The economy is composed of one household that decides between consuming non-durable goods, home-produced energy services produced from durable goods and energy, and sharing energy services, and investing in capital goods. Non-durable goods, capital goods and durable goods are provided by the final good firm, and sharing energy services are provided by the sharing firm. The final good firm rents labor and capital inputs to households, and the sharing firm rents durable goods to households during the time they do not use their durable good stock for home-produced energy services. Energy is imported via the Rest of the World.

The problem I face is that I'm trying the find the value of the rental price of durable goods, and the *non-arbitrage equation* between owning the durable good stock and fully using it for home-produced energy services and owning the durable good stock but renting a part of it to the sharing firm.

I highlighted the important equations.

List of variables

Variable	Name
p_t^{share}	Price of sharing services
Y_t^{share}	Sharing firm output
w_t^{share}	Labor wage of the sharing firm
L_t^{share}	Labor input of the sharing firm
p_t^e	Exogenous price of energy
E_t^{share}	Energy demand from the sharing firm
r_t^d	Rental price of durable goods
u_t	Utilization rate of durable goods by households
D_t	Durable goods stock
Z_t^{share}	Utilized durable goods – energy bundle of the sharing firm
w_t	Wage rate of the final good firm
L_t	Labor input of the final good firm
r_t^k	Rental price of capital
K_t	Capital stock

τ^c	Consumption tax
X_t	Non-durable goods
I_t^d	Durable goods investment
ES_t^{share}	Shared energy services demand from households
E_t^{home}	Energy demand for home-produced energy services from households
I_t^k	Capital investments
ES_t	Aggregate energy services demand from households
C_t	Aggregate consumption of households
ES_t^{home}	Home-produced energy services by households
$p_{es,t}$	Aggregate price of energy services
p_t^{home}	Home-produced energy services price
uc_t^d	User cost of durable goods
δ_d	Durable goods depreciation rate
δ_k	Capital goods depreciation rate
q_t^k	Tobin's q

Sharing services firm

The firm maximizes its following profits,

$$\max_{L_t^{share},(1-u_t)D_t,E} \pi_t = p_t^{share} Y_t^{share} - \left(w_t^{share} L_t^{share} + p_t^e E_t^{share} + r_t^d (1-u_t) D_t\right)$$

$$Y_t^{share} = {L_t^{share}}^{\alpha^{share}} {Z_t^{share}}^{1-\alpha^{share}}$$

$$Z_t^{share} = \left[\alpha_{d,share}((1-u_t)D_t)^{\frac{\sigma_{es,share}-1}{\sigma_{es,share}}} + \left(1-\alpha_{d,share}\right){E_t^{share}}^{((1-u_t)D_t)^{\frac{\sigma_{es,share}-1}{\sigma_{es,share}}}}\right]^{\frac{\sigma_{es,share}}{\sigma_{es,share}-1}}$$

1. Labor demand

$$\frac{w_t}{p_t^{share}} = \alpha^{share} \frac{Y_t^{share}}{L_t^{share}}$$

2. **Utilized durable goods demand (the firm does not choose Durable goods and utilization rate, only the household does)**

$$\frac{r_t^d}{p_t^{share}} = \left(1 - \alpha^{share}\right) \frac{Y_t^{share}}{Z_t^{share}} \alpha_{d,share} \left(\frac{Z_t^{share}}{(1-u_t)D_t}\right)^{\frac{1}{\sigma_{es,share}}}$$

3. Energy demand

$$\frac{p_t^e}{p_t^{share}} = (1 - \alpha^{share}) \frac{Y_t^{share}}{Z_t^{share}} (1 - \alpha_{d,share}) \left(\frac{Z_t^{share}}{E_t}\right)^{\frac{1}{\sigma_{es,share}}}$$

4. Marginal costs

$$p_t^{share} = \left(\frac{w_t}{\alpha^{share}}\right)^{\alpha^{share}} \left(\frac{\left[\alpha_{d,share}^{\sigma_{es,share}} (r_t^d)^{1-\sigma_{es,share}} + \left(1 - \alpha_{d,share}\right)^{\sigma_{es,share}} . (p_t^e)^{1-\sigma_{es,share}}\right]^{\frac{1}{1-\sigma_{es,share}}}}{1 - \alpha^{share}}\right)^{1-\alpha^{share}}$$

5. Sharing market clearing

$$Y_t^{share} = ES_t^{share}$$

Final good firm

The final good firm maximizes the following profit,

$$\max_{L,K} \pi_t = Y_t - (w_t L_t + r_t^k K_t)$$

$$Y_t = {K_t}^{\alpha^k} {L_t}^{1-\alpha^k}$$

6. Labor demand

$$w_t = (1 - \alpha^k) \frac{Y_t}{L_t}$$

7. Capital demand

$$r_t^k = \alpha^k \frac{Y_t}{K_t}$$

8. Final good market clearing

$$Y_t = X_t + I_t^k + I_t^d + p_t^e(E_t^{share} + E_t^{home})$$

9. Labor market clearing

$$1 = L_t + L_t^{share}$$

Households

The households maximizes their utility function under budget constraint. Their variable of decision are non-durable goods consumption, the investments in durable goods, durable goods, the utilization rate of durable goods and the consumption of shared energy services

$$\max_{X, I_t^d, ES_t^{share}, D, u} \sum_{t=0}^{\infty} \beta^t \frac{{C_t}^{1-\sigma} - 1}{1-\sigma}$$

$$s.t.\ (1+\tau^c)\left[X_t + I_t^d + p_t^{share} ES_t^{share}\right] + p_t^e E_t^{home} + I_t^k = w_t L_t + r_t^k K_t + r_t^d (1-u_t) D_t$$

