A model with Non-Tradable Goods

Valerio Nispi Landi*

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1 The Model[1]

The model presented in these lecture notes is an international business cycle model for a small open economy with non-tradable and tradable goods. With respect the Open Economy RBC Model in the Lecture Notes, I introduce a non-tradable good in order to give a role to the exchange rate. Specifically, households consume a consumption bundle that consists of a tradable and a non-tradable goods. The tradable good is the same in the small open economy and in the rest of the world.

1.1 Households

1.1.1 Intratemporal problem

The consumption bundle is defined as follows:

$$c_t = \left[\gamma^{\frac{1}{\eta}} c_{Tt}^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}},$$

where c_{Tt} and c_{Nt} denote consumption of tradable and non-tradable goods respectively. The investment bundle is defined analogously. The representative household decides how to allocate her consumption expenditure between the two goods. The problem is static and can be solved separately from the intertemporal optimization problem. The intratemporal problem is the following:

$$\max_{c_{Tt}, c_{Nt}} \left[\gamma^{\frac{1}{\eta}} c_{Tt}^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$s.t.\ p_{Tt} c_{Tt} + p_{Nt} c_{Nt} = Z_t,$$

*Harvard Kennedy School (Post-Doc Scholar) and Bank of Italy (Economist at the International Relations and Economics Directorate). Email: valerio.nispilandi@bancaditalia.it

[1] In these lecture notes I derive the equations used to simulate the model in the Dynare file ntg.mod and I explain how to compute the steady state. The notes are preliminary, if you find any error or inaccuracies, feel free to contact me. The views expressed in these notes are those of the author and do not necessarily reflect those of the Bank of Italy.

where p_{Tt} is the price of the tradable good, p_{Nt} is the price of the non-tradable good and Z_t is a given level of expenditure. Given that the domestic economy is sufficiently small with respect to the foreign economy, the price of the tradable good is exogenous. I normalized $p_T = 1$: the tradable good is the numeraire of the economy. Let ζ be the lagrangian multiplier. Foc wrt c_{Tt}:

$$\left[\gamma^{\frac{1}{\eta}} c_{Tt}^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}}\right]^{\frac{1}{\eta-1}} \gamma^{\frac{1}{\eta}} c_{Tt}^{-\frac{1}{\eta}} = \zeta$$
$$c_{Tt}^{-\frac{1}{\eta}} \gamma^{\frac{1}{\eta}} c_t^{\frac{1}{\eta}} = \zeta$$
$$c_{Ht} = \gamma \zeta^{-\eta} c_t.$$

Foc wrt c_{Nt}:

$$c_{Nt} = (1-\gamma)\left(\zeta P_{Nt}\right)^{-\eta} c_t.$$

Take the ratio:

$$\frac{c_{Tt}}{c_{Nt}} = \frac{\gamma}{1-\gamma} p_{Nt}^{\eta}$$
$$c_{Tt} = \frac{\gamma}{1-\gamma} p_{Nt}^{\eta} c_{Nt}.$$

Plug the latter expression in the consumption bundle:

$$c_t = \left[\gamma^{\frac{1}{\eta}} c_{Tt}^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$
$$c_t = \left[\gamma^{\frac{1}{\eta}} \gamma^{\frac{\eta-1}{\eta}} (1-\gamma)^{\frac{1-\eta}{\eta}} p_{Nt}^{\eta-1} c_{Nt}^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$
$$c_t^{\frac{\eta-1}{\eta}} = c_{Nt}^{\frac{\eta-1}{\eta}} \left[\gamma (1-\gamma)^{\frac{1-\eta}{\eta}} p_{Nt}^{\eta-1} + (1-\gamma)^{\frac{1}{\eta}}\right]$$
$$c_t^{\frac{\eta-1}{\eta}} = c_{Nt}^{\frac{\eta-1}{\eta}} p_{Nt}^{\eta-1} (1-\gamma)^{\frac{1-\eta}{\eta}} \left[\gamma + (1-\gamma) p_{Nt}^{1-\eta}\right]$$
$$c_t^{\frac{\eta-1}{\eta}} = c_{Nt}^{\frac{\eta-1}{\eta}} p_{Nt}^{\eta-1} (1-\gamma)^{\frac{1-\eta}{\eta}} p_t^{1-\eta}$$
$$c_{Nt} = (1-\gamma)\left(\frac{p_{Nt}}{p_t}\right)^{-\eta} c_t$$

