Macroeconomic Dynamics, **20**, 2016, 120–164. Printed in the United States of America.
doi:10.1017/S1365100514000273

INTEREST RATE EFFECTS OF DEMOGRAPHIC CHANGES IN A NEW KEYNESIAN LIFE-CYCLE FRAMEWORK

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This paper develops a small-scale DSGE model that embeds a demographic structure within a monetary policy framework. We extend the nonmonetary overlapping-generations model of Gertler and present a small synthesis model that combines the setup of Gertler with a New Keynesian structure, implying that the short-run dynamics related to monetary policy can be compared with that of the standard New Keynesian model. In sum, the model offers a New Keynesian platform that can be used to characterize the response of macroeconomic variables to demographic shocks, similarly to the responses to technology or monetary policy shocks. We offer such characterizations for flexible and sticky price equilibria. Empirically, we calibrate the model to demographic developments projected for the euro area. The main finding is that the projected slowdown in population growth and the increase in longevity contribute slowly over time to a decline in the equilibrium interest rate.

Keywords: Demographic Change, Monetary Policy, DSGE Modeling

1. INTRODUCTION

This paper starts out from the observation that most industrialized countries are subject to long-lasting demographic changes. Two key features of these changes, which are particularly pronounced in various European countries, are a secular slowdown in population growth and a substantial increase in longevity. As stressed

We would like to thank Neil Rankin, Massimo Rostagno, Jon Temple, Jean-Pierre Vidal and seminar participants at Bristol, the ECB, the European Economic Association Meeting (Milan, 2008), the North American Summer Meeting of the Econometric Society (Pittsburgh, 2008) and the Meeting of the German Economic Association (Muenchen, 2007) for their comments. Jasmin Sin provided excellent research assistance. Part of this paper was completed while Engin Kara was visiting the Monetary Policy Strategy Division at the European Central Bank and he thanks the ECB for its hospitality. The views expressed in this paper are those of the authors and do not necessarily reflect the views of the European Central Bank. Address correspondence to: Engin Kara, Department of Economics, University of Bristol, 8 Woodland Road, Bristol BS8 1TH, UK; e-mail: engin.kara@bristol.ac.uk.

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by Bean (2004), these developments are of relevance for monetary policy makers from a normative perspective, because the optimal monetary policy may depend on the age structure of an economy, reflecting the possibility that different age cohorts may tend to have different inflation preferences because of cohort-specific portfolio compositions. Moreover, they may also be of importance from a positive perspective. In particular, it is well known from economic growth theory that demographic variables are a key determinant of the equilibrium real interest rate, a variable that is important for judging the stance of monetary policy for any given inflation target. Yet, despite these insights, monetary policy is typically addressed in frameworks in which demographic changes are not explicitly modeled. In particular, going back to Clarida et al. (1999) and Woodford (2003), the canonical New Keynesian DSGE framework, which is widely used for monetary policy analysis, is based on the assumption of an infinitely lived representative household, thereby abstracting from realistic population dynamics, heterogeneity among agents, and individual life-cycle effects.

Against this background, this paper has the goal of developing a closed economy framework for monetary policy analysis that embeds a tractable demographic structure within an otherwise standard New Keynesian DSGE model. To this end, we build on the overlapping-generations model of Gertler (1999), which introduces life-cycle behavior by allowing two subsequently reached states of life of newborn agents, working age and retirement. This structure gives rise to two additional demographic variables besides the growth rate of newborn agents, namely the exit probabilities associated with the two states, which can be calibrated to match the average lengths of working age and retirement. Similarly to Blanchard (1985) and Weil (1989), these probabilities are assumed to be age-independent. This feature is the key to keeping the state space of the model small so that there exist closed-form aggregate consumption and savings relations despite the heterogeneity of agents at the micro level. We combine this tractable structure with New Keynesian supply-side features, characterized by capital accumulation, imperfect competition in the intermediate goods sector, and nominal price rigidities along the lines of Calvo (1983). These features give rise to a modified New Keynesian Phillips curve, implying that the short-run dynamics of the proposed framework can be compared with that of the standard model. Under the special assumption that workers are infinitely lived, this dynamics becomes identical to that of the standard model. In the absence of this special assumption, however, the New Keynesian dynamics becomes richer, because current and expected future consumption depend on demographic factors and individual life-cycle effects. Because the entire consumption trajectory matters for current and future values of output and inflation, this leads to a modification of the short-run dynamics that is relevant for the effects of monetary policy. To characterize this dynamics, we compare equilibria under flexible and sticky prices. This distinction matters for the response of real and nominal variables, and it facilitates a meaningful characterization of the short-run behavior of the economy in response to demographic shocks.

As a general feature, monetary and fiscal policies follow feedback rules in the spirit of Leeper (1991), anchoring the economy over time around target levels for the inflation rate and the government debt ratio. Reflecting the overlapping-generations structure, the dynamics of the model is critically affected by fiscal policy (which is, by construction, non-neutral) and, in particular, by the design of the pension system, which facilitates intergenerational transfers between workers and retirees and determines the strength of individual life-cycle effects. In sum, we offer an enlarged New Keynesian platform that can be used to investigate various macroeconomic questions.

We use our model, calibrated to euro area data, to examine selected macroeconomic implications of demographic changes from a positive perspective. In particular, we focus on the determinants of the equilibrium real interest rate. Our analysis proceeds in two steps.

First, we investigate qualitatively the long-run implications of a decrease in population growth and an increase in life expectancy. To this end, we abstract from inflationary (and any other) short-run dynamics and report comparative statics results from the flexible-price version of our framework. Our analysis reveals that the long-run effect on the equilibrium real interest rate depends critically on assumptions concerning the future course of the assumed PAYGO pension system. For illustration, we distinguish between two types of scenarios in which the rising old-age dependency ratio does or does not lead to changes in the replacement rate (defined as the ratio between individual pension benefits and wages).[1] For the first scenario type, the replacement rate declines endogenously so that the aggregate benefits–output ratio remains unchanged. This assumption amounts to a strengthening of privately funded elements, because it introduces a ceiling on tax-financed redistribution between workers and retirees. For the second scenario type, the replacement rate remains constant, leading to a rise in the aggregate benefits–output ratio. This assumption amounts to a “no reform” scenario that extrapolates the existing pension system into the future, leading to a higher tax burden on workers. The main finding is that under either scenario the decrease in population growth and the increase in life expectancy are two independent forces that contribute to the decline in the equilibrium real interest rate. Yet the predicted magnitude of this decline differs between the scenarios because of distinctly different incentives for individual savings. In particular, for the first scenario type, the decline is more pronounced because of additional savings undertaken by workers in anticipation of lower pension incomes in the future.

Second, we present a detailed macroeconomic scenario for the euro area until 2030 that also incorporates short- and medium-run effects. Our framework specifies all demographic processes as time-dependent. This assumption allows us to calibrate the model’s demographic parameters according to demographic projections for the euro area, as reported in European Economy (2009). Specifically, we take the annual demographic projections for population growth and life expectancy as a deterministic input and verify that the model matches the old-age dependency ratio projected until 2030. We then solve the nonlinear model numerically under perfect foresight. As a baseline for the pension system, we assume that the replacement rate declines endogenously, so that the aggregate benefits–output ratio remains unchanged. We first consider as a benchmark the flexible-price version, in which the central bank ensures price stability in all periods (so that the equilibrium real interest rate coincides with the natural rate).[2] According to our simulations, the decrease in population growth and the increase in life expectancy jointly contribute over the entire projection horizon to a smooth decline in the equilibrium real interest rate by about 100 basis points. Moreover, we consider an alternative scenario with sticky prices in which monetary policy follows a Taylor-type feedback rule. Assuming standard feedback coefficients, the equilibrium real interest rate falls by less, at the expense of prolonged deflation and a less favorable path of output. We take this finding as a confirmation that under meaningful demographic assumptions, combined with other frictions such as nominal rigidities, the design of optimal monetary policy deserves further analysis. Yet this requires a normative approach, which goes beyond the scope of this paper.

In related literature, Roeger (2005), Kilponen et al. (2006), and Ferrero (2010) consider nonmonetary versions of the Gertler setup that, similarly to ours, allow time-dependent demographic processes. Yet our paper differs from these studies in that we consider a closed economy setup in which the equilibrium interest rate is endogenously determined. Fujiwara and Teranishi (2008) offer a New Keynesian Gertler-type economy that is similar to ours in a number of respects. Yet the focus is distinctly different in that Fujiwara and Teranishi (2008) compare the effects of technology and monetary policy shocks in economies characterized by different steady-state age structures, whereas implications of time-varying demographic changes are not addressed. Moreover, in the absence of aging-related fiscal policy and social security aspects, the study does not explore links between demographic developments, pension systems, and monetary policy, as also pointed out by Ripatti (2008).[3] It is worth stressing that our benchmark predictions under flexible prices are in line with those obtained in large-scale nonmonetary settings. In particular, Miles (1999, 2002) uses a rich overlapping-generations framework for a closed economy, in the spirit of Auerbach and Kotlikoff (1987). Miles considers various specifications for pension systems and he reports, in simulations for the European economy, qualitative and quantitative predictions similar to ours. We find this encouraging because our model, because of its small state space and its parsimonious use of demographic variables, is computationally less demanding.

As stressed by INGENUE (2001), Batini et al. (2006), Boersch-Supan et al. (2006), and Krueger and Ludwig (2007), additional open-economy channels matter in global settings. In particular, to the extent that the euro area ages more rapidly than most OECD countries, closed-economy predictions for the decline in the interest rate tend to be overstated; i.e., capital mobility tends to moderate the pressure on factor price adjustments.[4]

This paper is structured as follows. Section 2 presents the model. Section 3 summarizes the general equilibrium conditions. Section 4 discusses the numerical assumptions that are used to calibrate the benchmark steady state to stylized features of the euro area. Moreover, it summarizes major demographic trends facing the euro area. Section 5 takes a comparative statics perspective and describes long-run effects under different policy assumptions concerning future pension systems. Section 6 presents the macroeconomic scenario for the euro area until 2030. Section 7 offers conclusions. Technical issues are relegated to three Appendices at the end of the paper.

2. THE MODEL

The model includes a number of features that are essential to analyze macroeconomic effects of demographic changes. The general modeling approach is to add tractable life-cycle features to an otherwise canonical New Keynesian DSGE model with monopolistic competition, price rigidities, and capital accumulation, as familiar from the monetary policy literature. The following exposition aims to outline the basic building blocks of the model, addressing in turn the demographic structure of the economy and the behavior of households, firms, and monetary and fiscal policy makers.

2.1. Demographic Structure

In the spirit of Gertler (1999), the population consists of two distinct groups of agents, workers (N^{w}) and retirees (N^{r}). Newborn agents enter directly the working-age population, which grows at rate n^{w}. Workers face a probability ω of remaining workers, whereas they retire with probability $(1-\omega)$. Similarly, retirees stay alive with probability γ, whereas $(1-\gamma)$ denotes the probability of death of retirees. Hence, the total lifespan of agents between birth and death is made up of two distinct states, working age and retirement age. These two states are successively reached by agents, giving rise to life-cycle patterns that are different from a standard representative agent economy. For tractability, ω and γ are assumed to be independent of the age of agents, similarly to Blanchard (1985) and Weil (1989). However, we assume that the three demographic variables of interest, namely n_t^{w}, ω_t, and γ_t, are time-dependent, similarly to Ferrero (2010), Kilponen et al. (2006), and Roeger (2005). The laws of motion for workers and retirees are given by

$$N_{t+1}^{\mathrm{w}} = (1-\omega_t+n_t^{\mathrm{w}})N_t^{\mathrm{w}} + \omega_t N_t^{\mathrm{w}} = (1+n_t^{\mathrm{w}})N_t^{\mathrm{w}},$$
$$N_{t+1}^{\mathrm{r}} = (1-\omega_t)N_t^{\mathrm{w}} + \gamma_t N_t^{\mathrm{r}}.$$

Let $\psi_t = N_t^{\mathrm{r}}/N_t^{\mathrm{w}}$ denote the ratio between retirees and workers, the so-called "old-age dependency ratio." Then the growth rate of retirees (n_t^{r}) satisfies the equation

$$N_{t+1}^{\mathrm{r}}/N_t^{\mathrm{r}} = (1+n_t^{\mathrm{r}}) = \frac{1-\omega_t}{\psi_t} + \gamma_t,$$

https://doi.org/10.1017/S1365100514000273 Published online by Cambridge University Press

whereas the law of motion for the dependency ratio can be calculated as

$$\psi_{t+1} = \frac{1-\omega_t}{1+n_t^{\mathrm{w}}} + \frac{\gamma_t}{1+n_t^{\mathrm{w}}}\psi_t.$$

Hence, any given specification of $\{n_t^{\mathrm{w}}, \omega_t, \gamma_t\}$ implies laws of motion for n_t^{r} and ψ_t. A demographic balanced growth path is characterized by $n_t^{\mathrm{w}} = n^{\mathrm{w}}$, $\omega_t = \omega$, and $\gamma_t = \gamma$, implying that

$$\psi = \frac{1-\omega}{1+n^{\mathrm{w}}-\gamma}; \tag{1}$$

i.e., the old-age dependency ratio (ψ) increases in the survival probability of retirees (γ) and in the retirement probability of workers ($1-\omega$), whereas it decreases in the growth rate of newborn agents (n^{w}). Finally, along a balanced growth path,

$$n^{\mathrm{r}} = n^{\mathrm{w}} = n;$$

i.e., the growth rates of the two groups coincide with the population growth rate.

2.2. Decision Problems of Retirees and Workers

The structure of the preferences of agents closely follows Gertler (1999). To align this structure with a monetary economy, we introduce real balances as an additional element in the utility function, leading to an additional first-order condition. This modifies the conjectures for the aggregate consumption function and the value functions associated with the two states. Otherwise, however, the procedure for solving the decision problems of retirees and workers is similar to that of Gertler (1999). Technical aspects are relegated to Appendix A.

Let V_t^z denote the value function associated with the two states of working age and retirement; i.e., $z = w, r$. Then

$$\begin{aligned} V_t^z &= \left\{\left[\left(c_t^z\right)^{v_1}\left(m_t^z\right)^{v_2}\left(1-l_t^z\right)^{v_3}\right]^\rho + \beta^z E_t\left[V_{t+1} \mid z\right]^\rho\right\}^{\frac{1}{\rho}}, \\ \beta^{\mathrm{w}} &= \beta, \beta^{\mathrm{r}} = \beta\gamma_t, \\ E_t\left[V_{t+1} \mid w\right] &= \omega_t V_{t+1}^{\mathrm{w}} + (1-\omega_t)V_{t+1}^{\mathrm{r}}, \\ E_t\left[V_{t+1} \mid r\right] &= V_{t+1}^{\mathrm{r}}, \end{aligned}$$

where the conditional expectations operator E_t depends on the states of working age and retirement, respectively.[5] Moreover, c_t, m_t, and $1-l_t$ denote consumption, real balances, and leisure, respectively. The parameter v_2 denotes the weight of real balances in the Cobb–Douglas flow utility of agents. If $v_2 \to 0$, preferences of agents converge to the economy with variable labor supply examined by Gertler (1999). The effective discount rates of the two types of agents differ because retirees face a positive probability of death, whereas workers, when leaving their state, stay alive and switch to retirement. Going back to Epstein and Zin (1989), this non-expected-utility specification can be used to separate risk aversion from intertemporal substitution aspects. For this particular functional form, as discussed in Farmer (1990), agents are risk-neutral with respect to income risk, whereas $\sigma = 1/(1-\rho)$ denotes the a priori unspecified intertemporal elasticity of substitution. The advantages of this specification become clear when it is considered together with the idiosyncratic risks faced by individuals and the (un)availability of insurance markets. There are two aspects to this. First, workers face an income risk when entering retirement. To allow for life-cycle behavior, there exists no insurance market against this risk, and the assumption of risk neutrality acts as a cushion to dampen the effects of this risk at the individual level. Second, retirees face the risk of death. To eliminate the uncertainty about the remaining lifetime horizon of retirees, there exists a perfect annuities market, similarly to Blanchard (1985). This market is operated by competitive mutual funds that collect the nonhuman wealth of retirees and pay in return to surviving retirees a return rate $(1+r)/\gamma$ that is above the pure real interest rate $(1+r)$.

