Banking Sector in Karadi & Nakov (2021)

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This document is intended to offer an overview about the financial intermediary's value function and the derivation of the banking sector FOCs found in Karadi & Nakov (2021).

1 Overview about the Banking Sector

The banking sector is characterized in aggregate by the following set of stochastic difference equations.

$$E_t\Lambda_{t,t+1}\Omega_{t+1}(R_{k,t+1} - R_{t+1}) = \theta\lambda_t \tag{1}$$
$$E_t\Lambda_{t,t+1}\Omega_{t+1}(R_{b,t+1} - R_{t+1}) = \Gamma\theta\lambda_t \tag{2}$$
$$\xi_t = \frac{E_t\Lambda_{t,t+1}(\Omega_{t+1} - 1)}{\zeta} \tag{3}$$
$$\Omega_t = 1 - \sigma + \sigma\eta_t \tag{4}$$
$$\eta_t = \frac{\nu_t}{1 - \lambda_t} \tag{5}$$
$$\nu_t = E_t\Lambda_{t,t+1}\left(\Omega_{t+1}R_{t+1} + \frac{(\Omega_{t+1} - 1)^2}{2\zeta}\right) \tag{6}$$
$$\lambda_t = \max\left(0, 1 - \frac{\nu_t}{\theta\phi_t}\right) \tag{7}$$
$$\phi_t = \frac{Q_tS_t + \Gamma q_tB_{b,t}}{N_t} \tag{8}$$
$$\bar{\phi}_t = \frac{E_t\Lambda_{t,t+1}\Omega_{t+1}R_{t+1}}{\theta - E_t\Lambda_{t,t+1}\Omega_{t+1}(R_{k,t+1} - R_{t+1})} \tag{9}$$
$$N_t = N_t^e + \omega_t \tag{10}$$
$$N_t^e = \sigma\left[(R_{k,t} - R_t)Q_{t-1}S_{t-1} + (R_{b,t} - R_t)q_{t-1}B_{b,t-1} + R_tN_{t-1} + \xi_{t-1}N_{t-1}\right] \tag{11}$$
$$\omega_t = \bar{\omega} + e_{\omega,t} \tag{12}$$

2 Derivations

First, let us recall the problem faced by each intermediary. An individual intermediary maximizes the terminal wealth (which can be considered as a stochastic dividend payment)

$$V_t = \max E_t\Lambda_{t,t+1}\left[(1 - \sigma)n_{t+1} + \sigma(W_{t+1} - e_t - \frac{\zeta}{2}\xi_t^2 n_t)\right] \tag{13}$$

with $\xi_t = e_t/n_t$ and where it needs to take into account the dynamic law of motion of its stock of net worth

$$n_t = (R_{k,t} - R_t)Q_{t-1}s_{t-1} + (R_{b,t} - R_t)q_{t-1}b_{t-1} + R_tn_{t-1} + e_{t-1} \tag{14}$$

as well as the incentive constraint stemming from the agency problem

$$V_t \geq \theta Q_ts_t + \Gamma\theta q_tb_t. \tag{15}$$

2.1 Guess and verify approach

Suppose the solution for the value function is given by $V_t = \mu_t^s Q_t s_t + \mu_t^b q_t b_t + \nu_t n_t$. Maximizing the value function subject to the incentive constraint yields:

$$\mathcal{L} = V_t + v_t(V_t - \theta(Q_t s_t + \Gamma q_t b_t)) \tag{16}$$
$$= (1+v_t)\left[\mu_t^s Q_t s_t + \mu_t^b q_t b_t + \nu_t n_t\right] - v_t\theta(Q_t s_t + \Gamma q_t b_t) \tag{17}$$
$$\frac{\partial \mathcal{L}}{\partial s_t} = (1+v_t)\mu_t^s Q_t - v_t\theta Q_t = 0 \implies \mu_t^s = \frac{v_t}{1+v_t}\theta \tag{18}$$
$$\frac{\partial \mathcal{L}}{\partial b_t} = (1+v_t)\mu_t^b q_t - v_t\theta\Gamma q_t = 0 \implies \mu_t^b = \frac{v_t}{1+v_t}\theta\Gamma \tag{19}$$
$$\frac{\partial \mathcal{L}}{\partial v_t} = \mu_t^s Q_t s_t + \mu_t^b q_t b_t + \nu_t n_t - \theta(Q_t s_t + \Gamma q_t b_t) = 0 \tag{20}$$
$$\implies Q_t s_t = \frac{\mu_t^b - \theta\Gamma}{\theta - \mu_t^s} q_t b_t + \frac{\nu_t}{\theta - \mu_t^s} n_t = \frac{\frac{-1}{1+v_t}\theta\Gamma}{\frac{1}{1+v_t}\theta} q_t b_t + \frac{\nu_t}{\frac{1}{1+v_t}\theta} n_t \tag{21}$$
$$\implies Q_t s_t = -\Gamma q_t b_t + (1+v_t)\frac{\nu_t}{\theta} n_t \tag{22}$$
$$V_t = \mu_t^s Q_t s_t + \mu_t^b q_t b_t + \nu_t n_t \tag{23}$$
$$= \frac{v_t}{1+v_t}\theta\left[-\Gamma q_t b_t + (1+v_t)\frac{\nu_t}{\theta} n_t\right] + \frac{v_t}{1+v_t}\theta\Gamma q_t b_t + \nu_t n_t \tag{24}$$
$$= v_t\nu_t n_t + \nu_t n_t = (1+v_t)\nu_t n_t \tag{25}$$
$$= \eta_t n_t \tag{26}$$