and

$$c_{Tt} = \gamma \left(p_t\right)^{\eta} c_t,$$

where $p_t \equiv \left[\gamma + (1-\gamma) p_{Nt}^{1-\eta}\right]^{\frac{1}{1-\eta}}$ is the domestic CPI. Indeed, given that the domestic CPI is defined as the price of one unit of consumption, it follows:

$$
\begin{aligned}
p_{Tt}c_{Tt} + p_{Nt}c_{Nt} &= Z_t \\
p_{Tt}c_{Tt} + p_{Nt}c_{Nt} &= p_t c_t \\
c_{Tt} + p_{Nt}c_{Nt} &= p_t c_t \\
\gamma p_t^{\eta} c_t + p_{Nt}\left(1-\gamma\right)\left(\frac{p_{Nt}}{p_t}\right)^{-\eta} c_t &= p_t c_t \\
\gamma p_t^{\eta} + \left(1-\gamma\right) p_t^{\eta} p_{Nt}^{-\eta} &= p_t \\
p_t^{1-\eta} &= \gamma + \left(1-\gamma\right) p_{Nt}^{1-\eta} \\
p_t &= \left[\gamma + \left(1-\gamma\right) p_{Nt}^{1-\eta}\right]^{\frac{1}{1-\eta}} .
\end{aligned}
$$

Suppose that in the rest of the world a similar equation holds:

$$
p_t^* = \left[\gamma^* + \left(1-\gamma^*\right) p_{Nt}^{*1-\eta}\right]^{\frac{1}{1-\eta}} .
$$

Assume that p_{Nt}^* is constant over time and equal to one. It turns out $p_t^* = p^* \forall t$. Hence we can interpret p_t as a measure of the real exchange rate: a higher p_t means that the small open economy is experiencing a real appreciation. To sum up, it holds:

$$
\begin{aligned}
p_t &= \left[\gamma + \left(1-\gamma\right) p_{Nt}^{1-\eta}\right]^{\frac{1}{1-\eta}} && (1) \\
c_{Tt} &= \gamma \left(p_t\right)^{\eta} c_t && (2) \\
c_{Nt} &= \left(1-\gamma\right)\left(\frac{p_{Nt}}{p_t}\right)^{-\eta} c_t && (3) \\
c_t &= \left[\gamma^{\frac{1}{\eta}} c_{Tt}^{\frac{\eta-1}{\eta}} + \left(1-\gamma\right)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} && (4) \\
i_{Tt} &= \gamma \left(p_t\right)^{\eta} i_t && (5) \\
i_{Nt} &= \left(1-\gamma\right)\left(\frac{p_{Nt}}{p_t}\right)^{-\eta} i_t && (6) \\
i_t &= \left[\gamma^{\frac{1}{\eta}} i_{Tt}^{\frac{\eta-1}{\eta}} + \left(1-\gamma\right)^{\frac{1}{\eta}} i_{Nt}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} && (7)
\end{aligned}
$$

Notice that (2), (3) and (4) imply equation (1):

$$
\begin{aligned}
c_t &= \left[\gamma^{\frac{1}{\eta}} c_{Tt}^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} c_{Nt}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \\
c_t^{\frac{\eta-1}{\eta}} &= \left[\gamma^{\frac{1}{\eta}} \gamma^{\frac{\eta-1}{\eta}} p_t^{\eta-1} c_t^{\frac{\eta-1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}} (1-\gamma)^{\frac{\eta-1}{\eta}} p_t^{\eta-1} c_t^{\frac{\eta-1}{\eta}} p_N^{1-\eta}\right] \\
1 &= \left[\gamma p_t^{\eta-1} + (1-\gamma) p_t^{\eta-1} p_N^{1-\eta}\right] \\
p_t &= \left[\gamma + (1-\gamma) p_{Nt}^{1-\eta}\right]^{\frac{1}{1-\eta}} .
\end{aligned}
$$

Hence one equation is redundant (I will exclude equation 4). The same holds for the investment bundle, and I will exclude (7) from the set of equilibrium conditions.