Decision problem of the representative retiree. The representative retiree (with index j) maximizes in period t the objective

$$V_t^{rj} = \left\{ \left[\left(c_t^{rj}\right)^{v_1} \left(m_t^{rj}\right)^{v_2} \left(1 - l_t^{rj}\right)^{v_3} \right]^{\rho} + \beta \gamma_t \left(V_{t+1}^{rj}\right)^{\rho} \right\}^{\frac{1}{\rho}},$$

subject to the flow budget constraint

$$c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj} + a_t^{rj} = \frac{1+r_{t-1}}{\gamma_{t-1}} a_{t-1}^{rj} + \xi w_t l_t^{rj} + e_t^j,$$

where a_{t-1}^{rj} denotes his predetermined stock of nonhuman wealth.[6] The retiree receives benefits e_t^j and faces an effective wage rate ξw_t. The parameter $\xi \in (0, 1)$ captures the productivity differential between retirees and workers, and in the equilibrium discussed later ξ will be adjusted so that the labor supply l_t^{rj} is zero. With i_t denoting the nominal interest rate, the term $\frac{i_t}{1+i_t} m_t^{rj}$ describes, in a sense, the "consumption level of real balances," reflecting the fact that real balances are dominated in return by interest-bearing assets. The decision problem gives rise to three first-order conditions. Consumption follows the intertemporal Euler equation

$$c_{t+1}^{rj} = \left[\beta (1+r_t) \left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_2 \rho} \left(\frac{w_t}{w_{t+1}} \right)^{v_3 \rho} \right]^{\sigma} c_t^{rj},$$

whereas upon appropriate substitutions the first-order conditions associated with leisure and real balances have a purely intratemporal representation, i.e.,

$$1 - l_t^{rj} = \frac{v_3}{v_1} \frac{c_t^{rj}}{\xi w_t},$$

$$m_t^{rj} = \frac{v_2}{v_1} \frac{1+i_t}{i_t} c_t^{rj}.$$

Let $\epsilon_t \pi_t$ denote the marginal propensity of retirees to consume out of wealth, where consumption is meant to include the term $\frac{i_t}{1+i_t} m_t^{rj}$. In other words, $\epsilon_t \pi_t$ corresponds to $c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj} = c_t^{rj}(1 + \frac{v_2}{v_1})$. Moreover, with d_t^{rj} and h_t^{rj} denoting the disposable income of a retiree and his stock of human capital, respectively, consider the recursive law of motion for human capital

$$h_t^{rj} = d_t^{rj} + \frac{\gamma_t}{1+r_t} h_{t+1}^{rj},$$

$$d_t^{rj} = \xi w_t l_t^{rj} + e_t^j,$$

which captures the retiree's surviving with probability γ_t. Then, in combination with the flow budget constraint, one can establish that the consumption function and the law of motion for $\epsilon_t \pi_t$ satisfy the relationships

$$c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj} = c_t^{rj} \left(1 + \frac{v_2}{v_1}\right) = \epsilon_t \pi_t \left(\frac{1+r_{t-1}}{\gamma_{t-1}} a_{t-1}^{rj} + h_t^{rj}\right)$$

and

$$\epsilon_t \pi_t = 1 - \left[\left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t}\right)^{v_2 \rho} \left(\frac{w_t}{w_{t+1}}\right)^{v_3 \rho}\right]^{\sigma} \beta^{\sigma} (1+r_t)^{\sigma-1} \gamma_t \frac{\epsilon_t \pi_t}{\epsilon_{t+1} \pi_{t+1}}. \quad \mathbf{(2)}$$

These expressions can be used to establish an analytical expression for the value function V_t^{rj}, which is a key input for the decision problem of the representative worker. In particular, the proportionality between m_t^{rj} and c_t^{rj} (which leads to the "gross" consumption term $c_t^{rj}(1 + \frac{v_2}{v_1})$) ensures that in Appendix A the conjectured solutions for the value functions of the monetary economy can be verified similarly to the nonmonetary model of Gertler.

Decision problem of the representative worker. Similarly, the representative worker maximizes in period t the objective

$$V_t^{wj} = \left\{\left[\left(c_t^{wj}\right)^{v_1} \left(m_t^{wj}\right)^{v_2} \left(1 - l_t^{wj}\right)^{v_3}\right]^{\rho} + \beta \left[\omega_t V_{t+1}^{wj} + (1-\omega_t) V_{t+1}^{rj}\right]^{\rho}\right\}^{\frac{1}{\rho}},$$

subject to the flow budget constraint

$$c_t^{wj} + \frac{i_t}{1+i_t} m_t^{wj} + a_t^{wj} = (1+r_{t-1}) a_{t-1}^{wj} + w_t l_t^{wj} + f_t^j - \tau_t^j,$$

which assumes that the worker was already in the workforce during the period $t-1$.[7] Notice that the return rate associated with a_{t-1}^{wj} is different from that in the previous section because of the discussed asymmetries of insurance possibilities in working age and retirement age. Moreover, the representative worker faces the full wage rate (w_t), receives profits (f_t^j) of imperfectly competitive firms in the intermediate goods sector, and pays lump-sum taxes (τ_t^j).[8] Again, the decision problem gives rise to three first-order conditions. The consumption Euler equation

$$\omega_t c_{t+1}^{wj} + (1-\omega_t)\left(\epsilon_{t+1}\right)^{\frac{\sigma}{1-\sigma}}\left(\frac{1}{\xi}\right)^{v_3} c_{t+1}^{rj}$$
$$= \left[\beta\left(1+r_t\right)\Omega_{t+1}\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_2\rho}\left(\frac{w_t}{w_{t+1}}\right)^{v_3\rho}\right]^{\sigma} c_t^{wj},$$

with associated

$$\Omega_{t+1} = \omega_t + (1-\omega_t)\,\epsilon_{t+1}^{1/(1-\sigma)}\left(\frac{1}{\xi}\right)^{v_3}, \tag{3}$$

is now more complicated, reflecting the possibility that the worker may switch into retirement in the next period. Specifically, the weighting term Ω_{t+1} (which is specific to the solution of the worker's problem) indicates that a worker, when switching into retirement, reaches a state that is characterized by a different effective wage rate (captured by ξ) and, as will become clear, by a different marginal propensity to consume (captured by ϵ_{t+1}). In contrast, the first-order conditions with respect to leisure (adjusted for the absence of ξ) and real balances are unchanged; i.e.,

$$1 - l_t^{wj} = \frac{v_3}{v_1}\frac{c_t^{wj}}{w_t},$$
$$m_t^{wj} = \frac{v_2}{v_1}\frac{1+i_t}{i_t}c_t^{wj}.$$

Let π_t denote the marginal propensity of workers to consume out of wealth, again, inclusive of the term $\frac{i_t}{1+i_t}m_t^{wj}$. Moreover, with d_t^{wj} and h_t^{wj} denoting the disposable income of a worker and his stock of human capital, consider the recursive law of motion

$$h_t^{wj} = d_t^{wj} + \frac{\omega_t}{\Omega_{t+1}}\frac{1}{1+r_t}h_{t+1}^{wj} + \left(1-\frac{\omega_t}{\Omega_{t+1}}\right)\frac{1}{1+r_t}h_{t+1}^{rj},$$
$$d_t^{wj} = w_t l_t^{wj} + f_t^j - \tau_t^j,$$

with h_t^{rj} following the law of motion defined earlier. Then, similar to the retiree's problem, one can verify that the worker's consumption function is given by

$$c_t^{wj} + \frac{i_t}{1+i_t}m_t^{wj} = c_t^{wj}\left(1+\frac{v_2}{v_1}\right) = \pi_t\left(\left(1+r_{t-1}\right)a_{t-1}^{wj} + h_t^{wj}\right).$$

Finally, these relationships are mutually consistent with each other if the marginal propensity to consume out of wealth, π_t, evolves according to

$$\pi_t = 1 - \left[\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{v_2\rho}\left(\frac{w_t}{w_{t+1}}\right)^{v_3\rho}\right]^{\sigma}\beta^{\sigma}\left[\left(1+r_t\right)\Omega_{t+1}\right]^{\sigma-1}\frac{\pi_t}{\pi_{t+1}}. \tag{4}$$

One can show that the marginal propensity of retiress to consume is higher than that of workers ($\epsilon > 1$), implying that $\Omega > 1$. This in turn indicates that workers discount future income streams at an effective interest rate $(1 + r_t)\,\Omega_{t+1}$, which is higher than the pure interest rate, reflecting the expected finiteness of life.

2.3. Aggregation over Retirees and Workers

To characterize aggregate variables, we use the notation introduced in the previous subsections but drop the index j. With the total numbers of retirees and workers in period t being given by N_t^{r} and N_t^{w}, respectively, aggregate labor supply schedules satisfy

$$l_t^{\mathrm{w}} = N_t^{\mathrm{w}} l_t^{wj} = N_t^{\mathrm{w}} \left(1 - \frac{\upsilon_3}{\upsilon_1} \frac{c_t^{wj}}{w_t}\right) = N_t^{\mathrm{w}} - \frac{\upsilon_3}{\upsilon_1} \frac{c_t^{\mathrm{w}}}{w_t}, \tag{5}$$

$$l_t^{\mathrm{r}} = N_t^{\mathrm{r}} l_t^{rj} = N_t^{\mathrm{r}} \left(1 - \frac{\upsilon_3}{\upsilon_1} \frac{c_t^{rj}}{\xi w_t}\right) = N_t^{\mathrm{r}} - \frac{\upsilon_3}{\upsilon_1} \frac{c_t^{\mathrm{r}}}{\xi w_t}, \tag{6}$$

$$l_t = l_t^{\mathrm{w}} + \xi l_t^{\mathrm{r}}. \tag{7}$$

The aggregate stocks of the human capital of retirees and of workers follow the recursive laws of motion

$$h_t^{\mathrm{r}} = d_t^{\mathrm{r}} + \frac{\gamma_t}{(1 + n_t^{\mathrm{r}})\,(1 + r_t)} h_{t+1}^{\mathrm{r}}, \tag{8}$$

$$\begin{aligned} h_t^{\mathrm{w}} = d_t^{\mathrm{w}} &+ \frac{\omega_t}{\Omega_{t+1}} \frac{1}{(1 + n_t^{\mathrm{w}})\,(1 + r_t)} h_{t+1}^{\mathrm{w}} \\ &+ \left(1 - \frac{\omega_t}{\Omega_{t+1}}\right) \frac{1}{(1 + n_t^{\mathrm{r}})\,(1 + r_t)} \frac{1}{\psi_t} h_{t+1}^{\mathrm{r}}, \end{aligned} \tag{9}$$

with the aggregate disposable income terms of the two groups being defined as

$$d_t^{\mathrm{r}} = d_t^{rj} N_t^{\mathrm{r}} = \left(\xi w_t l_t^{rj} + e_t^j\right) N_t^{\mathrm{r}} = \xi w_t l_t^{\mathrm{r}} + e_t, \tag{10}$$

$$d_t^{\mathrm{w}} = d_t^{wj} N_t^{\mathrm{w}} = \left(w_t l_t^{wj} + f_t^j - \tau_t^j\right) N_t^{\mathrm{w}} = w_t l_t^{\mathrm{w}} + f_t - \tau_t. \tag{11}$$

Compared with the laws of motion at the individual level, the two equations (8) and (9) feature the additional discounting terms $1 + n_t^{\mathrm{r}}$ and $1 + n_t^{\mathrm{w}}$, respectively. These terms ensure that the discounted income streams of currently alive retirees and workers do not incorporate contributions of agents who as of today do not yet belong to these two groups. Let a_{t-1}^{r} and a_{t-1}^{w} denote the predetermined levels of aggregate nonhuman wealth of retirees and workers in period t, resulting from savings decisions in period $t - 1$. Then, given the linear structure of individual consumption decisions, aggregate consumption levels of retirees and workers can be written as

$$c_t^{\mathrm{r}}\left(1+\frac{v_2}{v_1}\right)=\epsilon_t\pi_t\left[(1+r_{t-1})\,a_{t-1}^{\mathrm{r}}+h_t^{\mathrm{r}}\right], \tag{12}$$

$$c_t^{\mathrm{w}}\left(1+\frac{v_2}{v_1}\right)=\pi_t\left[(1+r_{t-1})\,a_{t-1}^{\mathrm{w}}+h_t^{\mathrm{w}}\right], \tag{13}$$

where the absence of the term γ_{t-1} in equation (12) reflects the competitive insurance of death probabilities of retirees. To aggregate these two expressions, let $a_t=a_t^{\mathrm{r}}+a_t^{\mathrm{w}}$ and $\lambda_t=a_t^{\mathrm{r}}/a_t$, where λ_t is introduced to summarize compactly the distribution of aggregate nonhuman wealth between retirees and workers. Using these definitions, aggregate consumption (c_t) and aggregate real balances (m_t) can be characterized by the expressions

$$c_t=c_t^{\mathrm{r}}+c_t^{\mathrm{w}}=\frac{1}{1+\frac{v_2}{v_1}}\pi_t\left\{\left[1+(\epsilon_t-1)\,\lambda_{t-1}\right](1+r_{t-1})\,a_{t-1}+\epsilon_t h_t^{\mathrm{r}}+h_t^{\mathrm{w}}\right\}, \tag{14}$$

$$m_t=m_t^{\mathrm{r}}+m_t^{\mathrm{w}}=\frac{1+i_t}{i_t}\frac{v_2}{v_1}c_t. \tag{15}$$

Finally, to characterize the law of motion for λ_t, notice that the aggregate nonhuman wealth of retirees evolves according to

$$\begin{aligned}\lambda_t a_t&=\lambda_{t-1}\,(1+r_{t-1})\,a_{t-1}+d_t^{\mathrm{r}}-c_t^{\mathrm{r}}-\frac{i_t}{1+i_t}m_t^{\mathrm{r}}\\&\quad+(1-\omega_t)\left[(1-\lambda_{t-1})\,(1+r_{t-1})\,a_{t-1}+d_t^{\mathrm{w}}-c_t^{\mathrm{w}}-\frac{i_t}{1+i_t}m_t^{\mathrm{w}}\right],\end{aligned}$$

whereas the aggregate nonhuman wealth of workers follows the law of motion

$$(1-\lambda_t)\,a_t=\omega_t\left[(1-\lambda_{t-1})\,(1+r_{t-1})\,a_{t-1}+d_t^{\mathrm{w}}-c_t^{\mathrm{w}}-\frac{i_t}{1+i_t}m_t^{\mathrm{w}}\right].$$

Combining these two expressions yields

$$\lambda_t a_t=\omega_t\left\{(1-\epsilon_t\pi_t)\left[\lambda_{t-1}\,(1+r_{t-1})\,a_{t-1}+h_t^{\mathrm{r}}\right]-\left(h_t^{\mathrm{r}}-d_t^{\mathrm{r}}\right)\right\}+(1-\omega_t)a_t. \tag{16}$$

2.4. Firms

The supply side of the economy has a simple New Keynesian structure, in the spirit of Clarida et al. (1999) and Woodford (2003). Specifically, we combine the assumption of monopolistic competition in the spirit of Dixit and Stiglitz (1977) with Calvo-type nominal rigidities in order to generate short-run dynamics

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comparable to a New Keynesian Phillips curve. Moreover, the production of capital goods is subject to adjustment costs, leading to a persistent reaction of investment dynamics to shocks hitting the economy.

