Graduate Macro Theory II: A Medium Scale DSGE Model

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1 Introduction

This set of notes formulates a "medium scale" DSGE model. This model incorporates essentially all of the twists to an RBC model that we have previously investigated. In particular, the model features capital, sticky prices and wages, habit formation in consumption, variable capital utilization, and adjustment costs to investment. Monetary policy is characterized by a Taylor rule, and there are stochastic shocks to the policy rule, neutral productivity, investment-specific productivity, and government spending.

This model is similar to the models in Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). The former is solely focused on accounting for the dynamics after a monetary policy shock, while the latter includes many shocks (many more shocks than I include in the model here). Like Smets and Wouters (2007), I estimate the model via Bayesian maximum likelihood, and then briefly examine the properties of the model in terms of moments and impulse responses.

2 Production

As in earlier models, we split production into two sectors: a competitive final goods sector that aggregates intermediate inputs, and a continuum of monopolistically competitive intermediate goods firms that produce output that is sold to the final good firm, which bundles it into a good available for households to consume. The only twist relative to what we did earlier is that output is produced using both capital and labor.

2.1 Final Good Firm

The final output good is a CES aggregate of a continuum of intermediates:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon_p - 1}{\epsilon_p}} \right)^{\frac{\epsilon_p}{\epsilon_p - 1}} \tag{1}$$

Here $\epsilon_p > 1$. Profit maximization by the final goods firm yields a downward-sloping demand curve for each intermediate:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t \tag{2}$$

This says that the relative demand for the j^{th} intermediate is a function of its relative price, with ϵ_p the price elasticity of demand. The price index (derived from the definition of nominal output as the sum of prices times quantities of intermediates) can be seen to be:

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon_p} dj\right)^{\frac{1}{1-\epsilon_p}} \tag{3}$$

2.2 Intermediate Goods Firms

There are a continuum of intermediate goods firms indexed by j. I normalize the mass of these firms to be 1. A typical intermediate producer produces output according to a constant returns to scale technology in labor and capital services, with a common productivity shock, A_t. Denote capital services by $\widehat{K}_t = u_t K_t$, where u_t is utilization, so that capital services is the product of utilization and physical capital. I assume that households make the capital accumulation and utilization decisions, and rent capital services to firms at nominal rental rate R_t^p. This seems perhaps a bit strange that households make utilization decisions, but it simplifies the analysis. Labor is paid nominal wage W_t^p. Firms take factor prices as given.

$$Y_t(j) = A_t \widehat{K}_t(j)^{\alpha} N_t(j)^{1-\alpha} \tag{4}$$

Firms are not free to update prices each period, but will choose inputs so as to minimize cost, given a price, subject to the constraint that it produce enough to meet demand. The cost-minimization problem is:

$$\min_{\widehat{K}_t(j), N_t(j)} \quad W_t^p N_t(j) + R_t^p \widehat{K}_t(j)$$

s.t.

$$A_t \widehat{K}_t(j)^{\alpha} N_t(j)^{1-\alpha} \geq \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$$

A Lagrangian is:

$$\mathcal{L} = -W_t^p N_t(j) - R_t^p \widehat{K}_t(j) + \varphi_t(j)\left(A_t \widehat{K}_t(j)^{\alpha} N_t(j)^{1-\alpha} - \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t\right)$$

The FOC are:

$$\frac{\partial \mathcal{L}}{\partial \widehat{K}_t(j)} = 0 \Leftrightarrow R_t^p = \varphi_t(j)\alpha A_t \widehat{K}_t(j)^{\alpha-1} N_t(j)^{1-\alpha}$$

$$\frac{\partial \mathcal{L}}{\partial N_t(j)} = 0 \Leftrightarrow W_t^p = \varphi_t(j)(1-\alpha) A_t \widehat{K}_t(j)^{\alpha} N_t(j)^{-\alpha}$$

We can combine these two to eliminate the multiplier, and get:

$$\frac{W_t^p}{R_t^p} = \frac{1-\alpha}{\alpha}\frac{\widehat{K}_t(j)}{N_t(j)}$$

Since firms face the same factor prices, it is obvious from above that they will hire capital and labor in the same ratio, which will in turn be equal to the aggregate ratio. We can also write the ratio of factor prices in terms of their real equivalents, with $w_t \equiv \frac{W_t^p}{P_t}$ and $R_t \equiv \frac{R_t^p}{P_t}$:

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha}\frac{\widehat{K}_t}{N_t} \tag{5}$$

Since firms hire capital and labor in the same ratio and face the same factor prices, we then can see that they have the same marginal cost, which I'll now write in real terms as $mc_t = \frac{\varphi_t}{P_t}$. This is implicitly defined by

$$w_t = mc_t(1-\alpha)A_t\left(\frac{\widehat{K}_t}{N_t}\right)^{\alpha} \tag{6}$$

This condition just has the interpretation that real marginal cost is the ratio of the real wage to the marginal product of labor. We could also have defined this in terms of the rental rate and the marginal product of capital:

$$R_t = mc_t\alpha A_t\left(\frac{\widehat{K}_t}{N_t}\right)^{\alpha-1} \tag{7}$$

This means that we can write real flow profit for the jth firm as:

$$\frac{\Pi_t^p(j)}{P_t} = \frac{P_t(j)}{P_t}Y_t(j) - mc_t(1-\alpha)A_t\widehat{K}_t(j)^{\alpha}N_t(j)^{1-\alpha} - mc_t(1-\alpha)A_t\widehat{K}_t(j)^{\alpha}N_t(j)^{1-\alpha}$$

Or:

$$\frac{\Pi_t^p(j)}{P_t} = \frac{P_t(j)}{P_t}Y_t(j) - mc_tY_t(j)$$

Plugging in the demand function, this is just:

$$\frac{\Pi_t^p(j)}{P_t} = P_t(j)^{1-\epsilon_p}P_t^{\epsilon_p-1}Y_t - mc_tP_t(j)^{-\epsilon_p}P_t^{\epsilon_p}Y_t$$

Firms are not freely able to adjust price each period. In particular, each period there is a fixed probability of $1-\phi_p$ that a firm can adjust its price. This means that the probability a firm will be stuck with a price one period is ϕ_p, for two periods is ϕ_p^2, and so on. Consider the pricing problem of a firm given the opportunity to adjust its price in a given period. Since there is a chance that the firm will get stuck with its price for multiple periods, the pricing problem becomes dynamic. Firms will discount profits s periods into the future by $\widetilde{M}_{t+s}\phi_p^s$, where $\widetilde{M}_{t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}$ is the stochastic discount factor, where λ_t is the marginal value of an extra unit of income. As a slight twist to what I had done before, I allow the possibility that firms can index their prices to lagged inflation at $\zeta_p \in (0,1)$. $\zeta_p = 0$ means no indexation, while $\zeta_p = 1$ means full indexation. Values in between are permitted and imply partial indexation. The price that a firm can charge in period $t+s$ if it is still charging a price set in period t is $\Pi_{t-1,t+s-1}^{\zeta_p} P_t(j)$, where $\Pi_{t-1,t+s-1}$ is cumulative gross inflation between $t-1$ and $t+s-1$, so $\frac{P_{t+s-1}}{P_{t-1}}$. When $s=0$, this is just 1. When $s=1$, it is $(1+\pi_t)$ where π_t is net inflation between t and $t-1$. When $s=2$, it is $(1+\pi_{t+1})(1+\pi_t) = \frac{P_{t+1}}{P_{t-1}}$, and so on.