1.1.2 Intertemporal problem

The representative household solves the following problem:

$$
\max_{\{c_t, i_t, h_t, k_t, b_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma} \left(c_t - \kappa_L \frac{h_t^{1+\varphi}}{1+\varphi}\right)^{1-\sigma}
$$

$$
\begin{cases}
p_t c_t + p_t i_t + b_t = r_t^k p_t k_{t-1} + r_{t-1}^* b_{t-1} + p_t w_t h_t - p_t t_t + p_t \Gamma_t - \frac{\kappa_D}{2} \left(b_t - \bar{b}\right)^2 \\
k_t = (1-\delta) k_{t-1} + \left[1 - \frac{\kappa_I}{2} \left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t,
\end{cases}
$$

where b_t denotes holding of a one-period foreign bond denominated in the tradable good, paying a real interest rate of r_t^*. The rest of the notation is standard. The foreign interest rate follows an exogenous process:

$$
r_t^* = (1-\rho^p) \frac{1}{\beta} + \rho^p r_{t-1}^* + v_t^p \tag{8}
$$

and $v_t^p \sim N\left(0, \sigma_p^2\right)$ is a foreign interest rate shock. Notice that domestic households pay a quadratic adjustment cost when they change their financial position with the rest of the world: this assumption ensures the existence of a determinate steady state and a stationary solution.[2] Rewrite the budget constraint in terms of the domestic CPI:

$$
c_t + i_t + \frac{b_t}{p_t} = r_t^k k_{t-1} + r_{t-1}^* \frac{b_{t-1}}{p_t} + w_t h_t - t_t + \Gamma_t - \frac{\kappa_D}{2} \frac{1}{p_t} \left(b_t - \bar{b}\right)^2 .
$$

The lagrangian function associated to the problem is the following:

$$
\begin{aligned}
\mathcal{L} \;=\; & \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{1-\sigma} \left(c_t - \kappa_L \frac{h_t^{1+\varphi}}{1+\varphi}\right)^{1-\sigma} - q_t \lambda_t \left\{ k_t - (1-\delta) k_{t-1} - \left[1 - \frac{\kappa_I}{2} \left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t \right\} - \right.\right. \\
& \left.\left. - \lambda_t \left[c_t + i_t + \frac{b_t - r_{t-1}^* b_{t-1}}{p_t} - r_t^k k_{t-1} - w_t h_t + t_t - \Gamma_t + \frac{\kappa_D}{2 p_t} \left(b_t - \bar{b}\right)^2 \right] \right\} \right\},
\end{aligned}
$$

[2]See Schmitt Grohe and Uribe (2003) and ? for a deep analysis on this issue.

where λ_t and q_t are lagrangian multipliers. Foc wrt consumption:

$$\lambda_t = \left(c_t - \kappa_L \frac{h_t^{1+\varphi}}{1+\varphi}\right)^{-\sigma}. \tag{9}$$

Foc wrt international bonds:

$$\lambda_t \left[1 + \kappa_D \left(b_t - \bar{b}\right)\right] = \beta \mathbb{E}_t \left(\lambda_{t+1} r_t^* \frac{p_t}{p_{t+1}}\right). \tag{10}$$

Foc wrt capital:

$$1 = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \frac{\left[r_{t+1}^k + (1-\delta) q_{t+1}\right]}{q_t} \right\}. \tag{11}$$

Foc wrt labor:

$$\kappa_L h_t^{\varphi} = w_t. \tag{12}$$

Foc wrt investment:

$$1 = q_t \left[1 - \frac{\kappa_I}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2 - \kappa_I \left(\frac{i_t}{i_{t-1}}\right)\left(\frac{i_t}{i_{t-1}} - 1\right)\right] + \kappa_I \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left[\left(\frac{i_{t+1}}{i_t}\right)^2 \left(\frac{i_{t+1}}{i_t} - 1\right)\right] \right\}. \tag{13}$$

1.2 Firms

There is a continuum of firms producing the tradable good and a continuum of firms producing the non-tradable good. These firms act in perfect competition. Capital and labor supplied by households are mobile across sectors, so they should cost the same price.

