Ramey and Shapiro (1998) - Model

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November 30, 2022

1 The Social Planner Problem

The model can be broken down into four blocks: preferences, production, capital accumulation and resource constraints.

Preferences: take the following form:

$$\sum_{t=0}^{\infty} \beta^t \cdot (\log(C_{1,t} + \theta \cdot \log C_{2,t} + \phi \cdot \log (T - L_{1,t} - L_{2,t})) \tag{1}$$

where:

- $C_{i,t}$: is consumption of good i at time t.
- T is the total time endowment.
- $L_{i,t}$ is total hours supplied by sector i.

Production: Goods 1 and 2 are produced according to the following Cobb-Douglas production function:[1]

$$Y_{i,t} = A \cdot L_{i,t}^{\alpha} \cdot \left(K_{i,t}^{*}\right)^{1-\alpha} \quad i = 1, 2 \tag{2}$$

where:

- $L_{i,t}$ is total hours worked.
- $K_{i,t}^{*}$ is capital stock available during period t.

[1] (!!!) Unlike the original paper, I use standard Dynare timing notation to make sure that the Blanchard-Kahn conditions are met. If you don't do so, capital becomes a forward-looking variable and the number of eigenvalues outside the unit circle is not enough to match the total number of forward looking variables (i.e. indeterminacy). This can be easily checked using the *check* command on Dynare.

Dynamics of Capital: The equations that specify the evolution of capital stocks are as follows:

$$
\begin{aligned}
K_{i,t} &= (1-\delta)\cdot K_{i,t-1} + \underbrace{(I_{i,t} - R_{i,t})}_{\text{Net Investment}}, \quad i = 1,2 \\
K^*_{i,t} &= K_{i,t-1} - R_{i,t}, \quad i = 1,2 \\
I_{1,t} &= X_{1,t} + (1-\gamma)\cdot R_{2,t} \\
I_{2,t} &= X_{2,t} + (1-\gamma)\cdot R_{1,t}
\end{aligned} \tag{3}
$$

where:

- $K_{i,t-1}$ is the stock of capital in sector i during period t,
- $I_{i,t}$ are purchases of new and used capital goods by sector i,
- $R_{i,t}$ are sales of capital by sector i,
- $X_{i,t}$ is production of new capital goods by sector i,
- productive capital, $K^*_{i,t}$, is given by the difference between the capital stock available for production, $K_{i,t-1}$, minus what is shifted to other sector, $R_{i,t}$.
- δ is the capital's rate of depreciation, and
- γ is a parameter between 0 and 1 which captures the loss in moving capital from one sector to another (e.g. aircraft wind tunnel example). γ is the fixed cost of shifting capital.

Notice that the new amount of capital produced in each sector - i.e. savings of sector i - is equal to:

$$
\begin{aligned}
X_{1,t} &= K_{1,t} - (1-\delta)\cdot K_{1,t-1} + R_{1,t} - (1-\gamma)\cdot R_{2,t} \\
X_{2,t} &= K_{2,t} - (1-\delta)\cdot K_{2,t-1} + R_{2,t} - (1-\gamma)\cdot R_{1,t}.
\end{aligned} \tag{4}
$$

Equilibrium: Under the assumption of complete markets and no distortions, the competitive equilibrium of this economy corresponds to the solution of the following social-planner problem: choose $\{C_{1,t}, C_{2,t}, L_{1,t}, L_{2,t}, K_{1,t+1}, K_{2,t+1}, R_{1,t}, R_{2,t} : t > 0\}$ to maximize (1) subject to equations (2), (3) and (4), and the initial position of the economy summarized by $K_{1,0}$ and $K_{2,0}$.