Final goods sector. There is a continuum of intermediate goods, indexed by $z \in [0, 1]$, that are transformed into a homogenous final good according to the technology

$$y_t = \left[\int_0^1 y_t(z)^{\frac{\theta-1}{\theta}} dz \right]^{\frac{\theta}{\theta-1}}.$$

The final goods sector is subject to perfect competition, giving rise to a demand function for the representative intermediate good z,

$$y_t(z) = \left[\frac{P_t(z)}{P_t} \right]^{-\theta} y_t,$$

where $P_t(z)$ and P_t denote the price of good z and the average price level of intermediate goods, respectively, and $\theta > 1$ denotes the price elasticity of demand. Reflecting the CES structure of the technology in the final goods sector, P_t is given by

$$P_t = \left[\int_0^1 P_t(z)^{1-\theta} dz \right]^{\frac{1}{1-\theta}}.$$

Intermediate goods sector. The representative firm produces the intermediate good z with the technology

$$y_t(z) = [X_t l_t(z)]^{\alpha} k_t(z)^{1-\alpha},$$

where $l_t(z)$ and $k_t(z)$ denote the input levels of labor and capital and X_t denotes the exogenously determined level of labor augmenting technical progress. For simplicity, we assume that X_t grows at a constant rate, i.e., $X_t = (1 + x)X_{t-1}$, with $x > 0$. Markets for the two inputs are competitive; i.e., the real wage rate w_t and the real rental rate r_t^k are taken as given in the production of good z. Cost minimization implies that

$$\frac{w_t l_t(z)}{\alpha y_t(z)} = \frac{r_t^k k_t(z)}{(1-\alpha) y_t(z)} = \mathrm{mc}_t,$$

where mc_t denotes real marginal costs, which are identical across firms. Profits of firm z are given by

$$f_t(z) = \left[\frac{P_t(z)}{P_t} - \mathrm{mc}_t \right] y_t(z).$$

Each firm has price-setting power in its output market. In line with Calvo (1983), in each period only a fraction $1 - \zeta$ of firms can reset their price optimally, whereas for a fraction ζ of firms the price remains unchanged. Let $P_t^*(z)$ denote the optimally reset price in period t for a firm that can change its price. Reflecting the forward-looking dimension of the price-setting decision under the Calvo constraint, $P_t^*(z)/P_t$ evolves over time according to[9]

$$\frac{P_t^*(z)}{P_t} = \frac{\theta}{(\theta - 1)} \frac{\sum_{i=0}^{\infty} (\zeta\beta)^i \left(\frac{1}{P_{t+i}}\right)^{1-\theta} y_{t+i} \mathrm{mc}_{t+i} \frac{P_{t+i}}{P_t}}{\sum_{i=0}^{\infty} (\zeta\beta)^i \left(\frac{1}{P_{t+i}}\right)^{1-\theta} y_{t+i}}.$$

Capital goods. There exists a continuum of capital-goods-producing firms, indexed by $u \in [0, 1]$, renting out capital to firms in the intermediate goods sector. In each period, after the production of intermediate and final goods is completed, the representative capital-goods-producing firm combines its existing capital stock $k_t(u)$ with investment goods $i_t^k(u)$ to produce new capital goods $k_{t+1}(u)$ according to the constant returns technology

$$k_{t+1}(u) = \phi \left[\frac{i_t^k(u)}{k_t(u)}\right] k_t(u) + (1 - \delta)k_t(u),$$

with $\phi'() > 0, \phi''() < 0$. Let $p_t^k = P_t^k/P_t$ denote the relative price of capital goods in terms of final output. Then the optimal choice of investment levels, $i_t^k(u)$, leads to the first-order condition

$$p_t^k \phi' \left[\frac{i_t^k(u)}{k_t(u)}\right] = 1.$$

Let $i^k(u)/k(u)$ denote the investment–capital ratio at the firm level along a balanced growth path. It is assumed that the function ϕ satisfies the relations

$$\phi \left[\frac{i^k(u)}{k(u)}\right] = \frac{i^k(u)}{k(u)}, \ \phi' \left[\frac{i^k(u)}{k(u)}\right] = 1,$$

which are well known from the q-theory of investment.

Aggregate relationships and resource constraint. At the aggregate level the capital stock is a predetermined variable, leading to

$$k_{t-1} = \int_0^1 k_t(z)dz = \int_0^1 k_t(u)du.$$

Moreover,

$$i_t^k = \int_0^1 i_t^k(u)du,$$

and

$$l_t = l_t^{\mathrm{w}} + \xi l_t^{\mathrm{r}} = \int_0^1 l_t(z)dz,$$

whereas aggregate output and profits are given by

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$$y_t = \left[\int_0^1 y_t(z)^{\frac{\theta-1}{\theta}} dz\right]^{\frac{\theta}{\theta-1}}, \quad \text{with } y_t(z) = [X_t l_t(z)]^{\alpha} k_t(z)^{1-\alpha}, \tag{17}$$

$$f_t = \int_0^1 f_t(z) dz = \int_0^1 \left[\frac{P_t(z)}{P_t} - \text{mc}_t\right] y_t(z) dz. \tag{18}$$

The capital–labor ratio in the intermediate goods sector will be identical across firms, implying that

$$\frac{k_{t-1}}{l_t} = \frac{w_t}{r_t^k} \frac{1-\alpha}{\alpha}, \tag{19}$$

whereas real marginal costs can be rewritten as

$$\text{mc}_t = \left(\frac{w_t}{\alpha X_t}\right)^{\alpha} \left[\frac{r_t^k}{(1-\alpha)}\right]^{1-\alpha}. \tag{20}$$

In any symmetric equilibrium, $P_t^*(z) = P_t^*$ must be identical across all firms that have the chance to adjust their prices, leading to

$$\frac{P_t^*}{P_t} = \frac{\theta}{(\theta-1)} \frac{\sum_{i=0}^{\infty} (\zeta\beta)^i \left(\frac{1}{P_{t+i}}\right)^{1-\theta} y_{t+i} \text{mc}_{t+i} \frac{P_{t+i}}{P_t}}{\sum_{i=0}^{\infty} (\zeta\beta)^i \left(\frac{1}{P_{t+i}}\right)^{1-\theta} y_{t+i}}, \tag{21}$$

whereas the evolution of the price level can be written as

$$P_t = \left[\zeta P_{t-1}^{1-\theta} + (1-\zeta) P_t^{*^{1-\theta}}\right]^{\frac{1}{1-\theta}}. \tag{22}$$

In the capital goods sector the investment–capital ratio is identical across firms, leading to

$$k_t = \phi\left(\frac{i_t^k}{k_{t-1}}\right) k_{t-1} + (1-\delta) k_{t-1}, \tag{23}$$

$$1 = p_t^k \phi'\left(\frac{i_t^k}{k_{t-1}}\right). \tag{24}$$

The aggregate resource constraint of the economy is given by

$$y_t = c_t + g_t + i_t^k, \tag{25}$$

where g_t denotes government expenditures in terms of the final output good.

2.5. Government

To discuss the role of the government sector, it is convenient to start with the flow budget constraint of the government in nominal terms (denoted by capital letters),

$$M_t + B_t = M_{t-1} + (1 + i_{t-1}) B_{t-1} + G_t + E_t - T_t.$$

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With

$$1 + i_t = (1 + r_t)\left(\frac{P_{t+1}}{P_t}\right), \quad \textbf{(26)}$$

the budget constraint can be rewritten in real terms as

$$b_t = (1 + r_{t-1})\left(b_{t-1} + \frac{1}{1 + i_{t-1}} m_{t-1}\right) + g_t + e_t - \tau_t - m_t. \quad \textbf{(27)}$$

Real government expenditures (g_t) are assumed to be exogenously given. The path of aggregate real benefits (e_t) is determined by the replacement rate μ_t between individual benefits and the real wage, i.e.,

$$\mu_t = \frac{e_t^j}{w_t} \Rightarrow e_t = e_t^j N_t^{\mathrm{r}} = \mu_t w_t N_t^{\mathrm{r}}. \quad \textbf{(28)}$$

Notice that the budget of the pension system is embedded in the overall budget constraint (27). In present value terms, the pension system is run on a PAYGO basis, because all benefits received by retirees are backed by taxes (which are entirely paid by workers), and not by proceeds from investments in the economy's capital stock. Real government debt (b_t) and real capital holdings ($p_t^k k_t$) are perceived as perfect substitutes by the private sector. This leads to the definition of total private sector nonhuman wealth

$$a_t = p_t^k k_t + b_t + \frac{m_t}{1 + i_t}, \quad \textbf{(29)}$$

which is supported by the arbitrage relationship between the return rates on real government debt and real capital holdings,

$$1 + r_t = \frac{r_{t+1}^k + p_{t+1}^k(1 - \delta)}{p_t^k}. \quad \textbf{(30)}$$

To close the system, we assume that fiscal policy follows a stylized feedback rule that stabilizes a certain target level b^* of the debt ratio (b_t/y_t) by variations in the remaining free fiscal instrument τ_t such that

$$\frac{\tau_t}{y_t} = \tau^* + \gamma_1\left(\frac{b_t}{y_t} - b^*\right) + \gamma_2\left(\frac{b_t}{y_t} - \frac{b_{t-1}}{y_{t-1}}\right), \quad \textbf{(31)}$$

where τ^* denotes the tax ratio (τ_t/y_t) that corresponds to b^* along a balanced growth path, γ_1 denotes the direct feedback parameter to counteract deviations of the debt ratio from its target, and γ_2 controls the smoothness of this process. This specification is in line with the broad discussion given in Mitchell et al. (2000) and may qualify as a simple benchmark. However, alternative fiscal closures (in terms of residual fiscal instruments as well as target variables) can be imagined that could easily replace (31).

As regards monetary policy, we consider as a benchmark the flexible-price version of our model, in which the central bank sets the nominal interest rate so that price stability prevails in all periods (and the equilibrium real interest rate coincides with the natural rate). Alternatively, we consider a scenario with sticky prices in which the behavior of monetary policy is modeled through a Taylor-type feedback rule, assuming that the target inflation rate is equal to zero ($\pi^{p*} = 0$).[10] According to this rule, the nominal interest rate is set as a function of the current inflation rate (π_t^p), the output gap ($\widetilde{y}_t = \ln(\overline{y_t}/\overline{y})$, where $\overline{y}$ denotes the steady-state level of the detrended economy established in the following), and the previous value of the nominal interest rate (with weight ρ), i.e.,

$$i_t = \rho i_{t-1} + (1-\rho)\left(r_t + \gamma_\pi \pi_t^p + \gamma_y \widetilde{y}_t\right). \tag{32}$$

3. GENERAL EQUILIBRIUM

At equilibrium, government actions and optimizing decisions of workers, retirees, and firms must be mutually consistent at the aggregate level. In sum, an *equilibrium* consists for all periods t of sequences of endogenous variables $\{\epsilon_t, \pi_t, \Omega_t, l_t^{\mathrm{w}}, l_t^{\mathrm{r}}, l_t, h_t^{\mathrm{w}}, h_t^{\mathrm{r}}, d_t^{\mathrm{w}}, d_t^{\mathrm{r}}, c_t^{\mathrm{w}}, c_t^{\mathrm{r}}, c_t, y_t, k_t, f_t, \mathrm{mc}_t, w_t, r_t^k, i_t^k, p_t^k, p_t^*, p_t, i_t, r_t, \lambda_t, a_t, b_t, m_t, \tau_t, e_t\}$ that satisfy the system of equations (2)–(32), taking as given exogenous sequences of policy-related variables $\{\tau^*,\ b^*,\ \mu_t\}$, demographic processes $\{n_t^{\mathrm{w}}, \omega_t, \gamma_t\}$, productivity growth x, and appropriate initial conditions for N_t^{w}, N_t^{r}, X_t, and all endogenous state variables.[11]

As long as one assumes $x > 0$, $n > 0$, the economy is subject to ongoing exogenous growth. Hence, we consider from now on a detrended version of (2)–(32) that expresses all unbounded variables in terms of efficiency units per worker. For the detrended equation system, we use the following notational conventions. Consider generic variables $v_t \in \{c_t, y_t, k_t, f_t, r_t^k, i_t^k, a_t, b_t, m_t, \tau_t, e_t\}$, $v_t^{\mathrm{w}} \in \{h_t^{\mathrm{w}}, d_t^{\mathrm{w}}, c_t^{\mathrm{w}}, \}$, and $v_t^{\mathrm{r}} \in \{ h_t^{\mathrm{r}}, d_t^{\mathrm{r}}, c_t^{\mathrm{r}}\}$. Then

$$\frac{v_t}{N_t^{\mathrm{w}} X_t} = \overline{v_t}, \quad \frac{v_t^{\mathrm{w}}}{N_t^{\mathrm{w}} X_t} = \overline{v_t^{\mathrm{w}}}, \quad \frac{v_t^{\mathrm{r}}}{N_t^{\mathrm{w}} X_t} = \frac{v_t^{\mathrm{r}}}{N_t^{\mathrm{r}} X_t}\frac{N_t^{\mathrm{r}}}{N_t^{\mathrm{w}}} = \overline{v_t^{\mathrm{r}}}\psi_t.$$

Moreover, reflecting the properties of labor-augmenting technical progress, we specify the real wage and the variables related to the labor supply as

$$\frac{w_t}{X_t} = \overline{w_t}, \quad \frac{l_t}{N_t^{\mathrm{w}}} = \overline{l_t}, \quad \frac{l_t^{\mathrm{w}}}{N_t^{\mathrm{w}}} = \overline{l_t^{\mathrm{w}}}, \quad \frac{l_t^{\mathrm{r}}}{N_t^{\mathrm{w}}} = \frac{l_t^{\mathrm{r}}}{N_t^{\mathrm{r}}}\frac{N_t^{\mathrm{r}}}{N_t^{\mathrm{w}}} = \overline{l_t^{\mathrm{r}}}\psi_t.$$

Appendix B summarizes the detrended counterparts of all equations (2)–(32) that make up the dynamic system we study from now on. Based on this representation, it is straightforward to characterize steady states of the detrended economy. A compact summary can be established if one invokes a number of steady-state features. Let the variables without time subscript refer to steady-state values. In particular, $P^* = P = P(z) = 1$, $\overline{y} = \overline{y(z)}$, $\overline{l} = \overline{l(z)}$, $\overline{k(z)} = \overline{k}/(1+n)(1+x)$, $\mathrm{mc} = (\theta - 1)/\theta$, and $\overline{f} = (1/\theta)\overline{y}$, as well as $p^k = 1$, $r = i$, and $r^k = r + \delta$, leading to the steady-state equations

$$\epsilon\pi = 1 - \left[\left(\frac{1}{1+x}\right)^{v_3\rho}\right]^{\sigma} \beta^{\sigma}(1+r)^{\sigma-1}\gamma,$$

$$\pi = 1 - \left[\left(\frac{1}{1+x}\right)^{v_3\rho}\right]^{\sigma} \beta^{\sigma}[(1+r)\Omega]^{\sigma-1},$$

$$\Omega = \omega + (1-\omega)\,\epsilon^{\frac{1}{1-\sigma}}\left(\frac{1}{\xi}\right)^{v_3},$$

$$\overline{l^{\mathrm{r}}} = 1 - \frac{v_3}{v_1}\frac{\overline{c^{\mathrm{r}}}}{\xi\overline{w}}, \quad \overline{l^{\mathrm{w}}} = 1 - \frac{v_3}{v_1}\frac{\overline{c^{\mathrm{w}}}}{\overline{w}}, \quad \overline{l} = \overline{l^{\mathrm{w}}} + \xi\overline{l^{\mathrm{r}}}\psi,$$

$$\overline{h^{\mathrm{r}}} = \xi\overline{w}\overline{l^{\mathrm{r}}} + \mu\overline{w} + \gamma\frac{1+x}{1+r}\overline{h^{\mathrm{r}}},$$

$$\overline{h^{\mathrm{w}}} = \overline{w}\overline{l^{\mathrm{w}}} + (1/\theta)\overline{y} - \overline{\tau} + \frac{\omega}{\Omega}\frac{1+x}{1+r}\overline{h^{\mathrm{w}}} + \left(1 - \frac{\omega}{\Omega}\right)\frac{1+x}{1+r}\overline{h^{\mathrm{r}}},$$

$$\overline{c^{\mathrm{r}}}\left(1 + \frac{v_2}{v_1}\right) = \epsilon\pi\left[\frac{1+r}{(1+n)(1+x)}\lambda\overline{a} + \overline{h^{\mathrm{r}}}\right],$$

$$\overline{c^{\mathrm{w}}}\left(1 + \frac{v_2}{v_1}\right) = \pi\left[\frac{1+r}{(1+n)(1+x)}(1-\lambda)\overline{a} + \overline{h^{\mathrm{w}}}\right],$$

$$\overline{c} = \overline{c^{\mathrm{w}}} + \psi\overline{c^{\mathrm{r}}}, \quad \overline{a} = \overline{k} + b^*\overline{y} + \frac{v_2}{v_1}\frac{1}{r}\overline{c},$$

$$\lambda\overline{a} = \omega\left\{(1-\epsilon\pi)\left[\frac{\lambda(1+r)}{(1+n)(1+x)}\overline{a} + \overline{h^{\mathrm{r}}}\psi\right]\right.$$

$$\left. - (\overline{h^{\mathrm{r}}} - \xi\overline{w}\overline{l^{\mathrm{r}}} - \mu\overline{w})\psi\right\} + (1-\omega)\overline{a},$$

$$\overline{y} = (\overline{l})^{\alpha}\left[\frac{\overline{k}}{(1+n)(1+x)}\right]^{1-\alpha} = \overline{c} + \overline{g} + \left[1 - \frac{1-\delta}{(1+n)(1+x)}\right]\overline{k},$$

$$\frac{\theta-1}{\theta} = \left(\frac{\overline{w}}{\alpha}\right)^{\alpha}\left(\frac{r+\delta}{1-\alpha}\right)^{1-\alpha}, \quad \frac{1}{(1+n)(1+x)}\frac{\overline{k}}{\overline{l}} = \frac{\overline{w}}{r+\delta}\frac{1-\alpha}{\alpha},$$

TABLE 1. Demographic parameters

Parameter	Value
Growth rate of working-age population, n	0.004
Retirement probability of workers, $1-\omega$	0.020
Implied average working period, $T^{\mathrm{w}} = 1/(1-\omega)$	50
Probability of death of retirees, $1-\gamma$	0.069
Implied average retirement period, $T^{\mathrm{r}} = 1/(1-\gamma)$	14.5
Implied old age dependency ratio, $\psi = \frac{1-\omega}{1+n-\gamma}$	0.27

$$b^{*}\overline{y} = \frac{(1+n)(1+x)}{1+r-(1+n)(1+x)}\left(\overline{\tau} - \overline{g} - \mu\overline{w}\psi\right) + \frac{(1+n)(1+x)-1}{1+r-(1+n)(1+x)}\frac{v_2}{v_1}\frac{1+r}{r}\overline{c}.$$

In sum, these equations constitute a system in 18 equations and 18 unknown endogenous variables, i.e., $\{\epsilon, \pi, \Omega, \overline{l^{\mathrm{r}}}, \overline{l^{\mathrm{w}}}, \overline{l}, \overline{h^{\mathrm{r}}}, \overline{h^{\mathrm{w}}}, \overline{c^{\mathrm{r}}}, \overline{c^{\mathrm{w}}}, \overline{c}, \overline{y}, \overline{a}, \overline{k}, \overline{w}, r, \lambda, \overline{\tau}\}$. Given the recursive structure of the system, it is computationally not demanding to solve for all steady-state values.