1.2.1 Tradable goods

The representative tradable good firm uses the following production function to produce the tradable good:

$$y_{Tt} = a_{Tt} k_{Tt}^{\alpha_T} h_{Tt}^{1-\alpha_T}, \tag{14}$$

where k_{Tt} and h_{Tt} are capital and labor employed in the tradable sector; a_{Tt} is the total factor productivity in the tradable sector, which follows an autoregressive process:

$$\log (a_{Tt}) = (1 - \rho_T) \log (\bar{a}_T) + \rho_T \log (a_{Tt-1}) + v_{Tt}^a \tag{15}$$

and $v_{Tt}^a \sim N\left(0, \sigma_{Ta}^2\right)$ is a technology shock specific to the tradable sector. Notice that capital has the index t, not the index $t-1$ as usual: in period $t-1$ households choose how much capital will be used by firms in period t; however, in this model with two sectors, the allocation across sectors is made in period t. The maximization problem of profit $p_t \Gamma_{Tt}$ is static and reads:

$$\max_{k_{Tt}, h_{Tt}} p_t \Gamma_{Tt} = \max_{k_{Tt}, h_{Tt}} \left(a_{Tt} k_{Tt}^{\alpha_T} h_{Tt}^{1-\alpha_T} - p_t w_t h_{Tt} - p_t r_t^K k_{Tt}\right).$$

Foc wrt to capital:

$$\alpha_T a_{Tt} k_{Tt}^{\alpha_T - 1} h_{Tt}^{1-\alpha_T} = p_t r_t^K$$
$$\alpha_T y_{Tt} = p_t r_t^K k_{Tt}. \tag{16}$$

Foc wrt to labor:

$$(1-\alpha_T)\, y_{Tt} = p_t w_t h_{Tt}. \tag{17}$$

1.2.2 Non-Tradable goods

The representative non-tradable good firm uses the following production function to produce the non-tradable good:

$$y_{Nt} = a_{Tt} k_{Nt}^{\alpha_N} h_{Nt}^{1-\alpha_N}, \tag{18}$$

where k_{Nt} and h_{Nt} are capital and labor employed in the non-tradable sector; a_{Nt} is the total factor productivity in the non-tradable sector, which follows an autoregressive process:

$$\log(a_{NTt}) = (1-\rho_N)\log(\overline{a}_N) + \rho_N \log(a_{Nt-1}) + v_{Nt}^a \tag{19}$$

and $v_{Tt}^a \sim N(0, \sigma_{Ta}^2)$ is a technology shock specific to the non-tradable sector. The maximization problem of profit $p_t \Gamma_{Nt}$ is static and reads:

$$\max_{k_{Nt}, h_{Nt}} p_t \Gamma_{Nt} = \max_{k_{Nt}, h_{Nt}} \left(p_{Nt} a_{Nt} k_{Nt}^{\alpha_N} h_{Nt}^{1-\alpha_N} - w_t p_t h_{Nt} - r_t^K p_t k_{Nt} \right).$$

Foc wrt to capital:

$$\alpha_N p_{Nt} a_{Nt} k_{Nt}^{\alpha_N - 1} h_{Nt}^{1-\alpha_N} = p_t r_t^K$$
$$\alpha_N y_{Nt} = \frac{p_t}{p_N} r_t^K k_{Nt}. \tag{20}$$

Foc wrt to labor:

$$(1-\alpha_N)\, y_{Nt} = \frac{p_t}{p_N} w_t h_{Nt}. \tag{21}$$

1.3 Policy

The public consumption bundle has the same composition of the consumption and investment bundles. It holds:

$$g_{Tt} = \gamma (p_t)^{\eta} g_t \tag{22}$$

$$g_{Nt} = (1-\gamma) \left(\frac{p_{Nt}}{p_t} \right)^{-\eta} g_t. \tag{23}$$

Government finances public expenditure g_t by raising lump-sum taxes:

$$g_t = t_t,$$

where g_t follows an autoregressive process:

$$\log(g_t) = (1-\rho_g)\log(\overline{g}) + \rho_g \log(g_{t-1}) + v_t^g \tag{24}$$

and $v_t^g \sim N\left(0, \sigma_g^2\right)$ is a public spending shock.