Formally, we can combine equations (2), (3) and (4) and rewrite the social planner problem as:

$$\max_{\{C_{1,t},C_{2,t},L_{1,t},L_{2,t},K_{1,t},K_{2,t},R_{1,t},R_{2,t}\}} \sum_{t=0}^{\infty} \beta^t \cdot (\log(C_{1,t} + \theta \cdot \log C_{2,t} + \phi \cdot \log(T - L_{1,t} - L_{2,t}))$$

Subject to:

$$\underbrace{A \cdot L_{1,t}^{\alpha} \cdot (K_{1,t-1} - R_{1,t})^{1-\alpha}}_{=Y_{1,t}} = C_{1,t} + \underbrace{K_{1,t} - (1-\delta) \cdot K_{1,t-1} + R_{1,t} - (1-\gamma) \cdot R_{2,t}}_{=X_{1,t}} + G_{1,t}$$

$$\underbrace{A \cdot L_{2,t}^{\alpha} \cdot (K_{2,t-1} - R_{2,t})^{1-\alpha}}_{=Y_{2,t}} = C_{2,t} + \underbrace{K_{2,t} - (1-\delta) \cdot K_{2,t-1} + R_{2,t} - (1-\gamma) \cdot R_{1,t}}_{=X_{2,t}} + G_{2,t}$$

$$G_{1,t} = (1-\rho_{1,G}) \cdot \gamma_1 + \rho_{1,G} \cdot G_{1,t-1} + \varepsilon_{1,t}^{G}$$

$$G_{2,t} = (1-\rho_{2,G}) \cdot \gamma_2 + \rho_{2,G} \cdot G_{2,t-1} + \varepsilon_{2,t}^{G}$$

$$R_{1,t} \geq 0$$

$$R_{2,t} \geq 0$$

Optimality conditions: it is clear from the setup of the problem that it is sub-optimal to shift capital from both sectors at the same time:

- In steady state we must have that $R_{1,t} = R_{2,t} = 0$. We do not shift capital.
- Outside of steady state, it is always sub-optimal to have $R_{1,t} > 0$ and $R_{2,t} > 0$. This is formally shown at the end of the document.
  Therefore, we will assume that purchases of capital can only happen when the government purchases new capital good in one sector. Since we assume that $dG_{2,t} > 0$, then $R_{2,t} = 0$ in any t while $R_{1,t} \geq 0$.
- Why it might be convenient for the planner to purchase capital from one sector to another? i.e. $R_{1,t} > 0$?
  Suppose there is an unexpected and persistent government spending shock to sector 2: $dG_{2,t} > 0$. The government needs to reduce $C_{2,t}$ to clear the resource constraint: $Y_{2,t} \uparrow = C_{2,t} \downarrow + X_{2,t} \uparrow + G_{2,t} \uparrow$. The planner can either give up consumption to expand the capital stock, i.e. $C_{2,t} \downarrow$ and $X_{2,t} \uparrow$, or, it can purchase used capital from sector 1, $R_{1,t}$, at the cost of reducing $C_{1,t}$. If the marginal utility of consumption of good 2 $\lambda_{2,t}$ is very high, it might be convenient for the planner to also pay the fixed cost γ to shift capital from sector 1 to sector 2, and dampen the fall in $C_{2,t}$.

The Competitive Equilibrium: Using the Lagrangean method, the optimality conditions which characterize the equilibrium path are:

$$[C_{1,t}]: \quad \frac{1}{C_{1,t}} = \lambda_{1,t}$$

$$[C_{2,t}]: \quad \frac{\theta}{C_{2,t}} = \lambda_{2,t}$$

$$[l_{1,t}]: \quad \frac{\phi}{T - l_{1,t} - l_{2,t}} = \lambda_{1,t} \cdot MPN_{1,t} \quad \text{with: } MPN_{1,t} = \alpha \cdot A \cdot \left(\frac{K_{1,t-1} - R_{1,t}}{l_{1,t}}\right)^{1-\alpha}$$

$$[l_{2,t}]: \quad \frac{\phi}{T - l_{1,t} - l_{2,t}} = \lambda_{2,t} \cdot MPN_{2,t} \quad \text{with: } MPN_{2,t} = \alpha \cdot A \cdot \left(\frac{K_{2,t-1}}{l_{2,t}}\right)^{1-\alpha}$$