The dynamic problem can be written:

$$\max_{P_t(j)} \quad E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{\lambda_{t+s}}{\lambda_t} \left(\frac{\Pi_{t-1,t+s-1}^{\zeta_p} P_t(j)}{P_{t+s}} \left(\frac{\Pi_{t-1,t+s-1}^{\zeta_p} P_t(j)}{P_{t+s}} \right)^{-\epsilon_p} Y_{t+s} - mc_{t+s} \left(\frac{\Pi_{t-1,t+s-1}^{\zeta_p} P_t(j)}{P_{t+s}} \right)^{-\epsilon_p} Y_{t+s} \right)$$

Multiplying out, we get:

$$\max_{P_t(j)} \quad E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{\lambda_{t+s}}{\lambda_t} \left(\Pi_{t-1,t+s-1}^{\zeta_p(1-\epsilon_p)} P_t(j)^{1-\epsilon_p} P_{t+s}^{\epsilon_p - 1} Y_{t+s} - mc_{t+s} \Pi_{t-1,t+s-1}^{-\zeta_p \epsilon_p} P_t(j)^{-\epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} \right)$$

The first order condition can be written:

$$(1-\epsilon_p) P_t(j)^{-\epsilon_p} E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{\zeta_p(1-\epsilon_p)} P_{t+s}^{\epsilon_p - 1} Y_{t+s} + \epsilon_p P_t(j)^{-\epsilon_p - 1} E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{-\zeta_p \epsilon_p} mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s} = 0$$

Simplifying:

$$P_t(j) = \frac{\epsilon_p}{\epsilon_p - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{-\zeta_p \epsilon_p} mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s}}{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \lambda_{t+s} \Pi_{t-1,t+s-1}^{\zeta_p(1-\epsilon_p)} P_{t+s}^{\epsilon_p - 1} Y_{t+s}}$$

First, note that since nothing on the right hand side depends on j, all updating firms will update to the same reset price, call it $P_t^{\#}$. We can write the expression more compactly as:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{X_{1,t}}{X_{2,t}} \tag{8}$$

Here:

$$X_{1,t} = \lambda_t mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta (1+\pi_t)^{-\zeta_p \epsilon_p} E_t X_{1,t+1} \tag{9}$$

$$X_{2,t} = \lambda_t P_t^{\epsilon_p - 1} Y_t + \phi_p \beta (1+\pi_t)^{\zeta_p (1-\epsilon_p)} E_t X_{2,t+1} \tag{10}$$

If $\phi_p = 0$, then the right hand side would reduce to $mc_t P_t = \varphi_t$. In this case, the optimal price would be a fixed markup, $\frac{\epsilon_p}{\epsilon_p - 1}$, over nominal marginal cost, φ_t.

3 Households

Households choose consumption, bond-holdings, wages, labor supply, capital accumulation, and capital utilization. Households supply differentiated labor input and are index by $l \in (0,1)$. Household labor input is "packed" into a bundled labor input that is sold to firms. Since household labor is imperfectly substitutable, there is a downward-sloping demand for each variety of labor, which gives the household some wage-setting power.

I first consider the problem of the labor packer, which generates a downward-sloping demand for labor and implies a wage index. Then I consider the problem of the household.

3.1 Labor Packer

Total labor input is equal to:

$$N_t = \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{11}$$

Here $\epsilon_w > 1$, and l indexes the differentiated labor inputs, which populate the unit interval. The profit maximization problem of the competitive labor packer is:

$$\max_{N_t(l)} \quad W_t^p \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - \int_0^1 W_t(l) N_t(l) dl$$

The first order condition for the choice of labor of variety l is:

$$W_t^p \frac{\epsilon_w}{\epsilon_w - 1} \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1} - 1} \frac{\epsilon_w - 1}{\epsilon_w} N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w} - 1} = W_t(l)$$

This can be simplified somewhat:

$$N_t(l)^{-\frac{1}{\epsilon_w}} \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{1}{\epsilon_w - 1}} = \frac{W_t(l)}{W_t^p}$$

Or:

$$N_t(l)\left(\int_0^1 N_t(l)^{\frac{\epsilon_w-1}{\epsilon_w}}dl\right)^{-\frac{\epsilon_w}{\epsilon_w-1}} = \left(\frac{W_t(l)}{W_t^p}\right)^{-\epsilon_w}$$

Or:

$$N_t(l) = \left(\frac{W_t(l)}{W_t^p}\right)^{-\epsilon_w} N_t \tag{12}$$

In a way exactly analogous to intermediate goods, the relative demand for labor of type l is a function of its relative wage, with elasticity ϵ_w. We can derive an aggregate wage index in a similar way to above, by defining:

$$W_t^p N_t = \int_0^1 W_t(l)N_t(l)dl = \int_0^1 W_t(l)^{1-\epsilon_w} W_t^{\epsilon_w} N_t dl$$

Or:

$$(W_t^p)^{1-\epsilon_w} = \int_0^1 W_t(l)^{1-\epsilon_w} dl$$

So:

$$W_t^p = \left(\int_0^1 W_t(l)^{1-\epsilon_w} dl\right)^{\frac{1}{1-\epsilon_w}} \tag{13}$$

3.2 Household Problem

Households are heterogenous and are indexed by $l \in (0,1)$, supplying differentiated labor input to the labor packer above. I'm going to assume that preferences are additively separable in consumption and labor, which turns out to be somewhat important. If wages are subject to frictions like the Calvo (1983) pricing friction, households will charge different wages, meaning they will work different hours, meaning they will have different incomes and therefore different consumption and saving. Erceg, Henderson, and Levin (2000, *JME*) show that if there exist state contingent claims that insure households against idiosyncratic wage risk, and if preferences are separable in consumption and leisure, households will be identical in their choice of consumption, capital accumulation, capital utilization, and bond-holdings, and will only differ in the wage they charge and labor supply. As such, in the notation below, I will suppress dependence on l for everything but wages and labor input.

The twists relative to earlier are the following. First, I allow for internal habit formation in consumption. I go ahead and assume that utility from consumption is logarithmic. Second, households can choose capital utilization, and end up paying a resource cost for that utilization. This is somewhat different than what I earlier showed, where the cost of utilization was faster depreciation. Third, I assume that there are adjustment costs to the flow rate of investment as in Christiano, Eichenbaum, and Evans (2005). These are a little different than what we had used before, and the cost shows up in the capital accumulation equation, not the resource constraint.

Third, I allow for an investment shock, Z_t, to the efficiency of transforming investment into new capital. The capital accumulation equation is:

$$K_{t+1} = Z_t \left(1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right) I_t + (1-\delta)K_t \tag{14}$$

What this adjustment cost specification does is that one unit of investment produces fewer units of new capital the more the gross growth rate of investment differs from one, with $\tau \geq 0$ governing the magnitude of the cost. The cost for utilization is quadratic in utilization relative to its normalized steady state value one, with the cost governed by the parameters χ_1 and χ_2, both ≥ 0, and proportional to the capital stock divided by the investment shock, Z_t. The flow budget constraint (written in real terms) is:

$$C_t + I_t + \frac{B_{t+1}}{P_t} \leq R_t u_t K_t + \frac{W_t(l)}{P_t} N_t(l) - \left(\chi_1(u_t - 1) + \frac{\chi_2}{2}(u_t-1)^2\right)\frac{K_t}{Z_t} + (1+i_{t-1})\frac{B_t}{P_t} + \frac{\Pi_t}{P_t} + T_t \tag{15}$$

A Lagrangian for the household with two constraints is:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty}\left(\ln(C_t - bC_{t-1}) - \psi\frac{N_t(l)^{1+\eta}}{1+\eta} + \mu_t\left(Z_t\left(1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right) I_t + (1-\delta)K_t - K_{t+1}\right)\right.$$
$$\left. + \lambda_t\left(R_t u_t K_t + \frac{W_t(l)}{P_t}N_t(l) - \left(\chi_1(u_t-1) + \frac{\chi_2}{2}(u_t-1)^2\right)\frac{K_t}{Z_t} + (1+i_{t-1})\frac{B_t}{P_t} + \frac{\Pi_t}{P_t} + T_t - C_t - I_t - \frac{B_{t+1}}{P_t}\right)\right)$$