1.4 Market clearing

The market clearing condition for the non-tradable good is the following:

$$y_{Nt} = c_{Nt} + i_{Nt} + g_{Nt}.$$

Using the demand function of the non-tradable good, I obtain:

$$y_{Nt} = (1-\gamma)\left(\frac{p_{Nt}}{p_t}\right)^{-\eta}(c_t + i_t + g_t). \tag{25}$$

Market clearing for the capital good:

$$k_{t-1} = k_{Tt} + k_{Nt}.$$

Notice that allocation of capital supplied in period $t-1$ is decided in period t. Market clearing for labor:

$$h_t = h_{Tt} + h_{Ht}.$$

Take the budget constraint and rearrange (using $\Gamma_t = \Gamma_{Nt} + \Gamma_{Tt}$):

$$c_t + i_t + \frac{b_t}{p_t} = r_t^k k_{t-1} + r_{t-1}^* \frac{b_{t-1}}{p_t} + w_t h_t - t_t + \Gamma_t - \frac{\kappa_D}{2}\frac{1}{p_t}\left(b_t - \overline{b}\right)^2$$

$$c_t + i_t + \frac{b_t}{p_t} = r_t^k k_{t-1} + r_{t-1}^* \frac{b_{t-1}}{p_t} + w_t h_t - g_t + \frac{p_{Nt}}{p_t} y_{Nt} - w_t h_{Nt} - r_t^K k_{Nt} +$$
$$+\frac{1}{p_t} y_{Tt} - w_t h_{Tt} - r_t^K k_{Tt} - \frac{\kappa_D}{2}\frac{1}{p_t}\left(b_t - \overline{b}\right)^2$$

$$c_t + i_t + g_t + \frac{b_t}{p_t} = r_{t-1}^* \frac{b_{t-1}}{p_t} + \frac{p_{Nt}}{p_t} y_{Nt} + \frac{1}{p_t} y_{Tt} - \frac{\kappa_D}{2}\frac{1}{p_t}\left(b_t - \overline{b}\right)^2$$

$$c_t + i_t + g_t + \frac{b_t}{p_t} - r_{t-1}^* \frac{b_{t-1}}{p_t} + \frac{\kappa_D}{2}\frac{1}{p_t}\left(b_t - \overline{b}\right)^2 = \frac{p_{Nt} y_{Nt} + y_{Tt}}{p_t}$$

$$c_t + i_t + g_t + tb_t = gdp_t, \tag{26}$$

where $gdp_t \equiv \frac{p_{Nt}y_{Nt}+y_{Tt}}{p_t}$ and $tb_t \equiv \frac{b_t}{p_t} - r^*_{t-1}\frac{b_{t-1}}{p_t} + \frac{\kappa_D}{2}\frac{1}{p_t}\left(b_t - \bar{b}\right)^2$is the trade balance. If the algebra is correct, also the tradable good market should clear:

$$y_{Tt} = c_{Tt} + i_{Tt} + g_{Tt} + b_t - r^*_{t-1}b_{t-1} + \frac{\kappa_D}{2}\left(b_t - \bar{b}\right)^2$$

$$y_{Tt} = c_{Tt} + i_{Tt} + g_{Tt} + p_t tb_t, \tag{27}$$

where the trade balance measures the amount of tradable good exported in the rest of the world. Rearrange the national account identity:

$$\frac{p_{Nt}y_{Nt}+y_{Tt}}{p_t} = c_t + i_t + g_t + tb_t$$

$$\frac{p_{Nt}y_{Nt}+y_{Tt}}{p_t} = \frac{1}{p_t}\left(p_{Nt}c_{Nt} + c_{Ht}\right) + \frac{1}{p_t}\left(p_{Nt}i_{Nt} + i_{Ht}\right) + \frac{1}{p_t}\left(p_{Nt}g_{Nt} + g_{Ht}\right) + tb_t$$

