+\frac{\partial\chi(I_2)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\chi(I_3)\right]\lambda_4 r_4+$$
$$\left[\frac{\partial\chi(I_4)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\chi(I_3)\chi(I_2)+\frac{\partial\chi(I_3)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\chi(I_4)\chi(I_2)+\frac{\partial\chi(I_2)}{\partial I_2}\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\chi(I_4)\chi(I_3)\right]\lambda_5 r_5+$$
$$\ldots\Bigg\}$$

Then

$$\frac{\lambda_1}{K_0}=\frac{\partial\chi(I_1)}{\partial I_1}\Bigg\{\lambda_2 r_2+\chi(I_2)\lambda_3 r_3+\chi(I_3)\chi(I_2)\lambda_4 r_4+\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5+.....\Bigg\}$$
$$+\chi(I_1)\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\Bigg\{\frac{\partial\chi(I_2)}{\partial I_2}\lambda_3 r_3+\left[\frac{\partial\chi(I_3)}{\partial I_2}\chi(I_2)+\frac{\partial\chi(I_2)}{\partial I_2}\chi(I_3)\right]\lambda_4 r_4+$$
$$\left[\frac{\partial\chi(I_4)}{\partial I_2}\chi(I_3)\chi(I_2)+\frac{\partial\chi(I_3)}{\partial I_2}\chi(I_4)\chi(I_2)+\frac{\partial\chi(I_2)}{\partial I_2}\chi(I_4)\chi(I_3)\right]\lambda_5 r_5+\ldots\Bigg\}$$

Which is simply

$$\frac{\lambda_1}{K_0}=\frac{\partial\chi(I_1)}{\partial I_1}\Bigg\{\lambda_2 r_2+\chi(I_2)\lambda_3 r_3+\chi(I_3)\chi(I_2)\lambda_4 r_4+\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5+.....\Bigg\}+\chi(I_1)\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\frac{\lambda_2}{K_1}$$

Now, let's find the analytical expressions for the the partial derivatives of interest:

$\chi(I_1)=\phi\left(\frac{I_1}{K_0}\right)+1-\delta$

$\frac{\partial\chi(I_1)}{\partial I_1}=\phi'_1 K_0^{-1}$

$\chi(I_2)=\phi\left(\frac{I_2}{K_1}\right)+1-\delta\Leftrightarrow\chi(I_2)=\phi\left(\frac{I_2}{\chi(I_1)K_0}\right)+1-\delta$

$\frac{\partial\chi(I_2)}{\partial I_1}=-\phi'_2\frac{I_2}{K_0}(\chi(I_1))^{-2}\frac{\partial\chi(I_1)}{\partial I_1}=-\phi'_2\frac{I_2}{K_0}(\chi(I_1))^{-2}\phi'_1 K_0^{-1}=-\phi'_1\phi'_2\frac{I_2}{K_1^2}$

Thereby:

$$\frac{\lambda_1}{K_0}=\frac{\phi'_1}{K_0}\Bigg\{\lambda_2 r_2+\chi(I_2)\lambda_3 r_3+\chi(I_3)\chi(I_2)\lambda_4 r_4+\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5+\ldots\Bigg\}+\chi(I_1)\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\frac{\lambda_2}{\chi(I_1)K_0}$$
$$\lambda_1=\phi'_1\Bigg\{\lambda_2 r_2+\chi(I_2)\lambda_3 r_3+\chi(I_3)\chi(I_2)\lambda_4 r_4+\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5+\ldots\Bigg\}-\chi(I_1)\phi'_1\phi'_2\frac{I_2}{K_1^2}\frac{K_1}{\phi'_2}\frac{\lambda_2}{\chi(I_1)}$$

Thereby:

$$\frac{\lambda_1}{K_0}=\frac{\phi'_1}{K_0}\Bigg\{\lambda_2 r_2+\chi(I_2)\lambda_3 r_3+\chi(I_3)\chi(I_2)\lambda_4 r_4+\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5+.....\Bigg\}+\chi(I_1)\frac{\partial\chi(I_2)}{\partial I_1}\frac{\partial I_2}{\partial\chi(I_2)}\frac{\lambda_2}{\chi(I_1)K_0}$$
$$\lambda_1=\phi'_1\Bigg\{\lambda_2 r_2+\chi(I_2)\lambda_3 r_3+\chi(I_3)\chi(I_2)\lambda_4 r_4+\chi(I_4)\chi(I_3)\chi(I_2)\lambda_5 r_5+\ldots\Bigg\}-\phi'_1\frac{I_2}{K_1}\lambda_2$$
$$\lambda_1=\phi'_1\lambda_2 r_2+\phi'_1\chi(I_2)\Bigg\{\lambda_3 r_3+\chi(I_3)\lambda_4 r_4+\chi(I_4)\chi(I_3)\lambda_5 r_5+\ldots\Bigg\}-\phi'_1\frac{I_2}{K_1}\lambda_2$$

thereby:

$$\frac{\lambda_1}{\phi'_1}=\lambda_2 r_2+\chi(I_2)\Bigg\{\lambda_3 r_3+\chi(I_3)\lambda_4 r_4+\chi(I_4)\chi(I_3)\lambda_5 r_5+\ldots\Bigg\}-\lambda_2\frac{I_2}{K_1}$$