Where the hybrid CD-CES speciation for the consumption bundle is,

$${C_t} = {X_t}^{\alpha_x} {ES_t}^{1-\alpha_x}$$

$$ES_t = \left[\alpha_{home} {ES_t^{home}}^{\frac{\sigma_{es}-1}{\sigma_{es}}} + (1-\alpha_{home}) {ES_t^{share}}^{\frac{\sigma_{es}-1}{\sigma_{es}}}\right]^{\frac{\sigma_{es}}{\sigma_{es}-1}}$$

$$ES_t^{home} = \left[\alpha_d (u_t D_t)^{\frac{\sigma_{home}-1}{\sigma_{home}}} + (1-\alpha_d) {E_t^{home}}^{\frac{\sigma_{home}-1}{\sigma_{home}}}\right]^{\frac{\sigma_{home}}{\sigma_{home}-1}}$$

$$D_{t+1} = (1-\delta_d) D_t + I_t^d$$

$$K_{t+1} = (1-\delta_k) K_t + I_t^k$$

8. Demand of non-durable goods

$$(1+\tau^c).\lambda_t = \underbrace{{C_t}^{-\sigma} \alpha_x \frac{C_t}{X_t}}_{UmX_t^i}$$

9. Demand of aggregate energy services

$$p_{es,t} = \frac{(1-\alpha_x)}{\alpha_x}\frac{X_t}{ES_t}$$

10. Aggregate price of energy services

$$p_{es,t} = \left[\alpha_{home}^{\sigma_{es}}.\left(p_t^{home}\right)^{1-\sigma_{es}} + (1-\alpha_{home})^{\sigma_{es}}.{p_t^{share}}^{1-\sigma_{es}}\right]^{\frac{1}{1-\sigma_{es}}}$$

11. Relative demand of home-produced durable services relative to shared services

$$\frac{p_t^{home}}{p_t^{share}} = \frac{\alpha_{home}}{(1-\alpha_{home})}\frac{ES_t^{share}}{ES_t^{home}}$$

12. Home-produced energy services price

$$p_t^{home} = \left[\alpha_d^{\sigma_{home}}.\left(\frac{uc_t^d}{u_t}\right)^{1-\sigma_{home}} + (1-\alpha_d)^{\sigma_{home}}.(p_t^e)^{1-\sigma_{home}}\right]^{\frac{1}{1-\sigma_{home}}}$$

13. Energy demand

$$\frac{p_t^e}{p_t^{home}} = (1-\alpha_d)\left(\frac{ES_t^{home}}{E_t}\right)^{\frac{1}{\sigma_{home}}}$$

14. Relative durable goods demand

$$uc_t^d\left(=\frac{UmD_{t+1}}{UmX_{t+1}}\right)$$

$$= \frac{(1-\alpha_x)}{\alpha_x}\frac{X_{t+1}}{ES_{t+1}}\alpha_{home}\left(\frac{ES_{t+1}}{ES_{t+1}^{home}}\right)^{\frac{1}{\sigma_{es}}}\alpha_d u_{t+1}\left(\frac{ES_{t+1}^{home}}{u_{t+1}D_{t+1}}\right)^{\frac{1}{\sigma_{home}}}$$

(would the user cost be $\frac{UmD_{t+1}}{UmX_{t+1}}$ or $\frac{Um(uD)_{t+1}}{UmX_{t+1}}$?)

15. Utilization rate decision

$$\frac{(1-\alpha_x)}{\alpha_x}\frac{X_t}{ES_t}\alpha_{home}\left(\frac{ES_t}{ES_t^{home}}\right)^{\frac{1}{\sigma_{es}}}\alpha_d\left(\frac{ES_t^{home}}{u_t D_t}\right)^{\frac{1}{\sigma_{home}}} - r_t^d = 0$$

16. No-arbitrage condition

$$\frac{UmD_t}{UmX_t} + (1-\delta_d) = r_t^d(1-u_t) + (1-\delta_d) \rightarrow r_t^d = \frac{uc_{t-1}^d}{1-u_t}$$

17. Durable goods Euler equation

$$\frac{UmX_t}{\beta} = UmD_{t+1} + UmX_{t+1}(1-\delta_d) + UmX_{t+1}r_{t+1}^d(1-u_{t+1})$$

$$\rightarrow uc_t^d \left(= \frac{UmD_{t+1}}{UmX_{t+1}}\right) = \frac{1}{\beta}\frac{UmX_t}{UmX_{t+1}} - \left(uc_t^d + 1 - \delta_d\right)$$

$$\rightarrow 2uc_t^d = \frac{1}{\beta}\frac{UmX_t}{UmX_{t+1}} - (1 - \delta_d)$$

However, the code does not work with that equation ($\boldsymbol{2uc_t^d = \frac{1}{\beta}\frac{UmX_t}{UmX_{t+1}} - (1 - \delta_d)}$), but works with the following equation that is the equation in the absence of rental market (or I have a mistake in my equations)

$$uc_t^d = \frac{1}{\beta}\frac{UmX_t}{UmX_{t+1}} - (1 - \delta_d)$$

18. Capital goods Euler equation

$$q_t^k = \beta \frac{UmX_{t+1}}{UmX_t}[r_{t+1}^k + (1 - \delta_k).q_{t+1}^k]$$