The FOC for non-labor choices are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \lambda_t = \frac{1}{C_t - bC_{t-1}} - \beta b E_t \frac{1}{C_{t+1} - bC_t} \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial u_t} = 0 \Leftrightarrow R_t = \frac{1}{Z_t}(\chi_1 + \chi_2(u_t - 1)) \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1+i_t)\frac{P_t}{P_{t+1}} \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Leftrightarrow \lambda_t = \mu_t Z_t\left(1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \tau\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right) + \beta E_t \mu_{t+1} Z_{t+1}\tau\left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2 \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \mu_t = \beta E_t\left(\lambda_{t+1}\left(R_{t+1}u_{t+1} - \frac{1}{Z_{t+1}}\left(\chi_1(u_{t+1}-1) + \frac{\chi_2}{2}(u_{t+1}-1)^2\right)\right) + \mu_{t+1}(1-\delta)\right) \tag{20}$$

Now let's think about wage-setting. As with pricing, households are not freely able to adjust their nominal wages each period. Each period there is a $1-\phi_w$ probability that they can adjust their wage. If they cannot adjust their wage, they can index to lagged inflation at $\zeta_w \in (0,1)$. Hence, in period $t+s$ a household that last adjusted its wage in period t has nominal wage $\Pi_{t,t+s-1}^{\zeta_w} W_t(l)$, where again $\Pi_{t,t+s-1}$ is cumulative gross price inflation between period $t-1$ and period $t+s-1$. When $s = 0$, this is 1. When $s = 1$, this is just $(1+\pi_t)$, and so on. For a non-updated household, the nominal wage it will have in period $t+s$ is given by:

$$W_{t+s}(l) = W_t(l)\Pi_{t-1,t+s-1}^{\zeta_w}$$

For example, suppose $s=0$. Then $\Pi_{t-1,t+s-1}=1$. When $s=1$, we have $\Pi_{t-1,t+s-1}=(1+\pi_t)$. This means that the household can adjust its wage in period $t+1$ by $(1+\pi_t)^{\zeta_w}$ relative to its nominal wage in period t, and so on. We also want to write this in real terms. So divide both sides by P_{t+s}:

$$\frac{W_{t+s}(l)}{P_{t+s}} = \frac{W_t(l)}{P_{t+s}}\Pi_{t-1,t+s-1}^{\zeta_w}$$

Multiply and divide the right hand side by P_t, and define $\Pi_{t,t+s}$ as gross cumulative inflation between t and $t+s$, or $\Pi_{t,t+s}=\frac{P_{t+s}}{P_t}$. Then we have:

$$w_{t+s}(l) = w_t(l)\Pi_{t,t+s}^{-1}\Pi_{t-1,t+s-1}^{\zeta_w}$$

Because of the probability that they will be stuck with a given wage going into the future, the problem of a household with the ability to adjust its wage becomes dynamic, and it discounts future (dis)utility flows from labor by $(\beta\phi_w)^s$, where $1-\phi_w$ is the probability that a firm can adjust its wage in any period. Eliminating labor as a choice by plugging in labor demand, we can re-produce the parts of the Lagrangian related to wage-setting as:

$$\mathcal{L} = E_t\sum_{s=0}^{\infty}(\beta\phi_w)^s\left(-\psi\frac{\left(\frac{w_t(l)\Pi_{t,t+s}^{-1}\Pi_{t-1,t+s-1}^{\zeta_w}}{w_{t+s}}\right)^{-\epsilon_w(1+\eta)}N_{t+s}^{1+\eta}}{1+\eta}+\dots\right.$$
$$\left.\dots+\lambda_{t+s}\left(w_t(l)\Pi_{t,t+s}^{-1}\Pi_{t-1,t+s-1}^{\zeta_w}\left(\frac{w_t(l)\Pi_{t,t+s}^{-1}\Pi_{t-1,t+s-1}^{\zeta_w}}{w_{t+s}}\right)^{-\epsilon_w}N_{t+s}\right)\right)$$

The first order condition is:

$$\frac{\partial\mathcal{L}}{\partial w_t(l)} = 0 \Leftrightarrow \epsilon_w w_t(l)^{-\epsilon_w(1+\eta)-1}E_t\sum_{s=0}^{\infty}(\beta\phi_w)^s\,\psi w_{t+s}^{\epsilon_w(1+\eta)}\Pi_{t,t+s}^{\epsilon_w(1+\eta)}\Pi_{t-1,t+s-1}^{-\zeta_w\epsilon_w(1+\eta)}N_{t+s}^{1+\eta}\dots$$
$$\dots+(1-\epsilon_w)w_t(l)\sum_{s=0}^{\infty}(\beta\phi_w)^s\,\lambda_{t+s}\Pi_{t,t+s}^{\epsilon_w-1}\Pi_{t-1,t+s-1}^{\zeta_w(1-\epsilon_w)}w_{t+s}^{\epsilon_w}N_{t+s}=0$$

Noting that nothing on the right hand side depends on l, and calling $w_t^{\#}$ the common reset price, this can be written:

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \, \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} \Pi_{t-1,t+s-1}^{-\zeta_w\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \, \lambda_{t+s} \Pi_{t,t+s}^{\epsilon_w-1} \Pi_{t-1,t+s-1}^{\zeta_w(1-\epsilon_w)} w_{t+s}^{\epsilon_w} N_{t+s}} \tag{21}$$

This can be written recursively as:

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{f_{1,t}}{f_{2,t}} \tag{22}$$

Where:

$$f_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w \beta (1+\pi_t)^{-\zeta_w\epsilon_w(1+\eta)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\eta)} f_{1,t+1} \tag{23}$$

$$f_{2,t} = \lambda_t w_t^{\epsilon_w} N_t + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} f_{2,t+1} \tag{24}$$

4 Policy and Exogenous Processes

I assume there exists a government that each period consumes a share of output. The share of output it consumes is ω_t^g and is stochastic:

$$G_t = \omega_t^g Y_t \tag{25}$$

$$\omega_t^g = (1-\rho_g)\omega^g + \rho_g \omega_{t-1}^g + \varepsilon_{g,t} \tag{26}$$

I assume that the government balances its budget each period with lump sum taxes. Since there are no distortionary taxes, the assumption of budget balance each period is innocuous because the mix between bond and tax finance is indeterminate:

$$T_t = G_t \tag{27}$$

Monetary policy follows an inertial Taylor rule that responds to inflation and output growth (which is easier to measure than the gap, and often turns out to have desirable normative properties):

$$i_t = (1-\rho_i)i + \rho_i i_{t-1} + (1-\rho_i)\left(\phi_\pi(\pi_t - \pi) + \phi_y(\ln Y_t - \ln Y_{t-1})\right) + \varepsilon_{i,t} \tag{28}$$

The exogenous processes for A_t and Z_t both follow mean 0 AR(1)s in the log:

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \tag{29}$$

$$\ln Z_t = \rho_z \ln Z_{t-1} + \varepsilon_{z,t} \tag{30}$$

5 Aggregation

Start with aggregate production. We have:

$$A_t \widehat{K}_t(j)^\alpha N_t(j)^{1-\alpha} = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$$

We can write this in terms of the capital-labor ratio, noting that all firms will hire capital and labor in the same ratio:

$$A_t \left(\frac{\widehat{K}_t}{N_t}\right)^\alpha N_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$$

Integrate over j:

$$A_t \left(\frac{\widehat{K}_t}{N_t}\right)^\alpha \int_0^1 N_t(j)dj = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj$$

Now, note that market-clearing for labor requires that total labor supply by the labor packer must equal the sum of demand from firms, or $\int_0^1 N_t(j)dj = N_t$. Define $v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj$. Then we have:

$$Y_t = \frac{A_t \widehat{K}_t^\alpha N_t^{1-\alpha}}{v_t^p} \tag{31}$$

Using the properties of Calvo pricing, we can write the price dispersion term as:

$$v_t^p = (1-\phi_p) P_t^{\#,-\epsilon_p} P_t^{\epsilon_p} + \int_{1-\phi_w}^1 (1+\pi_{t-1})^{-\zeta_p \epsilon_p} P_{t-1}(j)^{-\epsilon_p} P_t^{\epsilon_p} dj$$