By definition

$$\lambda_t \equiv \frac{v_t}{1+v_t} \implies 1-\lambda_t = \frac{1}{1+v_t} \implies 1+v_t = \frac{1}{1-\lambda_t}. \tag{27}$$

This implies equation (5)

$$\eta_t \equiv (1+v_t)\nu_t = \frac{\nu_t}{1-\lambda_t}. \tag{28}$$

Using (26), the definition in equation (4) and the law of motion of the bankers net worth (14) implies for the value function (13)

$$V_t = \max E_t\Lambda_{t,t+1}\left[(1-\sigma)n_{t+1} + \sigma(W_{t+1} - e_t - \frac{\zeta}{2}\xi_t^2 n_t)\right] \tag{29}$$
$$= E_t\Lambda_{t,t+1}\left((1-\sigma) + \sigma\eta_{t+1}\right)n_{t+1} - \sigma E_t\Lambda_{t,t+1}\left[e_t + \frac{\zeta}{2}\xi_t^2 n_t)\right] \tag{30}$$
$$= E_t\Lambda_{t,t+1}\Omega_{t+1}\left[(R_{k,t+1} - R_{t+1})Q_t s_t + (R_{b,t+1} - R_{t+1})q_t b_t + R_{t+1}n_t + e_t\right] \tag{31}$$
$$- \sigma E_t\Lambda_{t,t+1}\left[e_t + \frac{\zeta}{2}\xi_t^2 n_t)\right] \tag{32}$$
$$= E_t\Lambda_{t,t+1}\Omega_{t+1}\left[(R_{k,t+1} - R_{t+1})\right]Q_t s_t + E_t\Lambda_{t,t+1}\Omega_{t+1}\left[(R_{b,t+1} - R_{t+1})\right]q_t b_t \tag{33}$$
$$+ E_t\Lambda_{t,t+1}\Omega_{t+1}\left[R_{t+1}n_t + e_t\right] - \sigma E_t\Lambda_{t,t+1}\left[e_t + \frac{\zeta}{2}\xi_t^2 n_t)\right]. \tag{34}$$

Matching the last equation with the initial guess for the solution of the value function determines the values for the undetermined coefficients:

$$\mu_t^s = E_t\Lambda_{t,t+1}\Omega_{t+1}\left[(R_{k,t+1} - R_{t+1})\right] = \theta\lambda_t \tag{35}$$

$$\mu_t^b = E_t\Lambda_{t,t+1}\Omega_{t+1}\left[(R_{b,t+1} - R_{t+1})\right] = \theta\Gamma\lambda_t \tag{36}$$

$$\nu_t = ??? \tag{37}$$

$$\tag{38}$$

From the incentive constraint we derive expression (7) and (9).

$$V_t \geq \theta Q_t s_t + \Gamma\theta q_t b_t \tag{39}$$

$$\eta_t n_t \geq \theta(Q_t s_t + \Gamma q_t b_t) \tag{40}$$

$$\eta_t \geq \theta\frac{Q_t s_t + \Gamma\theta q_t b_t}{n_t} \tag{41}$$

$$\eta_t \geq \theta\frac{Q_t s_t + \Gamma\theta q_t b_t}{n_t} \tag{42}$$

$$\eta_t \geq \theta\phi_t \tag{43}$$

Using (5) implies

$$\frac{\nu_t}{1-\lambda_t} \geq \theta\phi_t \implies \lambda_t \geq 1 - \frac{\nu_t}{\theta\phi_t}. \tag{44}$$

Since in equilibrium excess returns are non-negative it must hold that $\lambda_t \geq 0$. This implies expression (7)

$$\lambda_t = \max\left(0, 1 - \frac{\nu_t}{\theta\phi_t}\right) \tag{45}$$

Alternatively, an upper bound for the leverage ratio can be derived from (43) once we impose equality.

$$\eta_t = \theta\bar{\phi}_t \tag{46}$$

$$\bar{\phi}_t = \frac{\eta_t}{\theta} \tag{47}$$

Using (5) and (1) implies

$$\bar{\phi}_t = \frac{\nu_t}{\theta - \theta\lambda_t} \tag{48}$$

$$= \frac{\nu_t}{\theta - E_t\Lambda_{t,t+1}\Omega_{t+1}(R_{k,t+1} - R_{t+1})}. \tag{49}$$

Comparing equation (49) with equation (11) in Karadi & Nakov (2021) shows that ν_t should be

$$\nu_t = E_t\Lambda_{t,t+1}\Omega_{t+1}R_{t+1} \tag{50}$$