4. CALIBRATION AND DEMOGRAPHIC TRENDS

4.1. Calibration

We calibrate the system of steady-state equations to match key features of annual euro area data, taking, in particular, recent demographic observations until 2008 as a benchmark, as provided by the comprehensive 2009 Ageing Report prepared by the European Commission and published in European Economy (2009). Tables 1–3 summarize our assumptions concerning the initial choices of demographic variables, the structural parameters, and the steady-state relevant policy-related variables, respectively. When used within the set of steady-state equations, these assumptions give rise to steady-state values of the endogenous variables (or ratios of them) as summarized in Table 4. Although all our assumptions are quantitatively in line with the related literature, it is worth making a number of comments that focus, in particular, on the demographic aspects of the model.

First, the demographic assumptions in Table 1 closely match euro area characteristics reported for the year 2008.[12] Because our model features only working age and retirement age, the choice of n corresponds to the growth rate of the working-age population, which is reported as 0.4%. Reflecting well-known properties of the geometric distribution, the total average lifetime (T) in our model is given by $T = 1/(1-\omega) + 1/(1-\gamma) = T^{\mathrm{w}} + T^{\mathrm{r}}$. In the data, working age covers the years 15–64, whereas retirement age is defined as 65 years and above. Life expectancy at birth is reported as 79.5 years, and our calibration of ω and γ corrects for the absence of young people below 15 in our model; i.e., $T^{\mathrm{w}} + T^{\mathrm{r}} = 64.5$.

TABLE 2. Structural parameters

Parameter	Value
Intertemporal elasticity of substitution, σ	1/3
Discount factor, β	0.99
Cobb–Douglas share of labor, α	2/3
Relative productivity of retirees, ξ	0.325
Depreciation rate of capital, δ	0.05
Growth rate of technological progress, x	0.01
Elasticity of demand (intermediate goods), θ	10
Preference parameter: consumption, υ_1	0.64
Preference parameter: real balances, υ_2	0
Preference parameter: leisure, υ_3	0.358

TABLE 3. Steady-state relevant policy parameters

Parameter	Value
Debt-to-output-ratio, b^*	0.7
Government spending share, g/y	0.18
Replacement rate, $\mu = e^j/w$	0.47

The (steady-state) old-age dependency ratio of $\psi = 0.27$ implied by the model matches exactly the old-age dependency ratio reported for the euro area in 2008.

Second, the relative productivity parameter ξ has been set to ensure that the participation rate of workers is 0.70, in line with the empirical value reported for 2008, whereas the implied participation rate of retirees is approximately zero. The latter result may seem restrictive, but it does respect the cutoff feature of the empirical data set, namely that all persons at age 65 or above are assumed to have retired. Third, to facilitate the comparison with related literature, all our quantitative results summarized in the following are obtained from the cashless limit of the model by setting $\upsilon_2 = 0$.[13] Fourth, in calibration exercises of this type there is some leeway to fix the long-run *level* of the real interest rate. Our numerical choices for the crucial parameters β, σ, x, δ are in line with the literature, as is the implied value of r, which amounts to 3.9%.

Fifth, concerning the fiscal closure of the model, we specify the share of government spending and the debt-to-output ratio as 0.18 and 0.7, respectively. Combined with a value of 0.47 for the replacement rate, this leads to a share of total retirement benefits in output (e/y) of 0.11, in line with euro area evidence.[14] Reflecting the residual role of taxes in our fiscal specification, these assumptions imply a share of taxes in output (τ/y) of 0.31.

Finally, the transitional dynamics depends on the reaction functions of monetary and fiscal policy, the assumed degree of nominal rigidity, and the specification of

TABLE 4. Endogenous variables

Parameter	Value
Real interest rate, r	0.039
Share of consumption in output, c/y	0.60
Share of investment in output, i^k/y	0.22
Share of taxes in output, τ/y	0.31
Share of total benefits in output, e/y	0.11
Capital–output ratio, k/y	3.50
Share of real balances in output, m/y	0.05
Distribution of wealth, λ	0.23
Participation rate of workers, $\overline{l^{\mathrm{w}}} = l^{\mathrm{w}}/N^{\mathrm{w}}$	0.70
Participation rate of retirees, $\overline{l^{\mathrm{r}}} = l^{\mathrm{r}}/N^{\mathrm{r}}$	0.01
Consumption share of workers, c^{w}/y	0.47
Consumption share of retirees, $\psi c^{\mathrm{r}}/y = c/y - c^{\mathrm{w}}/y$	0.13
Propensity to consume out of wealth (workers), π	0.05
Propensity to consume out of wealth (retirees), $\epsilon\pi$	0.09
Relative discount term, Ω	1.05

TABLE 5. Parameters responsible for adjustment dynamics

Parameter	Value
Direct adjustment parameter in debt rule, γ_1	0.04
Smoothing parameter in debt rule, γ_2	0.3
Inertial parameter in the Taylor rule, ρ	0.7
Inflation coefficient in the Taylor rule, γ_π	1.5
Output gap coefficient in the Taylor rule, γ_y	0
Calvo survival probability of contracts, ζ	0.2
Elasticity of investment function ($\eta = -\frac{\phi''(v)}{\phi'(v)}v$), η	0.25

the adjustment costs of the investment function. Table 5 summarizes the benchmark specifications that we use in the remainder of the paper.

Concerning the choices made in Table 5, four comments are worth making. First, fiscal feedback rules, in general, are much more difficult to pin down than monetary rules, reflecting the wide range of conceivable fiscal instruments and closure specifications. Specifically, in our particular fiscal specification, different pairs of γ_1 and γ_2 affect the speed of adjustment, although the shape of impulse responses is qualitatively robust to perturbations of the chosen parameter values. The particular numerical values of γ_1 and γ_2 are taken from the detailed analysis of Mitchell et al. (2000). Second, given the supply-side nature of demographic shocks and the slow materialization of their effects, the otherwise standard Taylor-type rule for monetary policy under sticky prices is specified as a pure inflation-targeting rule. Third, numerical specifications of the properties of the investment

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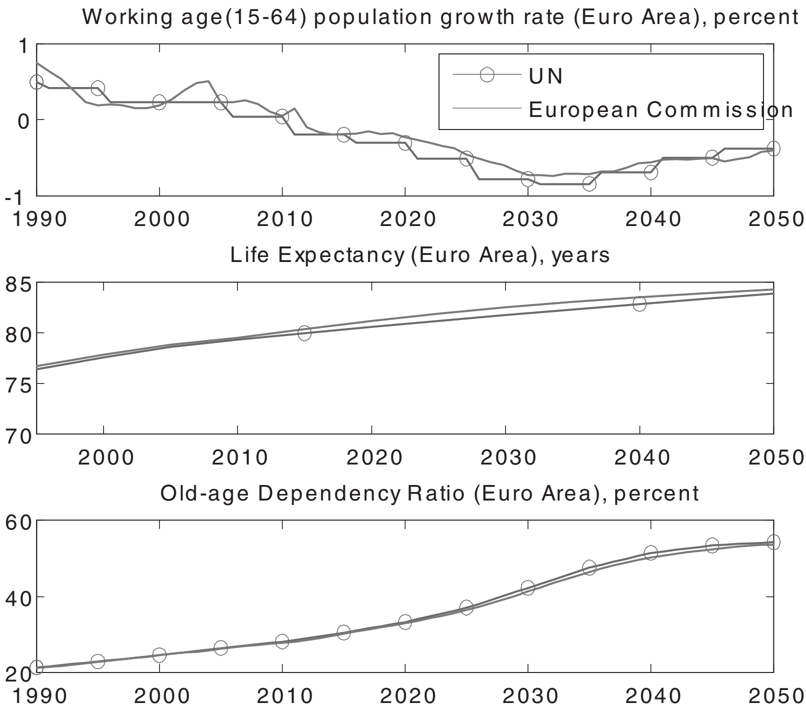

FIGURE 1. Past and projected demographic developments in the euro area.

function $\phi(.)$ differ greatly across the literature, as discussed, for example, in Gali et al. (2004). Our specification of $\eta = 0.25$ is in line with Dotsey (1999), but we comment on this choice in further detail later. Fourth, the assumption of $\zeta = 0.2$ implies an average duration of prices of 1.25 years, which is in accordance with euro area empirical evidence, as summarized in Altissimo et al. (2006).

4.2. Demographic Trends

From the stylized perspective of the model developed in Section 2, two aspects of projected demographic developments in the euro area are particularly noteworthy, as summarized in Figure 1.

First, the growth rate of the working-age population is predicted to decline significantly over the next decades. Second, the average life expectancy at birth is predicted to increase further in the future. The magnitudes of these predicted developments are substantial. When the years 2008 and 2030 (which defines the policy horizon considered in the final part of this paper) are compared, the growth rate of working-age population is projected to decline by one percentage point from $+0.4\%$ to -0.6%, whereas life expectancy is projected to increase by 3.4 years, reaching 82.9 years in 2030. In combination, these two developments lead to a significant increase in the old-age dependency ratio, which is projected to increase by 13 percentage points, reaching 40% in 2030. As verified in the following, our model matches this increase.[15]

5. COMPARATIVE STATICS EFFECTS OF DEMOGRAPHIC CHANGES

This section identifies general equilibrium repercussions of the previously summarized demographic changes, focusing on the implications for the long-run level of the equilibrium real interest rate. The section takes a comparative statics perspective and explains the key channels operating within the system of steady-state equations of our model by reporting long-run predictions for different scenarios. Our reasoning starts out from the insight in economic growth theory that demographic variables such as population growth rate are a key driving force for the long-run real interest rate. Whereas in models without a detailed life-cycle structure the relationship between demographics and the interest rate is rather mechanical, this is, by construction, different in our model. The two main demographic trends, namely the slowdown in population growth and the increase in longevity, tend to decrease the long-run interest rate. Yet the strength of this predicted decline depends sensitively on various features, such as the future design of pension payments, the retirement age of the working-age population, and the overall degree of fiscal consolidation.[16]

In particular, policy assumptions concerning future pension systems turn out to be important. To illustrate this insight, we distinguish between two different types of scenarios in which the rising old-age dependency ratio does or does not lead to changes in the existing PAYGO pension system (modeled via adjustments in the replacement rate). For the first scenario type, we assume that the replacement rate decreases endogenously so that the aggregate benefits–output ratio remains unchanged. This assumption can be interpreted as a strengthening of privately funded elements, resulting from a pension system reform that introduces a ceiling on the overall tax-financed redistribution between workers and retirees. For the second scenario type, we make the opposite assumption that the replacement rate remains constant, leading to a rise in the aggregate benefits–output ratio. This assumption models in a simple way a "no reform" scenario that extrapolates the existing pension system into the future, leading to a higher tax burden on workers. To distinguish between these two "extreme" policy scenarios is instructive because the distinctly different incentives for savings at the individual level generate sizably different degrees of downward pressure on the long-run real interest rate.

All reported comparative statics exercises are driven by exogenous demographic changes taken from a comparison between the euro area values projected for the year 2030 and the benchmark values corresponding to the year 2008. This is done for simple illustration only; Figure 1 indicates that there is no reason to believe that in 2030 a new steady state will be reached. In other words, this section illustrates qualitatively how the model works, before we quantify a particular empirical scenario in Section 6.

TABLE 6. Comparative statics effects of demographic changes

	Benchmark	I	II	III	IV	V	VI
n	0.004	−0.006	0.004	−0.006	−0.006	−0.006	−0.006
T^{w}	50	50	50	50	53.4	50	50
T^{r}	14.5	14.5	17.9	17.9	14.5	17.9	17.9
e/y	0.11	0.11	0.11	0.11	0.11	0.16	0.16
μ	0.47	0.41	0.39	0.33	0.43	0.47	0.47
x	0.01	0.01	0.01	0.01	0.01	0.01	0.01
b^*	0.7	0.7	0.7	0.7	0.7	0.7	1
ψ	0.27	0.32	0.33	0.40	0.30	0.40	0.40
r	0.039	0.030	0.036	0.028	0.029	0.034	0.036
c/y	0.60	0.61	0.59	0.61	0.61	0.62	0.63
c^{w}/y	0.47	0.47	0.44	0.44	0.47	0.43	0.43
$\psi c^{\mathrm{r}}/y$	0.13	0.15	0.15	0.17	0.14	0.20	0.20
i^k/y	0.22	0.21	0.23	0.21	0.21	0.20	0.19
τ/y	0.31	0.31	0.30	0.31	0.31	0.36	0.37
$l^{\mathrm{w}}/N^{\mathrm{w}}$	0.70	0.69	0.70	0.70	0.69	0.71	0.71
λ	0.23	0.24	0.26	0.28	0.22	0.29	0.30

5.1. Endogenous Replacement Rate

The likely decline in the long-run real interest rate is particularly pronounced in scenarios in which the replacement rate (μ) declines endogenously so that the aggregate benefits–output ratio (e/y) remains unchanged. To disentangle the contributions of the slowdown in working-age population growth and the rise in longevity, we consider first these two trends in isolation (Scenarios I, II) before we discuss them in combination (Scenario III). Finally, we also consider variations in the retirement age (Scenario IV).

Scenario I: Slowdown of working-age population growth. Scenario I isolates the effects resulting from the slowdown in working-age population growth. As summarized in Table 6, Scenario I has the key feature that, relative to the benchmark calibration, the decline in population growth by 1 percentage point reduces the equilibrium interest rate by about 0.9 percentage points, from 3.9% to 3.0%, with little movement in the other endogenous variables or ratios of variables reported in Table 6. Because in long-run comparison both groups of agents grow at the same (lower) rate, there is little scope for life-cycle effects in individual consumption and savings behavior. Although the share of the consumption of workers in total output (c^{w}/y) stays unchanged, there is a small increase in the aggregate consumption share (c/y). The latter effect is largely driven via equation (1) by the increase in the old-age dependency ratio (ψ) by 5 percentage points (which implies a shift of consumption toward agents with a higher propensity to consume), and not by changes in individual savings patterns. Because of these

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features, the findings are similar to predictions from infinite-horizon models with a single representative agent.

Scenario II: Increase in longevity (at unchanged retirement age). Scenario II isolates the effects resulting from a rise in longevity. Importantly, it is assumed that the expected number of years spent in the workforce remains unchanged (as captured by maintaining $T^{\mathrm{w}} = 50$), i.e., the expected retirement age is kept unchanged. Scenario II has the key feature that, relative to the benchmark, the increase in longevity by 3.4 years reduces the equilibrium interest rate by about 30 basis points, from 3.9% to 3.6%. By construction, this decline in r results entirely from the life-cycle features of the model. In particular, the increase in longevity leads to a significant increase in the old-age dependency ratio by about 6 percentage points, whereas the replacement rate falls substantially by 8 percentage points to keep the benefits–output ratio constant. This drop in the replacement rate strongly increases savings incentives at working age, as can be inferred from the decline of c^{w}/y by 3 percentage points and an associated increase in the investment share (i^k/y). In sum, these changes in consumption and savings patterns over the life cycle reduce the long-run interest rate, and the mechanism behind this reduction differs from that in Scenario I.