In Gali et.al. (2007), on page 237, the Euler equation for capital is given by:

$$Q_t = E_t\left\{\Lambda_{t,t+1}\left[R^k_{t+1} + Q_{t+1}\left(1-\delta+\phi_{t+1}-\frac{I_{t+1}}{K_{t+1}}\right)\phi'_{t+1}\right]\right\} \tag{1}$$

where $Q_t = \frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}$

OK, I normally work with the end-of-period notation, so the above equation will become:

$$Q_t = E_t\left\{\Lambda_{t,t+1}\left[R^k_{t+1} + Q_{t+1}\left(1-\delta+\phi_{t+1}-\frac{I_{t+1}}{K_t}\right)\phi'_{t+1}\right]\right\} \tag{2}$$

where $Q_t = \frac{1}{\phi'\left(\frac{I_t}{K_{t-1}}\right)}$

Now, I'd like to compare my result with this equation. If we transform the result from the paper into a period 1-period 2 problem (as I do in my own calculations) and work in a deterministic environment only, we get:

$$Q_1 = \Lambda_{1,2}\left[R^k_2 + Q_2\left(1-\delta+\phi_2-\frac{I_2}{K_1}\right)\phi'_2\right] \tag{3}$$

where $Q_1 = \frac{1}{\phi'\left(\frac{I_1}{K_0}\right)}$

Let's plug the expression for $Q_t$ into the equation:

$$\frac{1}{\phi'\left(\frac{I_1}{K_0}\right)} = \left\{\Lambda_{1,2}\left[R^k_2 + \frac{1}{\phi'\left(\frac{I_2}{K_1}\right)}\left(1-\delta+\phi_2-\frac{I_2}{K_1}\right)\phi'_2\right]\right\} \tag{4}$$

Now, let's split the discount factor into $\lambda_1$ and $\lambda_2$

$$\frac{\lambda_1}{\phi'\left(\frac{I_1}{K_0}\right)} = \lambda_2\left[R^k_2 + \frac{1}{\phi'\left(\frac{I_2}{K_1}\right)}\left(1-\delta+\phi_2-\frac{I_2}{K_1}\right)\phi'_2\right] \tag{5}$$

or

$$\frac{\lambda_1}{\phi'_1} = \lambda_2\left[R^k_2 + \frac{1}{\phi'_2}\left(1-\delta+\phi_2-\frac{I_2}{K_1}\right)\phi'_2\right] \tag{6}$$

or

$$\frac{\lambda_1}{\phi'_1} = \lambda_2\left[R^k_2 + \left(1-\delta+\phi_2-\frac{I_2}{K_1}\right)\right] \tag{7}$$

or

$$\frac{\lambda_1}{\phi'_1} = \lambda_2 R^k_2 + \lambda_2\left(1-\delta+\phi_2\right) - \lambda_2\frac{I_2}{K_1} \tag{8}$$

What I have obtained is

$$\frac{\lambda_1}{\phi'_1} = \lambda_2 r_2 + \chi(I_2)\left\{\lambda_3 r_3 + \chi(I_3)\lambda_4 r_4 + \chi(I_4)\chi(I_3)\lambda_5 r_5 + \ldots\right\} - \lambda_2\frac{I_2}{K_1} \tag{9}$$

which could be re-written as:

$$\frac{\lambda_1}{\phi_1'} = \lambda_2 r_2 + (1-\delta+\phi_2)\left\{\lambda_3 r_3 + \chi(I_3)\lambda_4 r_4 + \chi(I_4)\chi(I_3)\lambda_5 r_5 + \dots\right\} - \lambda_2 \frac{I_2}{K_1} \tag{10}$$

Finally, let's see both equations (Gali's paper and what I get):
Gali:

$$\frac{\lambda_1}{\phi_1'} = \lambda_2 R_2^k + \lambda_2 (1-\delta+\phi_2) - \lambda_2 \frac{I_2}{K_1} \tag{11}$$

My result:

$$\frac{\lambda_1}{\phi_1'} = \lambda_2 r_2 + (1-\delta+\phi_2)\left\{\lambda_3 r_3 + \chi(I_3)\lambda_4 r_4 + \chi(I_4)\chi(I_3)\lambda_5 r_5 + \dots\right\} - \lambda_2 \frac{I_2}{K_1} \tag{12}$$

The only way for them to be identical is if $\lambda_2$ is equal to the red part. But how can that be?