$$[K_{1,t}]: \quad \lambda_{1,t} = \beta \cdot \lambda_{1,t+1} \cdot (MPK_{1,t+1} + 1 - \delta) \quad \text{with: } MPK_{1,t} = A \cdot (1-\alpha) \cdot \left(\frac{l_{1,t}}{K_{1,t-1} - R_{1,t}}\right)^{\alpha}$$

$$[K_{2,t}]: \quad \lambda_{2,t} = \beta \cdot \lambda_{2,t+1} \cdot (MPK_{2,t+1} + 1 - \delta) \quad \text{with: } MPK_{2,t} = A \cdot (1-\alpha) \cdot \left(\frac{l_{2,t}}{K_{2,t-1}}\right)^{\alpha}$$

$$[\lambda_{1,t}]: \quad \underbrace{A \cdot l_{1,t}^{\alpha} \cdot (K_{1,t-1} - R_{1,t})^{1-\alpha}}_{=Y_{1,t}} - C_{1,t} - G_{1,t} - (K_{1,t} - (1-\delta) \cdot K_{1,t-1} + R_{1,t}) = 0$$

$$[\lambda_{2,t}]: \quad \underbrace{A \cdot l_{2,t}^{\alpha} \cdot K_{2,t-1}^{1-\alpha}}_{Y_{2,t}} - C_{2,t} - G_{2,t} - (K_{2,t} - (1-\delta) \cdot K_{2,t-1} - (1-\gamma) \cdot R_{1,t}) = 0$$

$$[R_{1,t}]: \quad R_{1,t} \cdot (\lambda_{1,t} \cdot (1 + MPK_{1,t}) - \lambda_{2,t} \cdot (1-\gamma)) = 0 \quad \text{(Complementary Slackness)}$$

Notice that this is a system of nine equations and nine variables. Moreover, that last equations is saying that if $R_{1,t} = 0$ then, the optimality condition of shifting capital does not necessarily need to hold. If $R_{1,t} > 0$, then, the optimality conditions must hold.

A Proofs

Sub-optimality of contemporaneous shift of capital in both sectors: Suppose that both $R_{1,t}$ and $R_{2,t}$ positive, then the following two optimally conditions must hold:

$$\begin{aligned} \lambda_{1,t} \cdot (1 + MPK_{1,t}) &= \lambda_{2,t} \cdot (1-\gamma) < \lambda_{2,t} = \beta \cdot \lambda_{2,t+1} \cdot (1 + MPK_{2,t+1}) \\ \underbrace{\lambda_{2,t} \cdot (1 + MPK_{2,t})}_{\text{Sales of Capital Optimality Conditions}} &= \lambda_{1,t} \cdot (1-\gamma) < \underbrace{\lambda_{1,t} = \beta \cdot \lambda_{1,t+1} \cdot (1 + MPK_{1,t+1})}_{\text{Euler Equations}} \end{aligned}$$

where $\lambda_{1,t}$ and $\lambda_{2,t}$ are the marginal utilities of consumption of good 1 and 2 respectively.

The left hand-side of the inequalities is saying that the cost of giving up one unit of capital of type i, i.e. $\lambda_{i,t} \cdot (1 + MPK_{i,t})$, must be equal to the benefit of shifting that unit of capital, i.e. $\lambda_{j,t} \cdot (1 - \gamma)$. In turns, the right hand-side of the inequality, is saying that we must be indifferent between saving an extra unit of capital and consuming it, i.e. euler equation. Notice that we cannot have that both lines hold true simultaneously. In fact, they lead to a contradiction:

$$\lambda_{2,t} < \lambda_{2,t} \cdot (1 + MPK_{2,t}) < \lambda_{1,t} < \lambda_{1,t} \cdot (1 + \underbrace{MPK_{1,t}}_{>0}) = \lambda_{2,t} \cdot (1 - \gamma) < \lambda_{2,t}.$$