Scenario III: Slowdown of working-age population growth and increase in longevity (at unchanged retirement age). Scenario III combines the two partial demographic scenarios of Scenarios I and II; i.e., it considers not only the slowdown in working-age population growth but also the projected increase in life expectancy, assuming again an unchanged retirement age. Not surprisingly, the combined effect of these two demographic changes on the interest rate is stronger than in either of the first two scenarios in isolation, leading to a decrease of r by 1.1 percentage points from 3.9% to 2.8% in comparison with the benchmark. This finding reflects that the model is capable of adding life-cycle aspects to the predictions from conventional infinite-horizon models.

Scenario IV: Slowdown of working-age population growth and combined increase in longevity and retirement age. To shed further light on the findings of Scenario III, Scenario IV maintains the assumptions concerning the slowdown in working-age population growth and the rise in longevity, but it is now assumed that the additional 3.4 years are entirely spent at working age. In other words, the retirement age is adjusted upward to keep the expected number of years spent in retirement identical to the benchmark. This measure undoes to a large extent the life-cycle effects on r discussed in Scenario II; i.e., the variables r, ψ, μ, and λ revert largely back to the values established in Scenario I of a pure slowdown in population growth.

To conclude this subsection, it is worth pointing out that in Scenarios I–IV, the assumption of an endogenously determined decline in the replacement rate stabilizes not only the share of aggregate benefits in output (e/y) but also the tax share (τ/y). In all scenarios, the government expenditure ratio (g/y) and the target level of the debt-to-output ratio (b^*) have been held constant. Because outstanding government debt needs to be backed by appropriate primary surpluses, the constancy of e/y also leaves the tax share τ/y unchanged.

5.2. Constant Replacement Rate

The implications of demographic changes are different under alternative scenarios in which the replacement rate remains unchanged. Given the rise in the old-age dependency ratio, this leads to a higher share of aggregate benefits in output and implies a higher tax burden on workers. In other words, the additional pension burden resulting from demographic changes is now financed out of additional transfers from workers to retirees within the unchanged PAYGO system, whereas in Scenarios I–IV it is financed from higher savings of workers in anticipation of lower pensions. This replacement of additional savings by additional transfers in scenarios with a constant replacement rate mitigates the effects of demographic changes on the long-run interest rate, as shown in Scenario V.

Scenario V: Slowdown of working-age population growth and increase in longevity (at unchanged retirement age). This scenario uses the same demographic assumptions as Scenario III. However, the full funding of pensions through the PAYGO system mitigates the effect on r substantially. Compared with the benchmark, r decreases from 3.9% to 3.4%; i.e., the decrease in r by 50 basis points is about half of the decrease reported in Scenario III. Notice that the consumption share of workers in total output (c^{w}/y) declines by about the same amount as in Scenario III. Differently from Scenario III, this decrease in the consumption of workers does not lead to higher savings but to an increase in the tax share by 5 percentage points, which funds the higher share of benefits in output. This increase in the tax burden on workers counteracts the life-cycle features in Scenario III, leading to overall smaller changes in the interest rate.

Finally, the model can be used to highlight another policy-related channel that is of key importance for the long-run effects of demographic changes on the real interest rate, namely the overall extent of fiscal consolidation in an aging society. For simple comparability, we add this channel, ceteris paribus, to Scenario V.

Scenario VI: Slowdown of working-age population growth and increase in longevity (at unchanged retirement age), combined with a lack of fiscal consolidation. Scenario VI relaxes the assumption that the steady-state debt ratio b^* can be stabilized at the benchmark value of 0.7. Instead we assume a higher value of $b^* = 1$. The additional government debt crowds out physical capital, as is to be inferred from the decline in the investment share. This crowding-out effect, ceteris paribus, further moderates the decline in r compared with Scenario III. To put it differently, this effect illustrates that Ricardian equivalence is not satisfied in

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our model because of life-cycle features, and this feature adds to the rich general equilibrium repercussions of demographic changes.

6. A SCENARIO FOR THE EURO AREA UNTIL 2030

Unlike typical shocks (such as productivity or government spending shocks), demographic developments are well predictable rather far into the future. Exploiting this feature, this section presents a detailed macroeconomic scenario for the euro area until 2030, which also incorporates short- and medium-run effects.[17] We use annual demographic projections for the euro area as a deterministic and time-varying input to our nonlinear model, which we solve numerically under perfect foresight. Specifically, we use the two series for working-age population growth and for life expectancy reported in European Economy (2009), as summarized in Figure 1, to specify the paths for n_t^{w} and γ_t exogenously. This leaves the old age dependency ratio ψ_t to be determined within the model, and we verify that the model matches the old-age dependency ratios projected until 2030. We assume for the pension system that the replacement rate declines endogenously so that the aggregate benefits–output ratio remains unchanged, assuming an unchanged retirement age. In view of Section 5 (which also explained the forces acting on the equilibrium interest rate under the opposite scenario of a constant replacement rate), it is clear that the decline in the equilibrium real interest rate projected in this section should be seen as a lower bound. According to our simulations (see Figure 2), all endogenous variables respond slowly over time and most of them reach their peak values at the end of the reported sample period. This pattern reflects that the demographic effects change slowly over time and do not die out over the reported horizon until 2030.

As a benchmark, we first consider the flexible-price version, in which the central bank ensures price stability in all periods (so that the equilibrium real interest rate coincides with the natural rate). Figure 2 shows that the decrease in population growth and the increase in life expectancy jointly contribute over the entire projection horizon to a smooth decline in the equilibrium real interest rate by about 100 basis points. As concerns the effects on other macroeconomic variables, Figure 2 reports an increase in the real wage (reflecting the inverse relationship of the two factor prices) and an increase in the labor supply of workers. Because the capital stock is predetermined, (detrended) output increases on impact. The positive output effect is reinforced over time through an increase in the investment share (consistent with the increase in Tobin's q). There is no lasting effect on the aggregate consumption share. Yet this combined effect masks significant effects of opposite sign of the consumption shares of workers and retirees, respectively, in line with the findings of Column III in Table 5, reported in Section 5.1. Finally, it is worth emphasizing that the replacement rate declines significantly. This is the key channel that supports the additional savings undertaken by workers in anticipation of lower pension incomes in the future, leading to a lasting increase in the capital–output ratio.

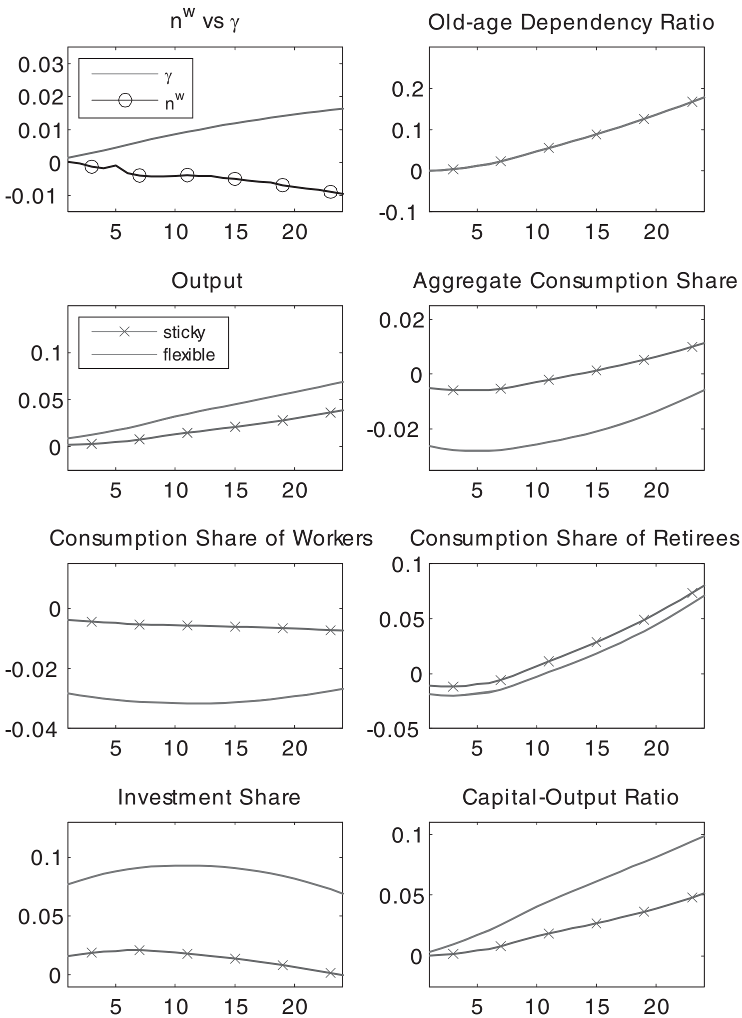

FIGURE 2. Effects of euro area demographic changes until 2030 with an endogenous replacement rate as a percentage of initial steady state. The series on inflation and the nominal and real interest rates report level changes in percentage points.

Figure 2 also summarizes findings from an alternative scenario with sticky prices in which monetary policy follows a Taylor rule, as discussed in Section 2.5. The main result is that under this alternative assumption, the equilibrium real interest rate falls by less. In other words, monetary policy remains relatively tight; i.e., the real interest rate does not fall enough to prevent the onset of deflation. In general, because factor prices react by less, this leads to more muted reactions of

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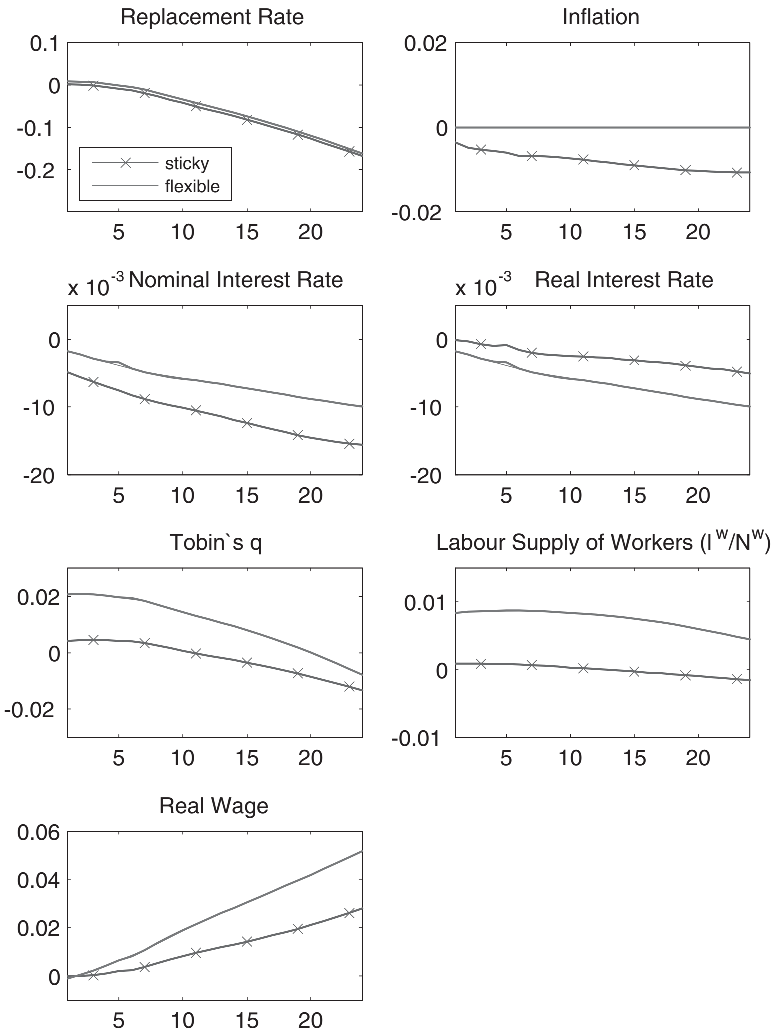

FIGURE 2. Continued.

the labor supply and investment dynamics, leading overall to a weaker increase of output.

A more detailed comparison of the two specifications of monetary policy is given in Figure 3, which reports gaps, defined as the differences between outcomes under sticky and flexible prices. Figure 3 shows not only the combined effects of the slowdown in population growth and the increase in longevity, but also the two effects in isolation. This decomposition is done with the aim of facilitating comparison with the qualitative discussion of comparative statics effects in Section 5.1 (i.e., Scenarios I, II, and III in Table 5). The main finding of Figure 3 is as

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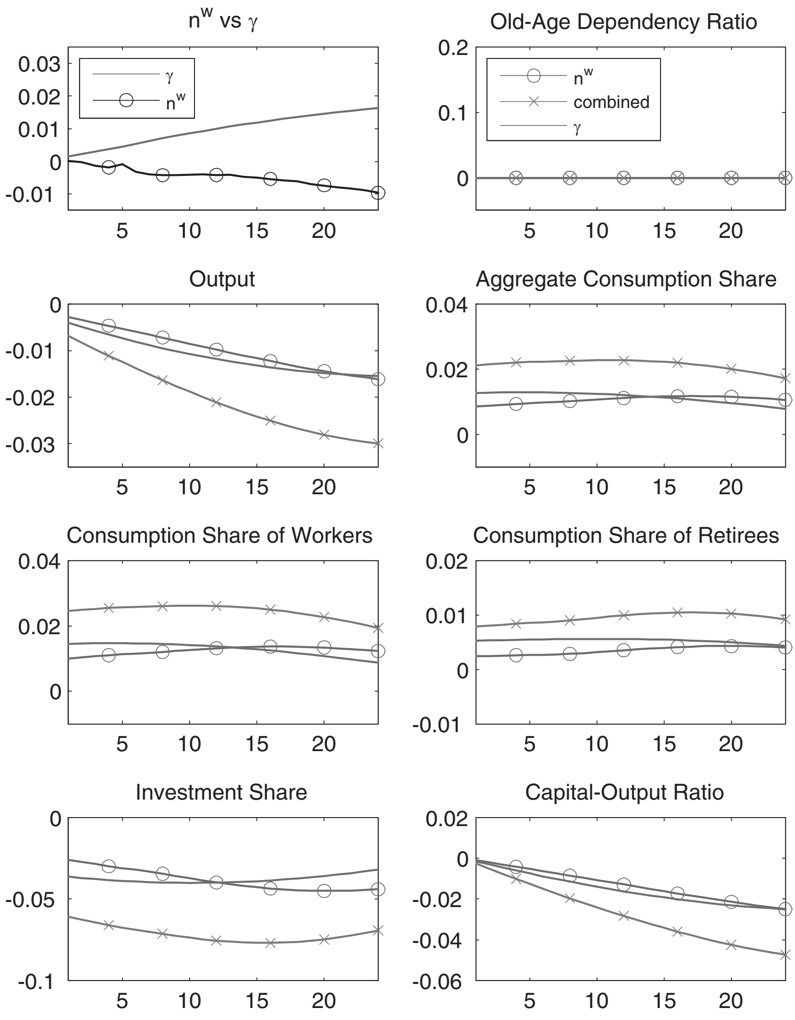

FIGURE 3. Effects of euro area demographic changes until 2030 with an endogenous replacement rate as a percentage of initial steady state in terms of gaps (i.e., the differences between outcomes under sticky and flexible prices). The series on inflation and the nominal and real interest rates report level changes in percentage points.

follows: For all variables the effects of the slowdown in population growth and the increase in longevity reinforce each other. This pattern is in line with Table 5 and it holds true, in particular, for the equilibrium real interest rate.