$$\frac{p_{Nt}y_{Nt}+y_{Tt}}{p_t} = \frac{p_{Nt}}{p_t}\left(c_{Nt} + i_{Nt} + g_{Nt}\right) + \frac{1}{p_t}\left(c_{Ht} + i_{Ht} + g_{Ht}\right) + tb_t$$

$$\frac{p_{Nt}y_{Nt}+y_{Tt}}{p_t} = p_{Nt}y_{Nt} + \frac{1}{p_t}\left(c_{Ht} + i_{Ht} + g_{Ht}\right) + tb_t$$

$$\frac{y_{Tt}}{p_t} = \frac{1}{p_t}\left(c_{Ht} + i_{Ht} + g_{Ht}\right) + tb_t$$

$$y_{Tt} = \left(c_{Ht} + i_{Ht} + g_{Ht}\right) + p_t tb_t,$$

which is exactly equation (27). Hence the algebra is correct, equation (27) is redundant by Walras Law and should not be included in the set of equilibrium conditions. Finally, define debt as follows:

$$d_t = -b_t.$$

2 Equilibrium

The equilibrium conditions of the model are the following:

$$
\begin{aligned}
\lambda_t &= \left(c_t - \kappa_L \frac{h_t^{1+\varphi}}{1+\varphi}\right)^{-\sigma} && \text{FOC for consumption} \\
\lambda_t \left[1 - \kappa_D \left(d_t - \bar{d}\right)\right] &= \beta \mathbb{E}_t \left(\lambda_{t+1} r_t^* \frac{p_t}{p_{t+1}}\right) && \text{FOC for international bonds} \\
\kappa_L h_t^{\varphi} &= w_t && \text{FOC for labor supply} \\
1 &= \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \frac{\left[r_{t+1}^k + (1-\delta) q_{t+1}\right]}{q_t} \right\} && \text{FOC for capital Tobin's Q} \\
1 &= q_t \left[1 - \frac{\kappa_I}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2 - \kappa_I \left(\frac{i_t}{i_{t-1}}\right)\left(\frac{i_t}{i_{t-1}} - 1\right)\right] + \\
&\quad + \kappa_I \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left[\left(\frac{i_{t+1}}{i_t}\right)^2 \left(\frac{i_{t+1}}{i_t} - 1\right)\right]\right\} && \text{FOC for investment} \\
k_t &= (1-\delta) k_{t-1} + \left[1 - \frac{\kappa_I}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t && \text{Law of motion for capital} \\
y_{Tt} &= a_{Tt} k_{Tt}^{\alpha_T} h_{Tt}^{1-\alpha_T} \\
\alpha_T y_{Tt} &= p_t r_t^K k_{Tt} \\
(1-\alpha_T) y_{Tt} &= p_t w_t h_{Tt} \\
y_{Nt} &= a_{Nt} k_{Nt}^{\alpha_N} h_{Nt}^{1-\alpha_N} \\
\alpha_N y_{Nt} &= \frac{p_t}{p_{Nt}} r_t^K k_{Nt} \\
(1-\alpha_N) y_{Nt} &= \frac{p_t}{p_{Nt}} w_t h_{Nt} \\
y_{Nt} &= (1-\gamma) \left(\frac{p_{Nt}}{p_t}\right)^{-\eta} (c_t + i_t + g_t) \\
k_{t-1} &= k_{Tt} + k_{Nt} \\
h_t &= h_{Tt} + h_{Ht} \\
c_t + i_t + g_t + tb_t &= gdp_t
\end{aligned}
$$

$$
\begin{aligned}
tb_t &= -\frac{d_t}{p_t} + r^*_{t-1}\frac{d_{t-1}}{p_t} + \frac{\kappa_D}{2}\frac{1}{p_t}\left(d_t - \overline{d}\right)^2 \\
gdp_t &= \frac{p_{Nt}y_{Nt} + y_{Tt}}{p_t} \\
p_t &= \left[\gamma + (1-\gamma)\, p_{Nt}^{1-\eta}\right]^{\frac{1}{1-\eta}} \\
\log(a_{Tt}) &= (1-\rho_T)\log(\overline{a}_T) + \rho_T \log(a_{Tt-1}) + v^a_{Tt} \\
\log(a_{NTt}) &= (1-\rho_N)\log(\overline{a}_N) + \rho_N \log(a_{Nt-1}) + v^a_{Nt} \\
\log(g_t) &= (1-\rho_g)\log(\overline{g}) + \rho_g \log(g_{t-1}) + v^g_t \\
r^*_t &= (1-\rho^p)\frac{1}{\beta} + \rho^p r^*_{t-1} + v^p_t .
\end{aligned}
$$