The right hand side follows because non-updating firms can index their price by $(1+\pi_{t-1})^{\zeta_p}$. We can simplify this as follows:

$$v_t^p = (1-\phi_p) P_t^{\#,-\epsilon_p} P_t^{\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p \epsilon_p} \int_{1-\phi_w}^1 P_{t-1}(j)^{-\epsilon_p} P_{t-1}^{\epsilon_p} P_{t-1}^{-\epsilon_p} P_t^{\epsilon_p} dj$$

Or:

$$v_t^p = (1-\phi_p) P_t^{\#,-\epsilon_p} P_t^{\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p \epsilon_p} (1+\pi_t)^{\epsilon_p} \int_{1-\phi_w}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon_p} dj$$

By the properties of Calvo pricing, the right hand side becomes:

$$v_t^p = (1-\phi_p) P_t^{\#,-\epsilon_p} P_t^{\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p \epsilon_p} (1+\pi_t)^{\epsilon_p} \phi_p v_{t-1}^p$$

The first part can be written in terms of inflation and reset price inflation $1+\pi_t^\# = \frac{P_t^\#}{P_{t-1}}$:

$$v_t^p = (1+\pi_t)^{\epsilon_p} \left((1-\phi_p)(1+\pi_t^\#)^{-\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p \epsilon_p} \phi_p v_{t-1}^p \right) \tag{32}$$

We can describe the evolution of aggregate prices as:

$$P_t^{1-\epsilon_p} = (1-\phi_p)P_t^{\#,1-\epsilon_p} + \int_{1-\phi_p}^{1} (1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)} P_{t-1}(j)^{1-\epsilon_p} dj$$

Or, by properties of Calvo pricing:

$$P_t^{1-\epsilon_p} = (1-\phi_p)P_t^{\#,1-\epsilon_p} + \phi_p(1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)} P_{t-1}^{1-\epsilon_p}$$

To write this in terms of inflation rates, divide both sides by $P_{t-1}^{1-\epsilon_p}$:

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi_p(1+\pi_{t-1})^{\zeta_p(1-\epsilon_p)} \tag{33}$$

To write the optimal pricing condition in terms of inflation rates, define $x_{1,t} \equiv \frac{X_{1,t}}{P_t^{\epsilon_p}}$ and $x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon_p - 1}}$. We get:

$$1+\pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1}(1+\pi_t)\frac{x_{1,t}}{x_{2,t}} \tag{34}$$

$$x_{1,t} = \lambda_t mc_t Y_t + \phi_p \beta (1+\pi_t)^{-\zeta_p \epsilon_p} E_t (1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \tag{35}$$

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta (1+\pi_t)^{\zeta_p(1-\epsilon_p)} E_t (1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1} \tag{36}$$

We can break up the aggregate nominal wage index by using properties of Calvo pricing:

$$(W_t^p)^{1-\epsilon_w} = (1-\phi_w)W_t^{\#,1-\epsilon_w} + \int_{1-\phi_w}^{1} (1+\pi_{t-1})^{\zeta_p(1-\epsilon_w)} W_{t-1}(l)^{1-\epsilon_w} dl$$

Now, to write this in real terms divide both sides by $P_t^{1-\epsilon_w}$:

$$w_t^{1-\epsilon_w} = (1-\phi_w)w_t^{\#,1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} \phi_w P_t^{\epsilon_w - 1} W_{t-1}^{1-\epsilon_w}$$

Or:

$$w_t^{1-\epsilon_w} = (1-\phi_w)w_t^{\#,1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w(1-\epsilon_w)} (1+\pi_t)^{\epsilon_w - 1} \phi_w w_{t-1}^{1-\epsilon_w} \tag{37}$$

6 Full set of Equilibrium Conditions

$$\lambda_t = \frac{1}{C_t - bC_{t-1}} - \beta b E_t \frac{1}{C_{t+1} - bC_t} \tag{38}$$

$$R_t = \frac{1}{Z_t}\left(\chi_1 + \chi_2(u_t - 1)\right) \tag{39}$$

$$\lambda_t = \beta E_t \lambda_{t+1}(1+i_t)(1+\pi_{t+1})^{-1} \tag{40}$$

$$\lambda_t = \mu_t Z_t \left(1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \tau\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right) + \beta E_t \mu_{t+1} Z_{t+1} \tau \left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2 \tag{41}$$

$$\mu_t = \beta E_t \left(\lambda_{t+1} \left(R_{t+1} u_{t+1} - \frac{1}{Z_{t+1}} \left(\chi_1 (u_{t+1} - 1) + \frac{\chi_2}{2} (u_{t+1} - 1)^2 \right) \right) + \mu_{t+1}(1-\delta) \right) \tag{42}$$

$$w_t^{\#, 1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{f_{1,t}}{f_{2,t}} \tag{43}$$

$$f_{1,t} = \psi w_t^{\epsilon_w (1+\eta)} N_t^{1+\eta} + \phi_w \beta (1+\pi_t)^{-\zeta_w \epsilon_w (1+\eta)} E_t (1+\pi_{t+1})^{\epsilon_w (1+\eta)} f_{1,t+1} \tag{44}$$

$$f_{2,t} = \lambda_t w_t^{\epsilon_w} N_t + \phi_w \beta (1+\pi_t)^{\zeta_w (1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} f_{2,t+1} \tag{45}$$

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#, 1-\epsilon_w} + (1+\pi_{t-1})^{\zeta_w (1-\epsilon_w)} (1+\pi_t)^{\epsilon_w - 1} \phi_w w_{t-1}^{1-\epsilon_w} \tag{46}$$

$$Y_t = \frac{A_t \widehat{K}_t^{\alpha} N_t^{1-\alpha}}{v_t^p} \tag{47}$$

$$v_t^p = (1+\pi_t)^{\epsilon_p} \left((1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p} + (1+\pi_{t-1})^{-\zeta_p \epsilon_p} \phi_p v_{t-1}^p \right) \tag{48}$$

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi_p (1+\pi_{t-1})^{\zeta_p (1-\epsilon_p)} \tag{49}$$

$$1+\pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} (1+\pi_t) \frac{x_{1,t}}{x_{2,t}} \tag{50}$$

$$x_{1,t} = \lambda_t mc_t Y_t + \phi_p \beta (1+\pi_t)^{-\zeta_p \epsilon_p} E_t (1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \tag{51}$$

$$x_{2,t} = \lambda_t Y_t + \phi_p \beta (1+\pi_t)^{\zeta_p (1-\epsilon_p)} E_t (1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1} \tag{52}$$

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha} \frac{\widehat{K}_t}{N_t} \tag{53}$$

$$w_t = mc_t (1-\alpha) A_t \left(\frac{\widehat{K}_t}{N_t} \right)^{\alpha} \tag{54}$$

$$i_t = (1-\rho_i) i + \rho_i i_{t-1} + (1-\rho_i) \left(\phi_\pi (\pi_t - \pi) + \phi_y (\ln Y_t - \ln Y_{t-1}) \right) + \varepsilon_{i,t} \tag{55}$$

$$Y_t = C_t + I_t + G_t + \left(\chi_1 (u_t - 1) + \frac{\chi_2}{2} (u_t - 1)^2 \right) \frac{K_t}{Z_t} \tag{56}$$

$$K_{t+1} = Z_t \left(1 - \frac{\tau}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right) I_t + (1-\delta) K_t \tag{57}$$

$$\widehat{K}_t = u_t K_t \tag{58}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \tag{59}$$

$$\ln Z_t = \rho_z \ln Z_{t-1} + \varepsilon_{z,t} \tag{60}$$

$$G_t = \omega_t^g Y_t \tag{61}$$

$$\omega_t^g = (1-\rho_g) \omega^g + \rho_g \omega_{t-1}^g + \varepsilon_{g,t} \tag{62}$$

$$q_t = \frac{\mu_t}{\lambda_t} \tag{63}$$

Here I have defined one more variable as Hayashi's q, which is the ratio of the Lagrange multiplier

on the capital accumulation equation to the Lagrange multiplier on the household budget constraint. In total, this is 26 equations in 26 variables:
$\left\{\lambda_t, C_t, R_t, Z_t, u_t, i_t, \pi_t, \mu_t, I_t, w_t^{\#}, f_{1,t}, f_{2,t}, w_t, Y_t, A_t, N_t, v_t^p, \pi_t^{\#}, x_{1,t}, x_{2,t}, mc_t, K_t, \widehat{K}_t, G_t, \omega_t^g, q_t\right\}$.