An interesting feature of the dynamic simulations reported in Figure 2 is the persistence of deflation under sticky prices, suggesting that the central bank may persistently undershoot its zero-inflation target. The undershooting is especially

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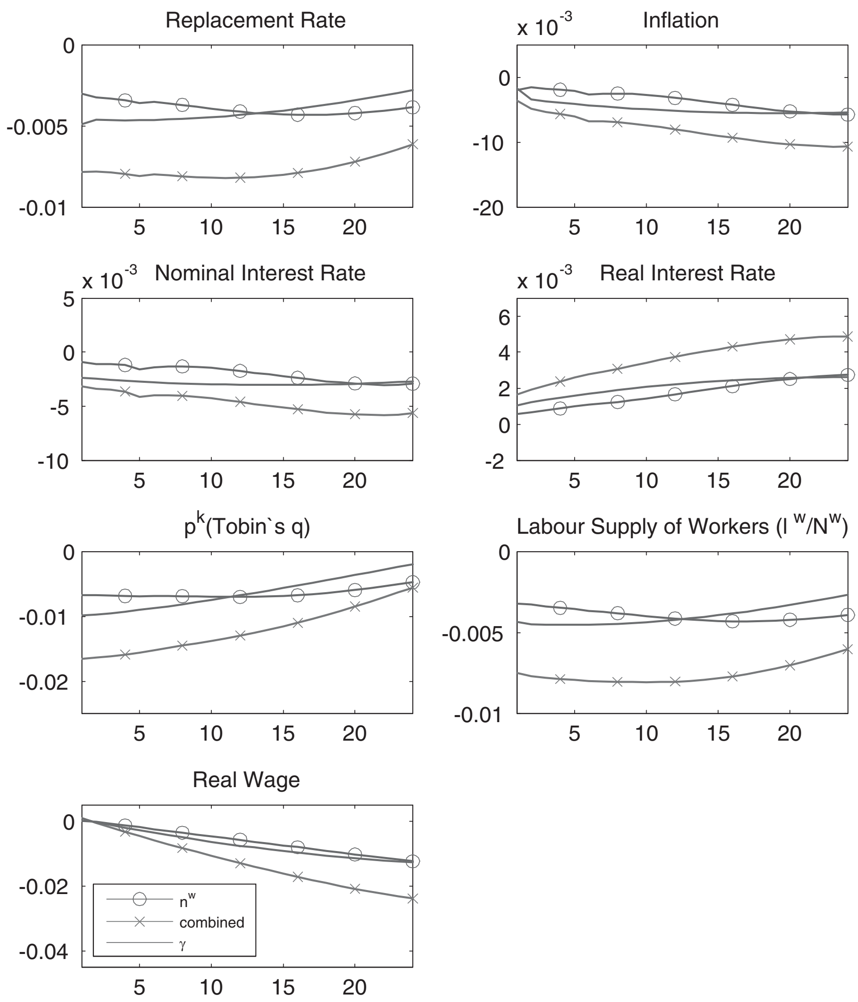

FIGURE 3. Continued.

pronounced in the specification that considers the combined effects of the slowdown in population growth and the increase in longevity. This observation suggests that, in our baseline scenario with sticky prices, the central bank does not reduce the real interest rate strongly enough to offset the deflationary consequences of the shocks. A more aggressive reaction of the central bank to falling prices would reduce the real interest rate more strongly than in the baseline scenario. This would stimulate output and counteract the decline in prices. Hence, inflation would be closer to its target. Indeed, Figure 4 confirms this conjecture. There, we report simulations for the real interest rate, output, and inflation with a higher coefficient on inflation in the monetary policy rule (i.e., $\gamma_\pi = 5$) for the combined shock

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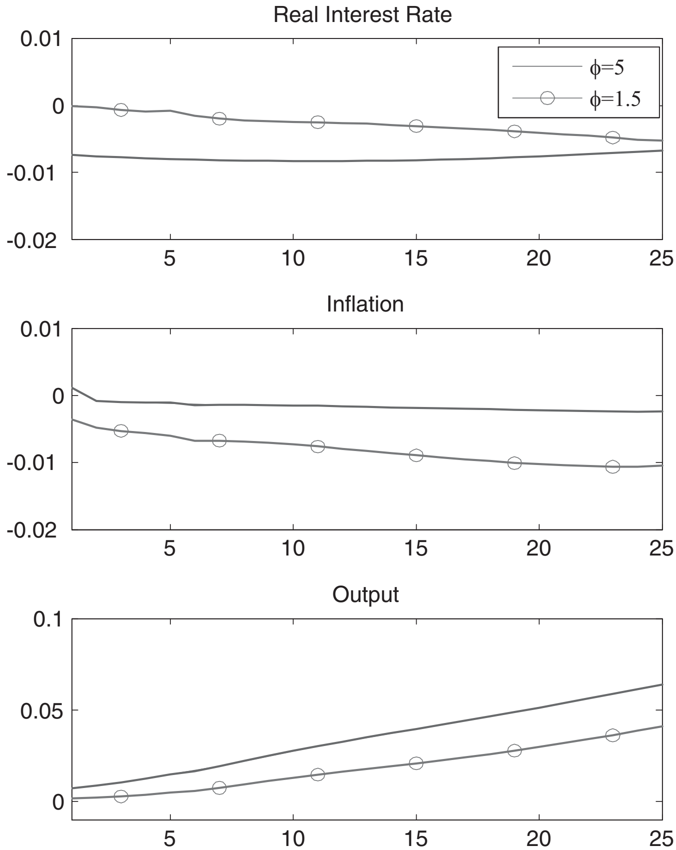

FIGURE 4. Effects of euro area demographic changes until 2030 with an endogenous replacement rate as a percentage of initial steady state. Comparison of different inflation coefficients ($\gamma_\pi = 1.5$ and $\gamma_\pi = 5$) under sticky prices. The series on inflation and the nominal and real interest rates report level changes in percentage points.

case. With $\gamma_\pi = 5$, the real interest rate is lower than in the baseline scenario of a standard Taylor coefficient on inflation of $\gamma_\pi = 1.5$, resulting in higher output and virtually no deflation. This finding suggests that, ceteris paribus, central banks may have to respond more aggressively to inflation in economic environments characterized by a slowdown in population growth and a substantial increase in longevity. We point out that even in the case with $\gamma_\pi = 5$, the dynamics of output and the interest rate in the sticky-price model are somewhat different from those in the flexible-price version of the model (reported in Figure 2).

Another reason for the persistence of deflation is the presence of interest rate smoothing in our benchmark policy rule. In this context, it should be borne in mind that the model implies a time-varying natural rate. This feature introduces

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a difference into a standard Taylor rule that includes a constant real interest rate term. To see what this implies, consider, similarly to Woodford (2003, p. 293), the special case in which there is no interest rate inertia, so that the time-varying natural rate enters the policy rule with a coefficient of one. It is straightforward to see that the real interest rate would adjust one for one with the natural rate. The flexible-price solution in Figure 2 indicates that the natural rate falls monotonically over the reported horizon until 2030. Hence, a policy rule that displays inertia brings the real interest rate in line with the natural rate only gradually, resulting in lower output and more persistent deflation. Thus, removing the interest rate smoothing brings inflation closer to its target. Findings from simulations that are not reported here confirm this intuition.

In sum, we take these findings as a confirmation that the design of optimal monetary policy under meaningful demographic assumptions and individual life-cycle effects**,** when combined with other frictions such as nominal rigidities, deserves further analysis.

7. CONCLUSION

This paper develops a small-scale DSGE model that embeds a demographic structure within a monetary policy framework. We extend the nonmonetary overlapping-generations model of Gertler (1999) and present a small synthesis model that combines the setup of Gertler with a New Keynesian structure, implying that the short-run dynamics related to monetary policy can be compared with that of the standard New Keynesian model, as summarized by Woodford (2003). Reflecting the underlying overlapping-generations structure, the dynamics of the model is critically affected by fiscal policy (including aspects of the social security system), as we show in a number of policy experiments. In sum, the model offers a New Keynesian platform that can be used to characterize the response of macroeconomic variables to demographic shocks, similar to technology or monetary policy shocks. We offer such characterizations for flexible and sticky price equilibria. Empirically, we calibrate the model to demographic developments projected for the euro area and discuss a macroeconomic scenario with a horizon of around 20 years. The main finding is that the projected slowdown in population growth and the increase in longevity contribute slowly over time to a decline in the equilibrium interest rate.

In future work, the analysis can be extended in a number of directions. We find four aspects particularly important. First, the model can be used to compare short-run features of New Keynesian models with and without life-cycle effects with other shocks (such as productivity or preference shocks). Such analysis would deemphasize the deliberate focus on long-run features taken in this paper and thereby assign a more significant role to the monetary margin. Second, the fiscal specification should be enriched by allowing for a broader set of taxes. In particular, it remains to be checked to what extent the quantitative predictions of our analysis will be affected in the presence of distortionary taxes, similarly to the analysis of Kilponen et al. (2006). Third, in line with the nonstandard requirements of the Gertler economy, the current labor supply specification is restrictive, and it would be interesting to redo the analysis with a more flexible specification of preferences. Fourth, from a normative perspective, optimal policies should be linked to the welfare of the representative agents in the economy. Because of the heterogeneity of agents, this raises interesting distributional questions that go beyond the analysis of optimal policies typically carried out in monetary policy frameworks.

NOTES

1. As will become clear later, we also allow for variations in the retirement age of workers.
2. Woodford (2003, pp. 247–252) refers to this rate as the "Wicksellian natural rate of interest."
3. Another core modeling difference concerns labor supply specifications. Differently from us, the labor supply of retirees in Fujiwara and Teranishi (2008) is not restricted to be zero. This feature leads to qualitatively different long-run predictions for the equilibrium interest rate, in the sense that a 'grayer' society may well be characterized by a higher equilibrium interest rate.
4. The models of Boersch-Supan et al. (2006) and Krueger and Ludwig (2007) are in the tradition of Auerbach and Kotlikoff (1987), whereas Batini et al. (2006) uses a large-scale extension of Blanchard (1985), assuming that agents face a constant probability of death.
5. As clarified later, in our model specification workers and retirees face no other source of uncertainty, because the future course of aggregate demographic developments is modeled as a deterministic input.
6. The budget constraint, if written like this, assumes that the retiree was already in retirement during the previous period $t-1$. For a complete description of the cohort-specific behavior of all agents, the decision problem would have to be conditioned on the year of birth and the age at which retirement takes place. However, this is not needed for the derivation of the aggregate behavior of retirees and workers, anticipating the linear structure of the decision rules derived later.
7. Newborn agents are assumed to enter the workforce with zero nonhuman wealth.
8. In a richer framework, it would be straightforward to modify the simplifying assumption that all taxes (profits) are paid (received) by workers and all benefits are received by retirees. Because the key results depend only on the net transfers made between the two groups, this simple specification, however, captures the main redistribution effects occurring in a life-cycle framework.
9. This equation holds in the model specification discussed later, in which we treat demographic developments as a deterministic input and abstract from any other shocks. Moreover, the valuation of profits is linked to the discount rate of workers. Because in our specification all profits are received by workers, this assumption turns out to be unimportant for our findings reported later.
10. For an early discussion of stylized feedback rules of fiscal and monetary policies in our spirit, see Leeper (1991). More detailed discussions can be found, for example, in Schmitt-Grohé and Uribe (2007) and Leith and von Thadden (2008).
11. Endogenous state variables with predetermined initial conditions relate, in particular, to the level of aggregate nonhuman wealth, its breakdown across assets, and its distribution between workers and retirees.
12. The benchmark calibration reproduces euro area data listed in the column for the year 2008 of the summary table "Main demographic and macroeconomic assumptions" for the euro area (EA 16) in the Statistical Annex to *European Economy* (2009, p. 174).
13. In the early working paper version [Kara and von Thadden (2010)], we used the calibration $v_2 = 0.02$. All quantitative implications of this assumption are negligible.
14. The value of $e/y = 0.11$ corresponds to the observation (reported for 2007) of "social security pensions as % of GDP" in the Statistical Annex to European Economy (2009, p. 174).

15. Projections are taken from the Statistical Annex to European Economy (2009, p. 174). As shown in Figure 1, they are very similar to projections provided by the United Nations (see World Population Prospects: The 2008 Revision, available at http://esa.un.org/unpp). For closely related discussions of demographic changes in the euro area, see also European Economy (2006) and Maddaloni et al. (2006).

16. Moreover, it is clear that the assumed trend path of labor productivity matters, which itself may well depend on the age structure of the economy.

17. The simulations use the demographic projections until 2050 as an input. We have checked that the reported results for the projection horizon until 2030 are not sensitive to the assumptions made for the years close to 2050 and beyond.

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APPENDIX A: DECISION PROBLEMS OF RETIREES AND WORKERS

A.1. PROBLEM OF THE REPRESENTATIVE RETIREE

Let

$$V_t^{rj} = \left\{ \left[\left(c_t^{rj}\right)^{v_1} \left(m_t^{rj}\right)^{v_2} \left(1 - l_t^{rj}\right)^{v_3} \right]^{\rho} + \beta \gamma_t \left(V_{t+1}^{rj}\right)^{\rho} \right\}^{\frac{1}{\rho}}, \tag{A.1}$$

subject to

$$c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj} + a_t^{rj} = \frac{1 + r_{t-1}}{\gamma_{t-1}} a_{t-1}^{rj} + \xi w_t l_t^{rj} + e_t^{j}. \tag{A.2}$$

First-order conditions:
(i) W.r.t. consumption c_t^{rj}:

$$v_1 \left(c_t^{rj}\right)^{v_1 \rho - 1} \left(m_t^{rj}\right)^{v_2 \rho} \left(1 - l_t^{rj}\right)^{v_3 \rho} = \beta \gamma_t \left(V_{t+1}^{rj}\right)^{\rho - 1} \frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}}. \tag{A.3}$$

To obtain $\frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}}$ invoke the envelope theorem, i.e.,

$$\frac{\partial V_t^{rj}}{\partial a_{t-1}^{rj}} = \left(V_t^{rj}\right)^{1-\rho} \upsilon_1 \frac{1+r_{t-1}}{\gamma_{t-1}} \left(c_t^{rj}\right)^{\upsilon_1\rho-1} \left(m_t^{rj}\right)^{\upsilon_2\rho} \left(1-l_t^{rj}\right)^{\upsilon_3\rho}.$$

Shifting this expression one period forward leads to

$$\frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} = \left(V_{t+1}^{rj}\right)^{1-\rho} \upsilon_1 \frac{1+r_t}{\gamma_t} \left(c_{t+1}^{rj}\right)^{\upsilon_1\rho-1} \left(m_{t+1}^{rj}\right)^{\upsilon_2\rho} \left(1-l_{t+1}^{rj}\right)^{\upsilon_3\rho}. \tag{A.4}$$

Combining (A.4) and (A.3) yields

$$\left(c_t^{rj}\right)^{\upsilon_1\rho-1} \left(m_t^{rj}\right)^{\upsilon_2\rho} \left(1-l_t^{rj}\right)^{\upsilon_3\rho} = \beta(1+r_t) \left(c_{t+1}^{rj}\right)^{\upsilon_1\rho-1} \left(m_{t+1}^{rj}\right)^{\upsilon_2\rho} \left(1-l_{t+1}^{rj}\right)^{\upsilon_3\rho}. \tag{A.5}$$

(ii) W.r.t. labor supply l_t^{rj} :

$$\upsilon_3 \left(c_t^{rj}\right)^{\upsilon_1\rho} \left(m_t^{rj}\right)^{\upsilon_2\rho} \left(1-l_t^{rj}\right)^{\upsilon_3\rho-1} = \beta\gamma_t \left(V_{t+1}^{rj}\right)^{\rho-1} \frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} \xi w_t.$$

Combine with (A.3) to obtain

$$1-l_t^{rj} = \frac{\upsilon_3}{\upsilon_1} \frac{c_t^{rj}}{\xi w_t}. \tag{A.6}$$

(iii) W.r.t. real balances m_t^{rj} :

$$\upsilon_2 \left(c_t^{rj}\right)^{\upsilon_1\rho} \left(m_t^{rj}\right)^{\upsilon_2\rho-1} \left(1-l_t^{rj}\right)^{\upsilon_3\rho} = \beta\gamma_t \left(V_{t+1}^{rj}\right)^{\rho-1} \frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} \frac{i_t}{1+i_t}.$$

Combine with (A.3) to obtain

$$m_t^{rj} = \frac{\upsilon_2}{\upsilon_1} \frac{1+i_t}{i_t} c_t^{rj}. \tag{A.7}$$

(iv) Derivation of Euler equation in c_t^{rj}-terms:
Combining (A.5), (A.6), and (A.7) yields

$$c_{t+1}^{rj} = \left[\beta(1+r_t) \left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{\upsilon_2\rho} \left(\frac{w_t}{w_{t+1}}\right)^{\upsilon_3\rho}\right]^{\sigma} c_t^{rj}, \tag{A.8}$$

where $\sigma = \frac{1}{1-\rho}$.