There are 23 equations for 23 endogenous variables:

$$X_t \equiv \left[\lambda_t, c_t, r^k_t, w_t, h_t, k_t, q_t, i_t, y_{Tt}, y_{Nt}, k_{Tt}, k_{NT}, h_{Tt}, h_{Nt}, d_t, p_t, p_{Nt}, tb_t, gdp_t, a_{Nt}, a_{Tt}, g_t, r^*_t\right].$$

The model features 4 exogenous shocks: $v_t \equiv \left[v^a_{Tt}, v^a_{Nt}, v^g_t, v^p_t\right]$.

3 Steady State

Variables without time index denote the steady state level. The four stochastic processes imply in the steady state:

$$a_T = \overline{a}_T$$

$$a_N = \overline{a}_N$$

$$r^* = \frac{1}{\beta}$$

$$g = \overline{g}.$$

Calibrate ex ante the yearly external debt GDP ratio:

$$D = \frac{\overline{d}}{4p \cdot gdp}$$

and compute $\overline{d}$ ex post. Calibrate ex ante the the public spending-GDP share:

$$G = \frac{\overline{g}}{4gdp}$$

and compute $\overline{g}$ ex post. In the steady state (13) implies:

$$q = 1,$$

which gives according to (11):

$$r^k = \frac{1}{\beta} - (1 - \delta).$$

Now I follow this strategy. Given that finding an analytical solution is not trivial, I simplify the model in a system of 4 equations in 4 unknowns, p_N, k_N, k_T, h_N. All the other equations will depend only on the 4 unknowns. Price level:

$$p = \left[\gamma + (1 - \gamma) p_N^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

Use capital market clearing:

$$k = k_T + k_N.$$

Use the law of motion of capital:

$$i = \delta k.$$

Use labor market clearing:

$$h_T = h - h_T.$$

Use the production functions:

$$y_T = a_T k_T^{\alpha_T} h_T^{1-\alpha_T}$$

$$y_N = a_N k_N^{\alpha_N} h_N^{1-\alpha_N}.$$

Use the capital demand in the non-tradable sector:

$$y_N = \frac{p}{\alpha_N p_N} r^K k_N.$$

GDP definition:

$$gdp = \frac{p_N y_N + y_T}{p}.$$

Debt:

$$d = 4D \cdot p \cdot gdp.$$

Public spending:

$$g = G \cdot gdp.$$

Use the trade balance definition:

$$tb = \frac{d}{p}\left(\frac{1}{\beta} - 1\right).$$

Resource constraint:

$$c = gdp - i - g - tb.$$

Use the labor demand in the tradable sector:

$$w = \frac{(1 - \alpha_T)\, y_T}{p h_T}.$$

We are left with 4 equations, which are function (after replacing the previous expressions) only of p_N, k_N, k_T, h_N:

$$\begin{aligned}
y_N &= (1 - \gamma) \left(\frac{p_N}{p} \right)^{-\eta} (c + i + g) \\
\alpha_N y_{Nt} &= \frac{p_t}{p_{Nt}} r_t^K k_{Nt} \\
(1 - \alpha_N)\, y_{Nt} &= \frac{p_t}{p_{Nt}} w_t h_{Nt} \\
\alpha_T y_{Tt} &= p_t r_t^K k_{Tt}
\end{aligned}$$

One can solve in Matlab this system of 4 equations and 4 unknowns. Finally, one can find κ_L by using the labor supply:

$$\kappa_L = \frac{w}{h^{\varphi}}.$$

and λ::

$$\lambda = \left(c - \kappa_L \frac{h^{1+\varphi}}{1 + \varphi} \right)^{-\sigma}.$$