It is helpful to re-write the expression for the reset wage. The reason why is that the exponents in the expression can get very large, which can lead to numerical problems. For convenience, the conditions relating to wage-setting are reproduced here:

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1}\frac{f_{1,t}}{f_{2,t}}$$

$$f_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w\beta(1+\pi_t)^{-\zeta_w\epsilon_w(1+\eta)} E_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)} f_{1,t+1}$$

$$f_{2,t} = \lambda_t w_t^{\epsilon_w} N_t + \phi_w\beta(1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t(1+\pi_{t+1})^{\epsilon_w - 1} f_{2,t+1}$$

The exponent $1+\epsilon_w\eta$ could be very large for reasonable parameterizations of the model. Let's try to re-write this where on the right hand side we have ratios of the actual wage to the reset wage. Divide both sides by $w_t^{\#,\epsilon_w(1+\eta)}$:

$$w_t^{\#,1-\epsilon_w} = \frac{\epsilon_w}{\epsilon_w - 1}\frac{f_{1,t}/w_t^{\#,\epsilon_w(1+\eta)}}{f_{2,t}}$$

Define $\widehat{f}_{1,t} \equiv \frac{f_{1,t}}{w_t^{\#,\epsilon_w(1+\eta)}}$. We have:

$$\widehat{f}_{1,t} = \psi\left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w\beta(1+\pi_t)^{-\zeta_w\epsilon_w(1+\eta)} E_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)} f_{1,t+1} w_t^{\#,-\epsilon_w(1+\eta)}$$

Multiply and divide the second term by $w_{t+1}^{\#,\epsilon_w(1+\eta)}$ and simplify:

$$\widehat{f}_{1,t} = \psi\left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w\beta(1+\pi_t)^{-\zeta_w\epsilon_w(1+\eta)} E_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)} \frac{f_{1,t+1}}{w_{t+1}^{\#,\epsilon_w(1+\eta)}} w_{t+1}^{\#,\epsilon_w(1+\eta)} w_t^{\#,-\epsilon_w(1+\eta)}$$

$$\widehat{f}_{1,t} = \psi\left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w\beta(1+\pi_t)^{-\zeta_w\epsilon_w(1+\eta)} E_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} \widehat{f}_{1,t+1}$$

Now, let's multiply both sides by $w_t^{\#,\epsilon_w}$. This yields:

$$w_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1}\frac{\widehat{f}_{1,t} w_t^{\#,\epsilon_w}}{f_{2,t}}$$

Now, define $\widehat{f}_{2,t} \equiv \frac{f_{2,t}}{w_t^{\#,\epsilon_w}}$. We have:

$$\widehat{f}_{2,t} = \lambda_t\left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_t + \phi_w\beta(1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t(1+\pi_{t+1})^{\epsilon_w - 1} f_{2,t+1} w_t^{\#,-\epsilon_w}$$

Multiply and divide the right hand side by $w_{t+1}^{\#,\epsilon_w}$ and simplify:

$$\widehat{f}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_t + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \frac{f_{2,t+1}}{w_{t+1}^{\#,\epsilon_w}} w_{t+1}^{\#,\epsilon_w} w_t^{\#,-\epsilon_w}$$
$$\widehat{f}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_t + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w} \widehat{f}_{2,t+1}$$

So the three wage-setting conditions are now:

$$w_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\widehat{f}_{1,t}}{\widehat{f}_{2,t}} \tag{64}$$

$$\widehat{f}_{1,t} = \psi \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \phi_w \beta (1+\pi_t)^{-\zeta_w \epsilon_w (1+\eta)} E_t (1+\pi_{t+1})^{\epsilon_w(1+\eta)} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} \widehat{f}_{1,t+1} \tag{65}$$

$$\widehat{f}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} N_t + \phi_w \beta (1+\pi_t)^{\zeta_w(1-\epsilon_w)} E_t (1+\pi_{t+1})^{\epsilon_w - 1} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w} \widehat{f}_{2,t+1} \tag{66}$$

7 Estimation

Rather than (somewhat arbitrarily) choosing parameters for the model, I instead estimate the parameters using Bayesian maximum likelihood. This is relatively straightforward to do in Dynare, though it's a bit of a black box and often times estimation fails.

To estimate the model, you have to have observed series from the data on some of the variables. You can use as many series as you have shocks. In the model I wrote down, there are four shocks, so I can use up to four data series to estimate the model. There is no obvious way to pick which data to use, but as a general guide you want to use series that are going to be sensitive to parameter values (e.g. to help you identify the parameters), and you don't want the data series you put in to be too closely related to one another (too highly collinear, meaning there isn't much new information in the second series that isn't already in the first).

The data series I use to estimate the model are the growth rates of output and investment, and the levels of the interest rate and inflation. I measure output using standard NIPA real GDP. Investment is defined as the sum of durable consumption expenditure and private non-residential fixed investment.[1] The interest rate is the 3 month T-Bill secondary market rate, converted from a monthly to quarterly frequency by averaging across months. The inflation series is the growth

[1]To create this series, I first create real series of durable consumption and non-residential fixed investment using NIPA quantities and own-price deflators. To combine into one series, I define the growth rate of real investment expenditures as the weighted sum of the real growth rates of the individual components, where the weights are one period lagged *nominal* ratios of each component relative to the total. In other words, the growth rate of real investment is: $\Delta \ln I_t = \left(\frac{D_{t-1}^p}{D_{t-1}^p + FI_{t-1}^p}\right) \Delta \ln D_t + \left(\frac{FI_{t-1}^p}{D_{t-1}^p + FI_{t-1}^p}\right) \Delta \ln FI_t$, where FI is fixed investment, D is durable consumption, and superscript p denotes a nominal value, whereas the absence of a superscript p denotes a real value.

rate of the GDP implicit price deflator. Because the model features no trend growth, and because I assume zero trend inflation, I demean the growth rates of output and investment and the inflation rate from the actual data. I leave the interest rate series in levels, though it's important to "de-annualize" it so that the interest rate measured in the data corresponds to the interest rate in the model (interest rates are always quoted at annualized rates, whereas in the model it's a quarterly rate).

You'll note that these series are basically scale-free, with the exception of the interest rate, whose mean is defined by $\frac{1}{\beta} - 1$. As a practical matter, I don't want to estimate parameters that govern long run scales, but rather parameters which govern cycles. So I fix several parameters prior to estimation. I set $\beta = 0.99$, $\psi = 2$, and $\epsilon_w = \epsilon_p = 10$. I also fix the average government share of output at $\omega^g = 0.2$. I fix $\alpha = 1/3$. As noted above, I assume zero trend inflation, so $\pi = 0$. So as to normalize steady state capital utilization to 1, I need to set $\chi_1 = \frac{1}{\beta} - (1 - \delta)$. It turns out that χ_2 is difficult to estimate, so most people follow Christiano, Eichenbaum, and Evans (2005) in setting it to a low value. I assume $\chi_2 = 0.01$.

The rest of the parameters are estimating using Bayesian maximum likelihood. Doing this requires specifying prior distributions. I'm not going to go into much detail on how to do this. It's typical for parameters restricted to be between 0 and 1 to use a beta distribution, for shock standard deviations to use an inverse gamma distribution, and for other parameters to use a normal distribution. The prior means and standard errors are somewhat arbitrary but are chosen to fit in line with what other folks have found.