Solution of V_t^{rj}:

(i) Conjecture for the combined consumption term $c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj}$:

$$c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj} = c_t^{rj}\left(1+\frac{\upsilon_2}{\upsilon_1}\right) = \epsilon_t \pi_t \left(\frac{1+r_{t-1}}{\gamma_{t-1}} a_{t-1}^{rj} + h_t^{rj}\right), \tag{A.9}$$

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using

$$h_t^{rj} = d_t^{rj} + \frac{\gamma_t}{1+r_t} h_{t+1}^{rj},$$
$$d_t^{rj} = \xi w_t l_t^{rj} + e_t^j.$$

Using (A.9) within the Euler equation (A.8) yields

$$a_t^{rj} + \frac{\gamma_t}{1+r_t} h_{t+1}^{rj} = \left[\beta \left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{v_2 \rho} \left(\frac{w_t}{w_{t+1}}\right)^{v_3 \rho}\right]^{\sigma} (1+r_t)^{\sigma-1} \gamma_t \frac{\epsilon_t \pi_t}{\epsilon_{t+1} \pi_{t+1}}$$
$$\times \left(\frac{1+r_{t-1}}{\gamma_{t-1}} a_{t-1}^{rj} + h_t^{rj}\right).$$

Use (A.9) to rewrite the budget constraint (A.2) as

$$a_t^{rj} + \frac{\gamma_t}{1+r_t} h_{t+1}^{rj} = [1 - \epsilon_t \pi_t] \left(\frac{1+r_{t-1}}{\gamma_{t-1}} a_{t-1}^{rj} + h_t^{rj}\right),$$

implying

$$\epsilon_t \pi_t = 1 - \left[\beta \left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{v_2 \rho} \left(\frac{w_t}{w_{t+1}}\right)^{v_3 \rho}\right]^{\sigma} (1+r_t)^{\sigma-1} \gamma_t \frac{\epsilon_t \pi_t}{\epsilon_{t+1} \pi_{t+1}}. \quad \textbf{(A.10)}$$

(ii) Conjecture for value function V_t^{rj}:

$$V_t^{rj} = \Delta_t^{\mathrm{r}} c_t^{rj} \left(\frac{v_2}{v_1} \frac{1+i_t}{i_t}\right)^{v_2} \left(\frac{v_3}{v_1} \frac{1}{\xi w_t}\right)^{v_3}.$$

This implies within (A.1)

$$\Delta_t^{\mathrm{r}} c_t^{rj} \left(\frac{v_2}{v_1} \frac{1+i_t}{i_t}\right)^{v_2} \left(\frac{v_3}{v_1} \frac{1}{\xi w_t}\right)^{v_3} = \left\{\left[c_t^{rj} \left(\frac{v_2}{v_1} \frac{1+i_t}{i_t}\right)^{v_2} \left(\frac{v_3}{v_1} \frac{1}{\xi w_t}\right)^{v_3}\right]^{\rho}\right.$$
$$\left. + \beta \gamma_t \left[\Delta_{t+1}^{\mathrm{r}} c_{t+1}^{rj} \left(\frac{v_2}{v_1} \frac{1+i_{t+1}}{i_{t+1}}\right)^{v_2} \left(\frac{v_3}{v_1} \frac{1}{\xi w_{t+1}}\right)^{v_3}\right]^{\rho}\right\}^{\frac{1}{\rho}}.$$

Use (A.8) to write this as

$$(\Delta_t^{\mathrm{r}})^{\rho} = 1 + \gamma_t (\Delta_{t+1}^{\mathrm{r}})^{\rho} (1+r_t)^{\sigma-1} \left[\beta \left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{v_2 \rho} \left(\frac{w_t}{w_{t+1}}\right)^{v_3 \rho}\right]^{\sigma},$$

where we have used $\sigma = \frac{1}{1-\rho} \Leftrightarrow \rho = 1 - \frac{1}{\sigma} = \frac{\sigma-1}{\sigma} \Rightarrow \rho\sigma = \sigma - 1 \Rightarrow (1+\rho\sigma)\rho = \rho\sigma$.

Let

$$(\Delta_t^{\mathrm{r}})^{\rho} = (\epsilon_t \pi_t)^{-1}.$$

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Then

$$\epsilon_t \pi_t = 1 - \gamma_t (1+r_t)^{\sigma-1} \left[\beta \left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}} \right)^{v_2 \rho} \left(\frac{w_t}{w_{t+1}} \right)^{v_3 \rho} \right]^{\sigma} \frac{\epsilon_t \pi_t}{\epsilon_{t+1} \pi_{t+1}},$$

which completes the conjectures for the consumption function and the value function V_t^{rj}.

A.2. PROBLEM OF THE REPRESENTATIVE WORKER

Let

$$V_t^{wj} = \left\{ \left[\left(c_t^{wj} \right)^{v_1} \left(m_t^{wj} \right)^{v_2} \left(1 - l_t^{wj} \right)^{v_3} \right]^{\rho} + \beta \left[\omega_t V_{t+1}^{wj} + (1-\omega_t) V_{t+1}^{rj} \right]^{\rho} \right\}^{\frac{1}{\rho}}, \qquad \textbf{(A.11)}$$

subject to

$$c_t^{wj} + \frac{i_t}{1+i_t} m_t^{wj} + a_t^{wj} = (1+r_{t-1}) a_{t-1}^{wj} + w_t l_t^{wj} + f_t^j - \tau_t^j. \qquad \textbf{(A.12)}$$

First-order conditions:
(i) W.r.t. consumption c_t^{wj} :

$$\begin{aligned} v_1 \left(c_t^{wj} \right)^{v_1 \rho - 1} \left(m_t^{wj} \right)^{v_2 \rho} \left(1 - l_t^{wj} \right)^{v_3 \rho} &= \beta \Big[\omega_t V_{t+1}^{wj} \\ &+ (1-\omega_t) V_{t+1}^{rj} \Big]^{\rho-1} \left[\omega_t \frac{\partial V_{t+1}^{wj}}{\partial a_t^{wj}} + (1-\omega_t) \frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} \right]. \end{aligned} \qquad \textbf{(A.13)}$$

To obtain $\partial V_{t+1}^{wj} / \partial a_t^{wj}$ and $\partial V_{t+1}^{rj} / \partial a_t^{rj}$ invoke the envelope theorem:

$$\frac{\partial V_t^{wj}}{\partial a_{t-1}^{wj}} = \left(V_t^{wj} \right)^{1-\rho} v_1 (1+r_{t-1}) \left(c_t^{wj} \right)^{v_1 \rho - 1} \left(m_t^{wj} \right)^{v_2 \rho} \left(1 - l_t^{wj} \right)^{v_3 \rho},$$

$$\frac{\partial V_t^{rj}}{\partial a_{t-1}^{rj}} = \left(V_t^{rj} \right)^{1-\rho} v_1 (1+r_{t-1}) (c_t^{rj})^{v_1 \rho - 1} \left(m_t^{rj} \right)^{v_2 \rho} \left(1 - l_t^{rj} \right)^{v_3 \rho}.$$

Shifting these expressions one period forward leads to

$$\frac{\partial V_{t+1}^{wj}}{\partial a_t^{wj}} = \left(V_{t+1}^{wj} \right)^{1-\rho} v_1 (1+r_t) \left(c_{t+1}^{wj} \right)^{v_1 \rho - 1} \left(m_{t+1}^{wj} \right)^{v_2 \rho} (1 - l_{t+1}^{wj})^{v_3 \rho},$$

$$\frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} = \left(V_{t+1}^{rj} \right)^{1-\rho} v_1 (1+r_t) \left(c_{t+1}^{rj} \right)^{v_1 \rho - 1} \left(m_{t+1}^{rj} \right)^{v_2 \rho} \left(1 - l_{t+1}^{rj} \right)^{v_3 \rho}.$$

Using these expressions within (A.13) yields

$$\begin{aligned} v_1 \left(c_t^{wj} \right)^{v_1 \rho - 1} \left(m_t^{wj} \right)^{v_2 \rho} \left(1 - l_t^{wj} \right)^{v_3 \rho} &= \beta \Big[\omega_t V_{t+1}^{wj} \\ &+ (1-\omega_t) V_{t+1}^{rj} \Big]^{\rho-1} \left[\omega_t \frac{\partial V_{t+1}^{wj}}{\partial a_t^{wj}} + (1-\omega_t) \frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} \right], \end{aligned} \qquad \textbf{(A.14)}$$

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with

$$\omega_t \frac{\partial V_{t+1}^{wj}}{\partial a_t^{wj}} + (1-\omega_t)\frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}} = \omega_t \left(V_{t+1}^{wj}\right)^{1-\rho} \upsilon_1(1+r_t)\left(c_{t+1}^{wj}\right)^{\upsilon_1\rho-1}\left(m_{t+1}^{wj}\right)^{\upsilon_2\rho}\left(1-l_{t+1}^{wj}\right)^{\upsilon_3\rho}$$
$$+(1-\omega_t)\left(V_{t+1}^{rj}\right)^{1-\rho}\upsilon_1(1+r_t)\left(c_{t+1}^{rj}\right)^{\upsilon_1\rho-1}\left(m_{t+1}^{rj}\right)^{\upsilon_2\rho}\left(1-l_{t+1}^{rj}\right)^{\upsilon_3\rho}.$$

(ii) W.r.t. labor supply l_t^{wj} :

$$\upsilon_3\left(c_t^{wj}\right)^{\upsilon_1\rho}\left(m_t^{wj}\right)^{\upsilon_2\rho}\left(1-l_t^{wj}\right)^{\upsilon_3\rho-1} = \beta\big[\omega_t V_{t+1}^{wj}$$
$$+(1-\omega_t)V_{t+1}^{rj}\big]^{\rho-1}\left[\omega_t\frac{\partial V_{t+1}^{wj}}{\partial a_t^{wj}}+(1-\omega_t)\frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}}\right]w_t.$$

Combine with (A.13) to obtain

$$1-l_t^{wj} = \frac{\upsilon_3}{\upsilon_1}\frac{c_t^{wj}}{w_t}. \tag{A.15}$$

(iii) W.r.t. real balances m_t^{wj} :

$$\upsilon_2\left(c_t^{wj}\right)^{\upsilon_1\rho}\left(m_t^{wj}\right)^{\upsilon_2\rho-1}\left(1-l_t^{wj}\right)^{\upsilon_3\rho} = \beta\big[\omega_t V_{t+1}^{wj}$$
$$+(1-\omega_t)V_{t+1}^{rj}\big]^{\rho-1}\left[\omega_t\frac{\partial V_{t+1}^{wj}}{\partial a_t^{wj}}+(1-\omega_t)\frac{\partial V_{t+1}^{rj}}{\partial a_t^{rj}}\right]\frac{i_t}{1+i_t}.$$

Combine with (A.13) to obtain

$$m_t^{wj} = \frac{\upsilon_2}{\upsilon_1}\frac{1+i_t}{i_t}c_t^{wj}. \tag{A.16}$$

(iv) Substituting out for l_t^{wj} and m_t^{wj}-terms:

Use (A.15) and (A.16) in (A.14) to simplify the consumption Euler equation:

$$\left(c_t^{wj}\right)^{\rho-1} = [\omega_t V_{t+1}^{wj}+(1-\omega_t)V_{t+1}^{rj}]^{\rho-1}\beta(1+r_t)\left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{\upsilon_2\rho}\left(\frac{w_t}{w_{t+1}}\right)^{\upsilon_3\rho}$$
$$\cdot\left[\omega_t\left(V_{t+1}^{wj}\right)^{1-\rho}\left(c_{t+1}^{wj}\right)^{\rho-1}+(1-\omega_t)\left(V_{t+1}^{rj}\right)^{1-\rho}\left(\frac{1}{\xi}\right)^{\upsilon_3\rho}\left(c_{t+1}^{rj}\right)^{\rho-1}\right]. \tag{A.17}$$

Solution of V_t^{wj}:

(i) Conjecture for value function V_t^{wj}:

$$V_t^{wj} = \Delta_t^{\mathrm{w}}c_t^{wj}\left(\frac{\upsilon_2}{\upsilon_1}\frac{1+i_t}{i_t}\right)^{\upsilon_2}\left(\frac{\upsilon_3}{\upsilon_1}\frac{1}{w_t}\right)^{\upsilon_3},\quad \Delta_t^{\mathrm{w}} = \pi_t^{-\frac{1}{\rho}}.$$

Moreover, recall from before that

$$V_t^{rj} = \Delta_t^{\mathrm{r}}c_t^{\mathrm{r}}\left(\frac{\upsilon_2}{\upsilon_1}\frac{1+i_t}{i_t}\right)^{\upsilon_2}\left(\frac{\upsilon_3}{\upsilon_1}\frac{1}{w_t\xi}\right)^{\upsilon_3},\quad \Delta_t^{\mathrm{r}} = (\epsilon_t\pi_t)^{-\frac{1}{\rho}}.$$

Use these conjectures within (A.17) to further simplify the consumption Euler equation:

$$\left(c_t^{wj}\right)^{\rho-1} = \left[\omega_t c_{t+1}^{wj} + (1-\omega_t)\left(\epsilon_{t+1}\right)^{-\frac{1}{\rho}}\left(\frac{1}{\xi}\right)^{\upsilon_3} c_{t+1}^{rj}\right]^{\rho-1} \beta(1+r_t)\left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{\upsilon_2\rho}$$
$$\left(\frac{w_t}{w_{t+1}}\right)^{\upsilon_3\rho}\left[\omega_t + (1-\omega_t)\left(\epsilon_{t+1}\right)^{-\frac{1-\rho}{\rho}}\left(\frac{1}{\xi}\right)^{\upsilon_3}\right].$$

Recall from before that $\sigma = \frac{1}{1-\rho} \Leftrightarrow \rho = 1 - \frac{1}{\sigma} = \frac{\sigma-1}{\sigma}$ and let

$$\Lambda_{t+1} = \frac{\Delta_{t+1}^{\mathrm{r}}}{\Delta_{t+1}^{\mathrm{w}}} = \left(\epsilon_{t+1}\right)^{-\frac{1}{\rho}} = \left(\epsilon_{t+1}\right)^{\frac{\sigma}{1-\sigma}},$$
$$\chi = \left(\frac{1}{\xi}\right)^{\upsilon_3},$$
$$\Omega_{t+1} = \omega_t + (1-\omega_t)\,\epsilon_{t+1}^{\frac{1}{1-\sigma}}\,\chi$$

to further simplify the consumption Euler equation, yielding

$$\omega_t c_{t+1}^{wj} + (1-\omega_t)\,\Lambda_{t+1}\chi c_{t+1}^{rj} = \left[\beta(1+r_t)\Omega_{t+1}\left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{\upsilon_2\rho}\left(\frac{w_t}{w_{t+1}}\right)^{\upsilon_3\rho}\right]^{\sigma} c_t^{wj}. \tag{A.18}$$

(ii) Conjecture for the combined consumption term $c_t^{wj} + \frac{i_t}{1+i_t} m_t^{wj}$:

Conjecture for the worker that $c_t^{wj} + \frac{i_t}{1+i_t} m_t^{wj}$ is given by

$$c_t^{wj} + \frac{i_t}{1+i_t} m_t^{wj} = c_t^{wj}\left(1 + \frac{\upsilon_2}{\upsilon_1}\right) = \pi_t\left[(1+r_{t-1})a_{t-1}^{wj} + h_t^{wj}\right], \tag{A.19}$$

whereas the "just retired person" consumes according to

$$c_t^{rj} + \frac{i_t}{1+i_t} m_t^{rj} = c_t^{rj}\left(1 + \frac{\upsilon_2}{\upsilon_1}\right) = \epsilon_t\pi_t\left[(1+r_{t-1})a_{t-1}^{wj} + h_t^{rj}\right], \tag{A.20}$$

because his financial assets are predetermined from his previous savings as a worker. Moreover, define the human capital of workers as follows:

$$h_t^{wj} = d_t^{wj} + \frac{\omega_t}{\Omega_{t+1}}\frac{h_{t+1}^{wj}}{1+r_t} + \left(1 - \frac{\omega_t}{\Omega_{t+1}}\right)\frac{h_{t+1}^{rj}}{1+r_t},$$
$$d_t^{wj} = w_t l_t^{wj} + f_t^j - \tau_t^j.$$

Substituting the conjectures for the consumption of workers (A.19) and (A.20) into the Euler equation (A.18) yields

$$\frac{a_t^{wj} + h_t^{wj} - d_t^{wj}}{(1+r_{t-1})a_{t-1}^{wj} + h_t^{wj}} = \frac{\pi_t}{\pi_{t+1}}\beta^{\sigma}\left[(1+r_t)\Omega_{t+1}\right]^{\sigma-1}\left[\left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{\upsilon_2\rho}\left(\frac{w_t}{w_{t+1}}\right)^{\upsilon_3\rho}\right]^{\sigma}.$$

Next, rewrite the budget constraint (A.12) to obtain an alternative expression for the LHS of the previous equation:

$$a_t^{wj} + h_t^{wj} - d_t^{wj} = (1-\pi_t)\left[(1+r_{t-1})a_{t-1}^{wj} + h_t^{wj}\right],$$

implying that

$$\pi_t = 1 - \left[\left(\frac{\frac{1+i_{t+1}}{i_{t+1}}}{\frac{1+i_t}{i_t}}\right)^{\upsilon_2\rho}\left(\frac{w_t}{w_{t+1}}\right)^{\upsilon_3\rho}\right]^{\sigma}\beta^{\sigma}[(1+r_t)\Omega_{t+1}]^{\sigma-1}\frac{\pi_t}{\pi_{t+1}},$$

which ensures that all conjectures add up to consistent solutions across all equations characterizing optimal decisions of retirees and workers.

APPENDIX B. SUMMARY OF DETRENDED EQUILIBRIUM CONDITIONS

As stressed in the main text, consider generic variables $\upsilon_t \in \{c_t, y_t, k_t, f_t, r_t^k, i_t^k, a_t, b_t, m_t, \tau_t, e_t\}$, $\upsilon_t^{\mathrm{w}} \in \{h_t^{\mathrm{w}}, d_t^{\mathrm{w}}, c_t^{\mathrm{w}}, \}$, and $\upsilon_t^{\mathrm{r}} \in \{ h_t^{\mathrm{r}}, d_t^{\mathrm{r}}, c_t^{\mathrm{r}}\}$. Then

$$\frac{\upsilon_t}{N_t^{\mathrm{w}}X_t} = \overline{\upsilon_t}, \quad \frac{\upsilon_t^{\mathrm{w}}}{N_t^{\mathrm{w}}X_t} = \overline{\upsilon_t^{\mathrm{w}}}, \quad \frac{\upsilon_t^{\mathrm{r}}}{N_t^{\mathrm{w}}X_t} = \frac{\upsilon_t^{\mathrm{r}}}{N_t^{\mathrm{r}}X_t}\frac{N_t^{\mathrm{r}}}{N_t^{\mathrm{w}}} = \overline{\upsilon_t^{\mathrm{r}}}\psi_t.$$

Moreover, let variables related to the labor market be detrended as follows:

$$\frac{w_t}{X_t} = \overline{w_t}, \quad \frac{l_t}{N_t^{\mathrm{w}}} = \overline{l_t}, \quad \frac{l_t^{\mathrm{w}}}{N_t^{\mathrm{w}}} = \overline{l_t^{\mathrm{w}}}, \quad \frac{l_t^{\mathrm{r}}}{N_t^{\mathrm{w}}} = \frac{l_t^{\mathrm{r}}}{N_t^{\mathrm{r}}}\frac{N_t^{\mathrm{r}}}{N_t^{\mathrm{w}}} = \overline{l_t^{\mathrm{r}}}\psi_t.$$

Using these definitions, the detrended counterpart of the system (2)–(32) can be summarized as follows, where we report for all equations first the general and then the steady-state version.

Propensity to consume of retirees [equation (2)]:

$$\epsilon_t\pi_t = 1 - \left[\left(\frac{1+i_{t+1}}{i_{t+1}}\frac{i_t}{1+i_t}\right)^{\upsilon_2\rho}\left(\frac{\overline{w_t}}{\overline{w_{t+1}}}\frac{1}{1+x}\right)^{\upsilon_3\rho}\right]^{\sigma}\beta^{\sigma}(1+r_t)^{\sigma-1}\gamma_t\frac{\epsilon_t\pi_t}{\epsilon_{t+1}\pi_{t+1}},$$

$$\epsilon\pi = 1 - \left[\left(\frac{1}{1+x}\right)^{\upsilon_3\rho}\right]^{\sigma}\beta^{\sigma}(1+r)^{\sigma-1}\gamma.$$

Definition of Ω [equation (3)]:

$$\Omega_{t+1} = \omega_t + (1-\omega_t)\,\epsilon_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\xi}\right)^{\upsilon_3},$$

$$\Omega = \omega + (1-\omega)\,\epsilon^{\frac{1}{1-\sigma}}\left(\frac{1}{\xi}\right)^{\upsilon_3}.$$

Propensity of workers to consume [equation (4)]:

$$\pi_t = 1 - \left[\left(\frac{1+i_{t+1}}{i_{t+1}} \frac{i_t}{1+i_t} \right)^{v_2 \rho} \left(\frac{\overline{w_t}}{\overline{w_{t+1}}} \frac{1}{1+x} \right)^{v_3 \rho} \right]^{\sigma} \beta^{\sigma} ((1+r_t)\Omega_{t+1})^{\sigma-1} \frac{\pi_t}{\pi_{t+1}},$$

$$\pi = 1 - \left[\left(\frac{1}{1+x} \right)^{v_3 \rho} \right]^{\sigma} \beta^{\sigma} ((1+r)\Omega)^{\sigma-1}.$$

Aggregate labor supply of retirees, workers, and total labor force [equations (5)–(7)]:

$$1 - \overline{l_t^{\mathrm{r}}} = \frac{v_3}{v_1} \frac{\overline{c_t^{\mathrm{r}}}}{\xi \overline{w_t}}, \qquad 1 - \overline{l_t^{\mathrm{w}}} = \frac{v_3}{v_1} \frac{\overline{c_t^{\mathrm{w}}}}{\overline{w_t}}, \qquad \overline{l_t} = \overline{l_t^{\mathrm{w}}} + \xi \overline{l_t^{\mathrm{r}}} \psi_t,$$

$$1 - \overline{l^{\mathrm{r}}} = \frac{v_3}{v_1} \frac{\overline{c^{\mathrm{r}}}}{\xi \overline{w}}, \qquad 1 - \overline{l^{\mathrm{w}}} = \frac{v_3}{v_1} \frac{\overline{c^{\mathrm{w}}}}{\overline{w}}, \qquad \overline{l} = \overline{l^{\mathrm{w}}} + \xi \overline{l^{\mathrm{r}}} \psi.$$

Aggregate human capital of retirees [equation (8)]:

$$\overline{h_t^{\mathrm{r}}} = \overline{d_t^{\mathrm{r}}} + \gamma_t \frac{1+x}{1+r_t} \overline{h_{t+1}^{\mathrm{r}}},$$

$$\overline{h^{\mathrm{r}}} = \overline{d^{\mathrm{r}}} + \gamma \frac{1+x}{1+r} \overline{h^{\mathrm{r}}}.$$

Aggregate human capital of workers [equation (9)]:

$$\overline{h_t^{\mathrm{w}}} = \overline{d_t^{\mathrm{w}}} + \frac{\omega_t}{\Omega_{t+1}} \frac{1+x}{1+r_t} \overline{h_{t+1}^{\mathrm{w}}} + \left(1 - \frac{\omega_t}{\Omega_{t+1}} \right) \frac{1+x}{1+r_t} \overline{h_{t+1}^{\mathrm{r}}},$$

$$\overline{h^{\mathrm{w}}} = \overline{d^{\mathrm{w}}} + \frac{\omega}{\Omega} \frac{1+x}{1+r} \overline{h^{\mathrm{w}}} + \left(1 - \frac{\omega}{\Omega} \right) \frac{1+x}{1+r} \overline{h^{\mathrm{r}}}.$$

Aggregate disposable income of retirees and workers [equations (10) and (11)]:

$$\overline{d_t^{\mathrm{r}}} = \xi \overline{w_t} \overline{l_t^{\mathrm{r}}} + \frac{\overline{e_t}}{\psi_t}, \qquad \overline{d_t^{\mathrm{w}}} = \overline{w_t} \overline{l_t^{\mathrm{w}}} + \overline{f_t} - \overline{\tau_t},$$

$$\overline{d^{\mathrm{r}}} = \xi \overline{w} \overline{l^{\mathrm{r}}} + \frac{\overline{e}}{\psi}, \qquad \overline{d^{\mathrm{w}}} = \overline{w} \overline{l^{\mathrm{w}}} + \overline{f} - \overline{\tau}.$$

Aggregate consumption of retirees [equation (12)]:

$$\overline{c_t^{\mathrm{r}}} \left(1 + \frac{v_2}{v_1} \right) = \epsilon_t \pi_t \left[\frac{1+r_{t-1}}{\left(1+n_{t-1}^{\mathrm{r}}\right)(1+x)} \lambda_{t-1} \overline{a_{t-1}} + \overline{h_t^{\mathrm{r}}} \right],$$

$$\overline{c^{\mathrm{r}}} \left(1 + \frac{v_2}{v_1} \right) = \epsilon \pi \left[\frac{1+r}{(1+n)(1+x)} \lambda \overline{a} + \overline{h^{\mathrm{r}}} \right].$$

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Aggregate consumption of workers [equation (13)]:

$$\overline{c_t^{\mathrm{w}}}\left(1+\frac{v_2}{v_1}\right)=\pi_t\left[\frac{1+r_{t-1}}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)}(1-\lambda_{t-1})\overline{a_{t-1}}+\overline{h_t^{\mathrm{w}}}\right],$$

$$\overline{c^{\mathrm{w}}}\left(1+\frac{v_2}{v_1}\right)=\pi\left[\frac{1+r}{(1+n)(1+x)}(1-\lambda)\overline{a}+\overline{h^{\mathrm{w}}}\right].$$

Aggregate consumption and aggregate real balances [equations (14) and (15)]:

$$\overline{c_t}=\overline{c_t^{\mathrm{w}}}+\overline{c_t^{\mathrm{r}}}\psi_t,\qquad \overline{m_t}=\overline{m_t^{\mathrm{w}}}+\overline{m_t^{\mathrm{r}}}\psi_t=\frac{1+i_t}{i_t}\frac{v_2}{v_1}\overline{c_t},$$

$$\overline{c}=\overline{c^{\mathrm{w}}}+\overline{c^{\mathrm{r}}}\psi,\qquad \overline{m}=\overline{m^{\mathrm{w}}}+\overline{m^{\mathrm{r}}}\psi=\frac{1+i}{i}\frac{v_2}{v_1}\overline{c}.$$

Evolution of aggregate wealth of retirees [equation (16)]:

$$\lambda_t\overline{a_t}=\omega_t\left\{(1-\epsilon_t\pi_t)\left[\lambda_{t-1}\frac{(1+r_{t-1})}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)}\overline{a_{t-1}}+\overline{h^{\mathrm{r}}}\psi_t\right]-(\overline{h_t^{\mathrm{r}}}-\overline{d_t^{\mathrm{r}}})\psi_t\right\}+(1-\omega_t)\overline{a_t},$$

$$\lambda\overline{a}=\omega\left\{(1-\epsilon\pi)\left[\frac{\lambda(1+r)}{(1+n)(1+x)}\overline{a}+\overline{h^{\mathrm{r}}}\psi\right]-(\overline{h^{\mathrm{r}}}-\overline{d^{\mathrm{r}}})\psi\right\}+(1-\omega)\overline{a}.$$

Output [equation (17)]:

$$\overline{y_t}=\left[\int_0^1\overline{y_t(z)}^{\frac{\theta-1}{\theta}}dz\right]^{\frac{\theta}{\theta-1}},\quad \text{with } \overline{y_t(z)}=\left(\overline{l_t(z)}\right)^{\alpha}\left(\overline{k_t(z)}\right)^{1-\alpha},$$

$$\overline{y(z)}=\overline{y}=\overline{l}^{\alpha}\left[\frac{\overline{k}}{(1+n)(1+x)}\right]^{1-\alpha},\qquad \text{using } \overline{l(z)}=\overline{l} \text{ and } \overline{k(z)}=\frac{\overline{k}}{(1+n)(1+x)}.$$

Profits [equation (18)]:

$$\overline{f_t}=\int_0^1\left[\frac{P_t(z)}{P_t}-\mathrm{mc}_t\right]\overline{y_t(z)}dz,$$

$$\overline{f}=(1-\mathrm{mc})\,\overline{y}.$$

Aggregate capital–labor ratio and marginal costs [equations (19) and (20)]:

$$\frac{1}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)}\frac{\overline{k_{t-1}}}{\overline{l_t}}=\frac{\overline{w_t}}{r_t^k}\frac{1-\alpha}{\alpha},\qquad \mathrm{mc}_t=\left(\frac{\overline{w_t}}{\alpha}\right)^{\alpha}\left(\frac{r_t^k}{1-\alpha}\right)^{1-\alpha},$$

$$\frac{1}{(1+n)(1+x)}\frac{\overline{k}}{\overline{l}}=\frac{\overline{w}}{r^k}\frac{1-\alpha}{\alpha},\qquad \mathrm{mc}=\left(\frac{\overline{w}}{\alpha}\right)^{\alpha}\left(\frac{r^k}{1-\alpha}\right)^{1-\alpha}.$$

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Optimal price setting and evolution of aggregate price level [equations (21) and (22)]:

$$\frac{P_t^*}{P_t} = \frac{\theta}{\theta-1}\frac{E_t \sum_{i=0}^{\infty} (\zeta\beta)^i \left(\frac{1}{P_{t+i}}\right)^{1-\theta} \overline{y_{t+i}} N_{t+i}^{\mathrm{w}} X_{t+i} \mathrm{mc}_{t+i} \frac{P_{t+i}}{P_t}}{E_t \sum_{i=0}^{\infty} (\zeta\beta)^i \left(\frac{1}{P_{t+i}}\right)^{1-\theta} \overline{y_{t+i}} N_{t+i}^{\mathrm{w}} X_{t+i}},$$

$$P_t = \left[\zeta P_{t-1}^{1-\theta} + (1-\zeta) P_t^{*^{1-\theta}}\right]^{\frac{1}{1-\theta}}.$$

$$\mathrm{mc} = \frac{\theta-1}{\theta} \quad \text{using } P^*/P = 1.$$

Aggregate capital stock dynamics [equation (23)]:

$$\overline{k_t} = \phi\left[\frac{\overline{i_t^k}}{\overline{k_{t-1}}}\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)\right]\cdot\frac{\overline{k_{t-1}}}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)} + (1-\delta)\frac{\overline{k_{t-1}}}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)},$$

$$\overline{k} = \overline{i^k} + (1-\delta)\frac{\overline{k}}{(1+n)(1+x)}, \quad \text{using } \overline{i^k} = \overline{i^k(u)} \text{ and } \frac{\overline{k}}{(1+n)(1+x)} = \overline{k(u)}.$$

Price of capital, resource constraint, and interest rate relations [equations (24)–(26)]:

$$1 = p_t^k \phi'\left[\frac{\overline{i_t^k}}{\overline{k_{t-1}}}\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)\right], \quad \overline{y_t} = \overline{c_t} + \overline{g_t} + \overline{i_t^k}, \quad 1+i_t = (1+r_t)\left(\frac{P_{t+1}}{P_t}\right),$$

$$1 = p^k, \qquad \overline{y} = \overline{c} + \overline{g} + \overline{i^k}, \qquad i = r.$$

Consolidated public sector budget constraint [equation (27)]:

$$\overline{b_t} = (1+r_{t-1})\left[\frac{\overline{b_{t-1}}}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)} + \frac{\frac{1}{1+i_{t-1}}\overline{m_{t-1}}}{\left(1+n_{t-1}^{\mathrm{w}}\right)(1+x)}\right] + \overline{g_t} + \overline{e_t} - \overline{\tau_t} - \overline{m_t},$$

$$\overline{b} = (1+r)\left[\frac{\overline{b}}{(1+n)(1+x)} + \frac{\frac{1}{1+i}\overline{m}}{(1+n)(1+x)}\right] + \overline{g} + \overline{e} - \overline{\tau} - \overline{m}.$$

Aggregate pension benefits and aggregate nonhuman wealth [equations (28) and (29)]:

$$\overline{e_t} = \mu_t \overline{w_t} \psi_t, \qquad \overline{a_t} = p_t^k \overline{k_t} + \overline{b_t} + \frac{1}{1+i_t}\overline{m_t},$$

$$\overline{e} = \mu \overline{w} \psi, \qquad \overline{a} = p^k \overline{k} + \overline{b} + \frac{1}{1+i}\overline{m}.$$

No-arbitrage relationship [equation (30)]:

$$1 + r_t = \frac{r_{t+1}^k + p_{t+1}^k(1-\delta)}{p_t^k},$$

$$r = r^k - \delta.$$

Fiscal and monetary policy rules [equations (31) and (32)]:

$$\frac{\overline{\tau_t}}{\overline{y_t}} = \tau^* + \gamma_1 \left(\frac{\overline{b_t}}{\overline{y_t}} - b^* \right) + \gamma_2 \left(\frac{\overline{b_t}}{\overline{y_t}} - \frac{\overline{b_{t-1}}}{\overline{y_{t-1}}} \right),$$

$$\frac{\overline{\tau}}{\overline{y}} = \tau^*, \quad \frac{\overline{b}}{\overline{y}} = b^*, \quad i = r.$$