Here is the Dynare code that I use to estimate the model:

```
% medium scale model graduate macro

var lam C R Z u int infl mu I wsharp f1 f2 w Y A N vp pisharp x1 x2 mc K
Khat G omegag q dY dC dI dN;
varexo ea ez eg ei;

parameters psi beta phip phiw alpha eta b chi1 chi2 tau Δ epsw epsp
zetap zetaw rhoi phipi phiy rhog rhoa rhoz seg sea sez sei omega pistar;

load parameter_medium_scale;
set_param_value('psi',psi);
set_param_value('phip',phip);
set_param_value('phiw',phiw);
set_param_value('eta',eta);
set_param_value('beta',beta);
set_param_value('alpha',alpha);
set_param_value('b',b);
set_param_value('chi1',chi1);
set_param_value('chi2',chi2);
set_param_value('tau',tau);
set_param_value('Δ',Δ);
```

```
set_param_value('epsw',epsw);
set_param_value('epsp',epsp);
set_param_value('zetap',zetap);
set_param_value('zetaw',zetaw);
set_param_value('rhoi',rhoi);
set_param_value('phipi',phipi);
set_param_value('phiy',phiy);
set_param_value('rhog',rhog);
set_param_value('rhoa',rhoa);
set_param_value('rhoz',rhoz);
set_param_value('seg',seg);
set_param_value('sea',sea);
set_param_value('sez',sez);
set_param_value('sei',sei);
set_param_value('omega',omega);
set_param_value('pistar',pistar);

model;

% (1) marginal utility
exp(lam) = (exp(C) - b*exp(C(-1)))^(-1) - beta*b*(exp(C(+1)) -
b*exp(C))^(-1);

% (2) FOC on utilization
exp(R) = exp(Z)^(-1)*(chi1 + chi2*(exp(u) - 1));

% (3) Euler equation
exp(lam) = beta*exp(lam(+1))*(1+int)*(1+infl(+1))^(-1);

% (4) FOC on investment
exp(lam) = exp(mu)*exp(Z)*(1 - (tau/2)*(exp(I)/exp(I(-1)) - 1)^2 -
tau*(exp(I)/exp(I(-1)) - 1)*(exp(I)/exp(I(-1)))) +
beta*exp(mu(+1))*exp(Z(+1))*tau*(exp(I(+1))/exp(I)-1)*(exp(I(+1))/exp(I))^2;

% (5) FOC on capital
exp(mu) = beta*(exp(lam(+1))*(exp(R(+1))*exp(u(+1)) -
exp(Z(+1))^(-1)*(chi1*(exp(u(+1)) - 1) + (chi2/2)*(exp(u(+1)) - 1)^2))
+ exp(mu(+1))*(1-Δ));

% (6) Reset wage
exp(wsharp) = (epsw/(epsw-1))*exp(f1)/exp(f2);

% (7) f1
exp(f1) = psi*(exp(w)/exp(wsharp))^(epsw*(1+eta))*exp(N)^(1+eta) +
phiw*beta*(1+infl)^(-zetaw*epsw*(1+eta))*(1+infl(+1))^(epsw*(1+eta))*
(exp(wsharp(+1))/exp(wsharp))^(epsw*(1+eta))*exp(f1(+1));

% (8) f2
exp(f2) = exp(lam)*(exp(w)/exp(wsharp))^(epsw)*exp(N) +
```

```
phiw*beta*(1+infl)^(zetaw*(1-epsw))*(1+infl(+1))^(epsw-1)*
(exp(wsharp(+1))/exp(wsharp))^(epsw)*exp(f2(+1));

% (9) Real wage index
exp(w)^(1-epsw) = (1-phiw)*exp(wsharp)^(1-epsw) +
(1+infl(-1))^(zetaw*(1-epsw))*(1 + infl)^(epsw-1)*phiw*exp(w(-1))^(1-epsw);

% (10) Production function
exp(Y) = exp(A)*exp(Khat)^(alpha)*exp(N)^(1-alpha)/exp(vp);

% (11) Price dispersion
exp(vp) = (1+infl)^(epsp)*((1-phip)*(1+pisharp)^(-epsp) +
(1+infl(-1))^(-zetap*epsp)*phip*exp(vp(-1)));

% (12) Price evolution
(1+infl)^(1-epsp) = (1-phip)*(1+pisharp)^(1-epsp) +
phip*(1+infl(-1))^(zetap*(1-epsp));

% (13) Reset price
1+pisharp = (epsp/(epsp-1))*(1+infl)*exp(x1)/exp(x2);

% (14) x1
exp(x1) = exp(lam)*exp(mc)*exp(Y) +
phip*beta*(1+infl)^(-zetap*epsp)*(1+infl(+1))^(epsp)*exp(x1(+1));

% (15) x2
exp(x2) = exp(lam)*exp(Y) +
phip*beta*(1+infl)^(zetap*(1-epsp))*(1+infl(+1))^(epsp-1)*exp(x2(+1));

% (16) Factor prices
exp(w)/exp(R) = ((1-alpha)/alpha)*exp(Khat)/exp(N);

% (17) Marginal cost
exp(w) = exp(mc)*(1-alpha)*exp(A)*exp(Khat)^(alpha)*exp(N)^(-alpha);

% (18) Taylor rule
int = (1-rhoi)*(1/beta - 1)*(1+pistar) + rhoi*int(-1) +
(1-rhoi)*(phipi*(infl - pistar) + phiy*(Y - Y(-1))) + ei;

% (19) Aggregate resource
exp(Y) = exp(C) + exp(I) + exp(G) + (chi1*(exp(u) - 1) +
(chi2/2)*(exp(u) - 1)^2)*(exp(K(-1))/exp(Z));

% (20) Capital accumulation
exp(K) = exp(Z)*(1 - (tau/2)*(exp(I)/exp(I(-1)) - 1)^2)*exp(I) +
(1-Δ)*exp(K(-1));

% (21) Capital services
exp(Khat) = exp(u)*exp(K(-1));
```

```
% (22) Process for A
A = rhoa*A(-1) + ea;

% (23) Process for Z
Z = rhoz*Z(-1) + ez;

% (24) Government spend
exp(G) = omegag*exp(Y);

% (25) Process for omegag
omegag = (1-rhog)*omega + rhog*omegag(-1) + eg;

% (26) q
exp(q) = exp(mu)/exp(lam);

% (27) Output growth
dY = Y - Y(-1);

% (28) Consumption growth
dC = C - C(-1);

% (29) Investment growth
dI = I - I(-1);

% (30) Hours growth
dN = N - N(-1);

end;

initval;
A = 0;
Z = 0;
int = (1/beta - 1)*(1+pistar);
N = log(0.5);
K = log(15);
Khat = log(15);
u = 0;
vp = 0;
Y = log(15^(alpha)*0.5^(1-alpha));
C = log(0.6*15^(alpha)*0.5^(1-alpha));
lam = -log(0.6*15^(alpha)*0.5^(1-alpha)) - log(1-b) + log(1-beta*b);
I = log(Δ*15);
R = log(1/beta - 1 + Δ);
mc = log(epsp/(epsp-1));
w = log(((1-alpha)/alpha)*30*(1/beta - 1 + Δ));
infl = pistar;
pisharp = pistar;
wsharp = log(((1-alpha)/alpha)*30*(1/beta - 1 + Δ));
```

```
f1 = log(psi*0.5^(1+eta)/(1-phiw*beta));
f2 = log((0.5/(0.6*15^(alpha)*0.5^(1-alpha)))/(1-phiw*beta));
mu = -log(0.6*15^(alpha)*0.5^(1-alpha)) - log(1-b) + log(1-beta*b);
dY = 0;
dC = 0;
dI = 0;
dN = 0;
end;

varobs dY int dI infl;

estimated_params;
b, beta_pdf, 0.7, 0.1;
phiw, beta_pdf, 0.5, 0.1;
phip, beta_pdf, 0.5, 0.1;
zetaw, beta_pdf, 0.5, 0.2;
zetap, beta_pdf, 0.5, 0.2;
eta, normal_pdf, 1, 0.25;
tau, normal_pdf, 2, 0.5;
rhoi, beta_pdf, 0.9, 0.05;
rhoa, beta_pdf, 0.9, 0.05;
rhoz, beta_pdf, 0.9, 0.05;
rhog, beta_pdf, 0.9, 0.05;
phiy, normal_pdf, 0.125, 0.05;
phipi, normal_pdf, 1.5, 0.1;
stderr ea, inv_gamma_pdf, 0.01, 0.002;
stderr ez, inv_gamma_pdf, 0.01, 0.002;
stderr ei, inv_gamma_pdf, 0.002, 0.002;
stderr eg, inv_gamma_pdf, 0.005, 0.002;
end;

estimated_params_init(use_calibration);
stderr ea, 0.01;
stderr ez, 0.01;
stderr eg, 0.01;
stderr ei, 0.01;
end;

estimation(datafile=estimation_data_short,mh_replic=20000,mh_jscale=0.5,
mode_compute=4);
```

Basically, the first part of the .mod file is the same as it usually is. To do estimation, you replace the "stoch_simul" commant with the "estimation" command, with a prior block given prior distributions for the parameters to be estimated. Dynare does the rest.

The following table shows the estimation results. It gives a posterior mode, mean, and standard error. The parameter values end up looking pretty reasonable. There is a good bit of habit formation ($b = 0.72$), prices and wages are both sticky, though prices seem to be stickier. The data want a little bit of indexation in wages ($\zeta_w = 0.38$), but virtually no price indexation. The implied Frisch labor supply elasticity is a little less than 1 (the inverse of η). The investment adjustment cost parameter τ is near 2. The monetary policy rule has smoothing parameter of about 0.8, response to inflation of about 1.3, and response to output growth of about 0.3. The autocorrelation parameters in the other shock process are all high (the highest is for A, with ρ_a near 1). The shock standard deviations are as shown.

Table 1: Estimated Parameters

	Prior			Posterior		
Parameter	Dist.	Mean	SE	Mode	Mean	SE
b	Beta	0.70	0.10	0.72	0.64	0.0944
ϕ_w	Beta	0.50	0.10	0.43	0.42	0.0754
ϕ_p	Beta	0.50	0.10	0.71	0.71	0.0271
ζ_w	Beta	0.50	0.20	0.38	0.44	0.2568
ζ_p	Beta	0.50	0.20	0.03	0.05	0.0237
η	Normal	1.00	0.25	1.23	1.23	0.2241
τ	Normal	2.00	0.50	1.87	1.93	0.2868
ρ_i	Beta	0.50	0.20	0.79	0.78	0.0170
ρ_a	Beta	0.90	0.05	0.99	0.99	0.0038
ρ_z	Beta	0.90	0.05	0.90	0.90	0.0180
ρ_g	Beta	0.90	0.05	0.96	0.95	0.0205
ϕ_π	Normal	1.50	0.10	1.35	1.31	0.0927
ϕ_y	Normal	0.125	0.05	0.32	0.32	0.0412
s_a	Inv. Gamma	0.010	0.002	0.0074	0.0076	7.933e-04
s_z	Inv. Gamma	0.010	0.002	0.0091	0.0092	8.4717e-04
s_g	Inv. Gamma	0.010	0.002	0.0038	0.0039	2.8828e-04
s_i	Inv. Gamma	0.002	0.002	0.0013	0.0013	9.8624e-05

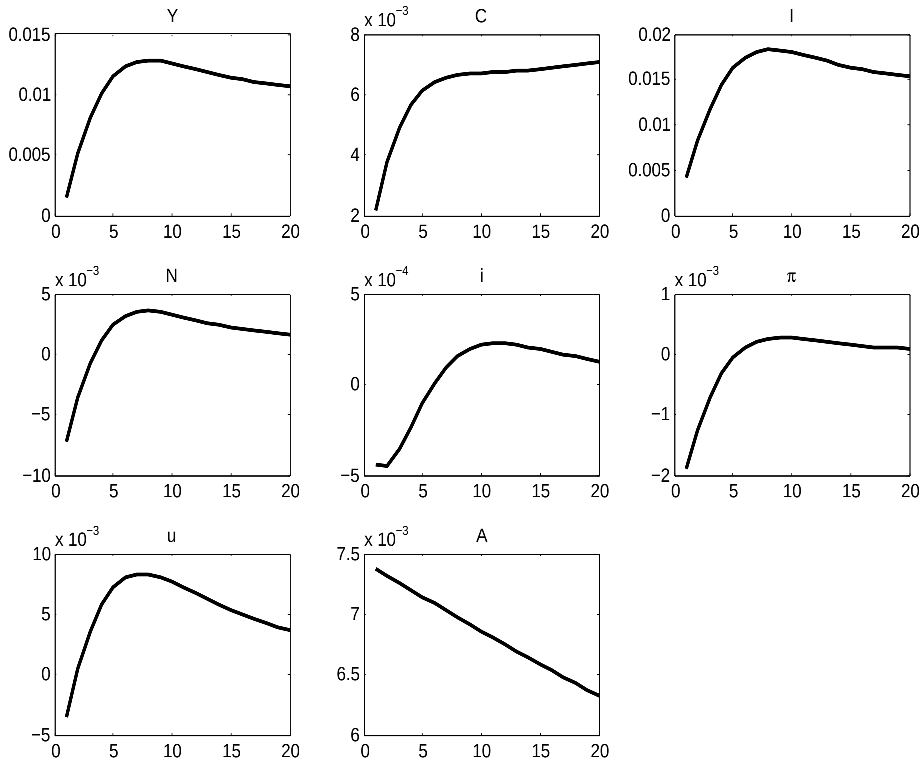
To get a sense for how the model works, I have a separate Dynare file which uses the estimated parameters and does the regular old "stoch_simul" command. From this I can get moments of the data, a variance decomposition, and impulse responses. In terms of a variance decomposition, productivity shocks, investment shocks, and government spending shocks are all estimated to account for about 30 percent of the variance of output growth, while monetary shocks only account for about 6 percent. Investment shocks dominate the variance decomposition of investment growth and q, while consumption growth is mostly explained by the neutral productivity shock.

The table below gives some business cycle moments from the simulated model. The model does reasonably well. It generates too much output volatility relative to the data, though the relative volatilities of consumption, investment, and hours are pretty close to what they are in the data. It does pretty well at matching co-movements with output, though it misses the sign on the correlations of both the nominal interest rate and inflation with output growth. It does pretty well on the estimated autocorrelations of output, but produces autocorrelations in consumption and investment that are too high. It completely whiffs on the autocorrelation of hours growth. It does pretty well on the autocorrelations of interest rates and inflation.

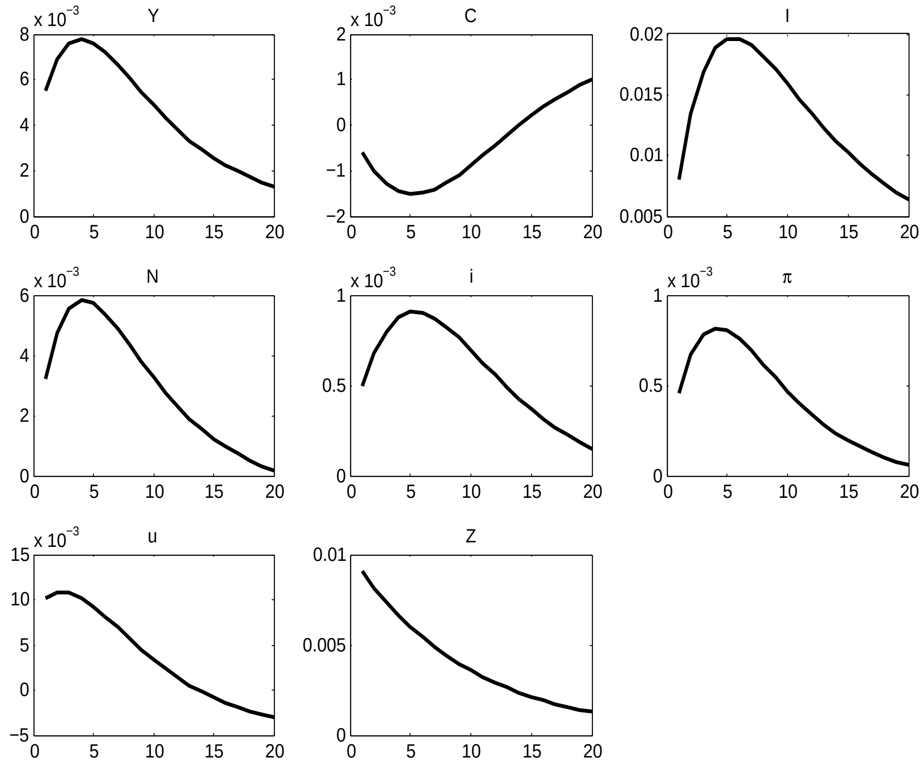
Table 2: Data vs. Model

Moment	Data	Model
$\sigma(\Delta Y_t)$	0.0060	0.0105
$\sigma(\Delta C_t)$	0.0030	0.0035
$\sigma(\Delta I_t)$	0.0077	0.0145
$\sigma(\Delta N_t)$	0.0073	0.0119
$\sigma(\pi_t)$	0.0024	0.0039
$\sigma(i_t)$	0.0062	0.0036
$\rho(\Delta Y_t, \Delta C_t)$	0.2509	0.2378
$\rho(\Delta Y_t, \Delta I_t)$	0.6063	0.7133
$\rho(\Delta Y_t, \Delta N_t)$	0.6505	0.6867
$\rho(\Delta Y_t, \pi_t)$	-0.1427	0.0481
$\rho(\Delta Y_t, i_t)$	-0.1232	0.2220
$\rho(\Delta Y_t, \Delta Y_{t-1})$	0.4243	0.3510
$\rho(\Delta C_t, \Delta C_{t-1})$	0.1968	0.7002
$\rho(\Delta I_t, \Delta I_{t-1})$	0.3674	0.7100
$\rho(\Delta N_t, \Delta N_{t-1})$	0.7108	0.0010
$\rho(\pi_t, \pi_{t-1})$	0.6267	0.8087
$\rho(i_t, i_{t-1})$	0.9850	0.9236

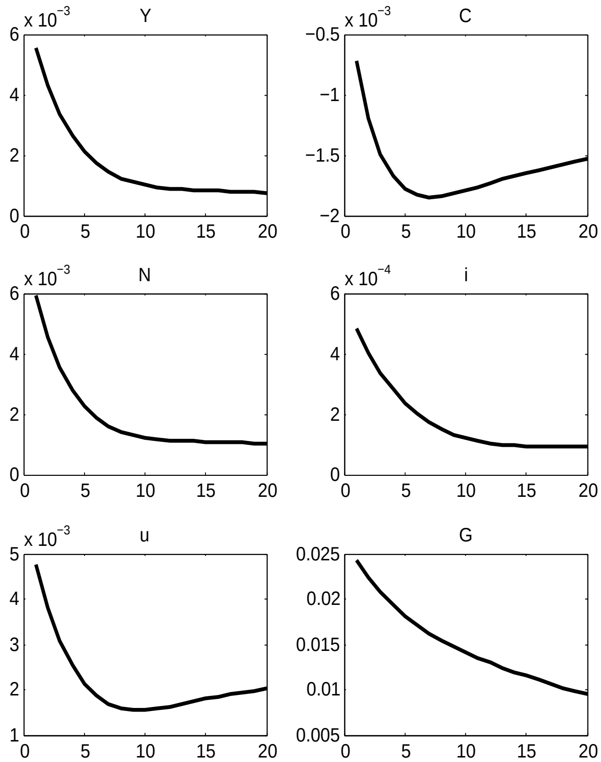
Next, I show impulse responses to each of the four shocks. I start with the neutral productivity shock. It leads to persistent increases in output, consumption, and investment. As in the simpler NK model, hours falls initially. The shock is also deflationary.



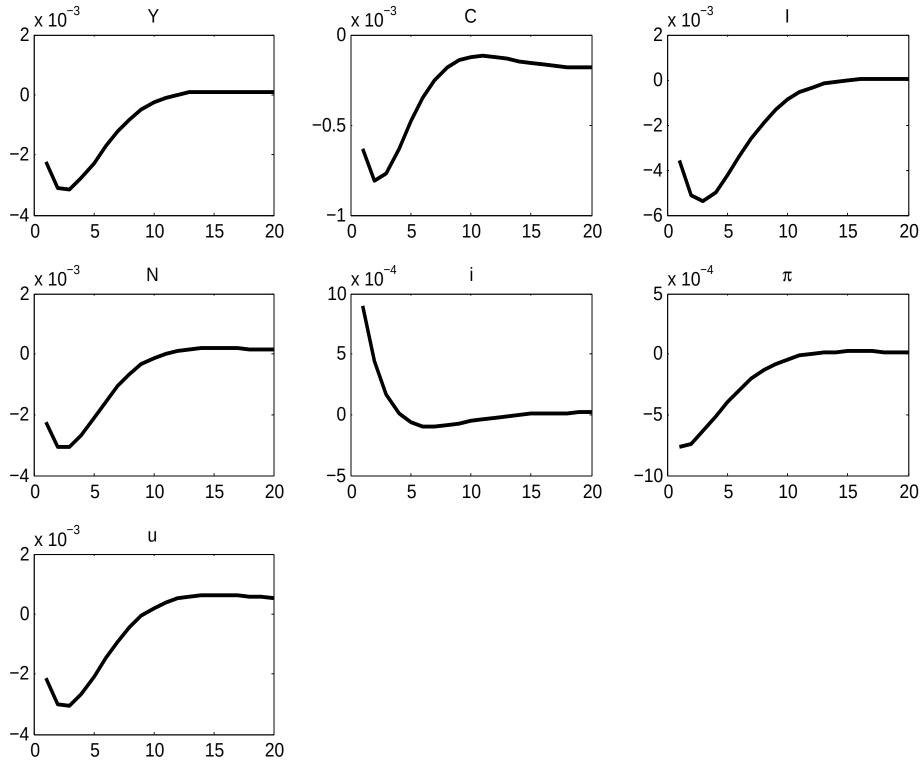
Next, I look at the impulse responses to an investment-specific shock, Z_t. This leads to a hump-shaped expansion of output and investment and hours. Consumption initially declines. This shock turns out to be inflationary given the parameterization.



The government spending shock, which is really a shock to the share of government spending, is quite large, resulting in a 2.5 percent increase in G on impact. This leads to an expansion in output and hours, but crowds out private expenditure, witch consumption and investment declining. Inflation goes up. The output multiplier comes out to be 1.14, which suggests that output rises by more than the increase in government spending. How does this happen when both private consumption and investment fall? It's a consequence of the somewhat peculiar way in which we modeled the cost of utilization, which show up as a resource cost. The rise in utilization on the right hand side of the resource constraint is sufficient to overcome the declines in C_t and I_t which makes total non-government expenditure go up, allowing the multiplier to exceed 1.



Finally, I look at the impulse responses to the monetary policy shock. Output, consumption, investment, and hours all go down, and follow a bit of a hump-shape, which is an important feature of these models, since most estimated VAR responses feature hump-shapes of this sort.



Y
x 10^−3
C
x 10^−3
I
x 10^−3
N
x 10^−3
i
x 10^−4
π
x 10^−4
u
x 10^−3