Part I

Fiscal Policy in the RBC Model with Heterogeneous agents

1 The Model's Specifications

1.1 Capitalis and Workers

It is assumed that there are N^k identical capitalists and N^w identical workers and the total population size is given by N. $N^k + N^w = N$ The population shares of the two groups are assumed to be: $n^k \equiv N^k/N$ and $n^w \equiv N^w/N = 1 - n^k$ Each capitalist owns a single firm, and hence the number of firms is equal to the number of capitalists, $N^f = N^k$ The capitalist and worker are indexed by superscript k and w, respectively, while firm is indexed by f.

The functional forms employed is a CRRA utility function.

$$u\left(c_t, 1-h_t\right) = \frac{\left(c_t^{\theta}\left(1-h_t\right)^{1-\theta}\right)^{1-\sigma}}{1-\sigma} \tag{3.1}$$

We fix $h_t = 0$ for the capitalists in their utility fuction. The objective of the representative capitalist is to maximise its lifetime utility:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_t^k\left(c_t^k\right) \text{ with } u_t^k\left(c_t^k\right) = \frac{\left(c_t^k\right)^{\theta(1-\sigma)}}{1-\sigma} \tag{3.2}$$

$$s.t.\ c_t^k + k_{t+1}^k - (1-\delta)\, k_t^k = \left(1-\tau_{kt}\right) r_t k_t^k$$

The Lagrangian function of the capitalist is as follows:

$$\Lambda^k = \sum_{t=0}^{\infty} \left\{\beta^t u_t^k\left(c_t^k\right) + \lambda_t \left\{\left(1-\tau_{kt}\right) r_t k_t^k - c_t^k - \left[k_{t+1}^k - (1-\delta)\, k_t^k\right]\right\}\right\}$$

The FOC with respect to $\left\{c_t^k, k_{t+1}^k\right\}$ are given by

$$\frac{\partial \Lambda}{\partial c_t^k} = \beta^t \theta \left(c_t^k\right)^{\theta(1-\sigma)-1} - \lambda_t = 0 \tag{3.3}$$

$$\frac{\partial \Lambda}{\partial k_{t+1}^k} = -\lambda_t + \lambda_{t+1}\left[\left(1-\tau_{kt+1}\right) r_{t+1} + (1-\delta)\right] = 0 \tag{3.4}$$

Combine these two equations, and then yield the consumption Euler equation of capitalist.

$$\beta \left(c_{t+1}^{k}\right)^{\theta(1-\sigma)-1}\left[\left(1-\tau_{kt+1}\right) r_{t+1}+(1-\delta)\right]=\left(c_{t}^{k}\right)^{\theta(1-\sigma)-1} \tag{3.5}$$

The workers do not save so that they do not face intertemporal problem. The optimality for them is static.

At time t, the objective function of the representative worker is given by:

$$\max_{c_t, h_t} u_{t}^{w}\left(c_{t}^{w}, 1-h_{t}^{w}\right) \text{ with } u_{t}^{w}\left(c_{t}^{w}, 1-h_{t}^{w}\right)=\frac{\left(c_{t}^{\theta}\left(1-h_{t}\right)^{1-\theta}\right)^{1-\sigma}}{1-\sigma} \tag{3.6}$$

$$s.t.\ c_{t}^{w}=\left(1-\tau_{ht}\right) w_{t} h_{t}^{w}$$

The time constraint of the worker is

$$h_{t}^{w}+l_{t}^{w}=1$$

where l_{t}^{w} denotes the leisure of the worker, h_{t}^{w} denotes the labor supply.

The Lagrangian function of the worker is shown as follows:

$$\Lambda^{w}=u_{t}^{w}\left(c_{t}^{w}, 1-h_{t}^{w}\right)+\phi_{t}\left[\left(1-\tau_{ht}\right) w_{t} h_{t}^{w}-c_{t}^{w}\right]$$

where ϕ_{t} is the Lagrangian multiplier on the worker's budget constraint.

The FOC with respect to $\{c_{t}^{w}, h_{t}^{w}\}$ are given by

$$\frac{\partial \Lambda}{\partial c_{t}^{w}} = \theta \frac{\left(c_{t}^{\theta}\left(1-h_{t}\right)^{1-\theta}\right)^{1-\sigma}}{c_{t}}-\phi_{t}=0 \tag{3.7}$$

$$\frac{\partial \Lambda}{\partial h_{t}^{w}} = -(1-\theta) \frac{\left(c_{t}^{\theta}\left(1-h_{t}\right)^{1-\theta}\right)^{1-\sigma}}{1-h_{t}}+\phi_{t}\left(1-\tau_{ht}\right) w_{t}=0 \tag{3.8}$$

Consolidating these two FOCs yields

$$\frac{(1-\theta)}{\theta} \frac{c_{t}^{w}}{1-h_{t}^{w}}=\left(1-\tau_{ht}\right) w_{t} \tag{3.9}$$

1.2 Firm

A representative firm's output is obtained from using capital and labor input using a neoclassical constant returns production function $F\left(k_{t}^{f}, h_{t}^{f}\right)=A\left(k_{t}^{f}\right)^{\alpha}\left(h_{t}^{f}\right)^{1-\alpha}$

The firm has to maximize its profits at time t:

$$Y=N y_{t}^{f}$$

$$\max_{h_t^f, k_t^f} \pi_t^f = y_t^f - r_t k_t^f - w_t h_t^f \tag{3.10}$$

$$s.t.\ \ y_t^f = F\left(k_t^f, h_t^f\right)\ \forall t \geq 0$$

The FOCs with respect to k_t^f and h_t^f are given by

$$\alpha A \left(k_t^f\right)^{\alpha-1} \left(h_t^f\right)^{1-\alpha} = r_t \tag{3.11}$$

$$(1-\alpha) A \left(k_t^f\right)^{\alpha} \left(h_t^f\right)^{-\alpha} = w_t \tag{3.12}$$

1.3 Government

At time t, the aggregate budget constraint of the government is:

$$N \cdot g_t = N^k \tau_{kt} r_t k_t^k + N^w \tau_{nt} w_t h_t^w$$

$$g_t = n^k \tau_{kt} r_t k_t^k + \left(1-n^k\right) \tau_{ht} w_t h_t^w \tag{3.13}$$

g_t here is per capita public spending, which is allowed to be residually determined to balance the government budget constraint in each period.

1.4 Equilibrium Conditions

Now I will summarize the equilibrium conditions.

$$\beta \left(c_{t+1}^k\right)^{\theta(1-\sigma)-1} \left[\left(1-\tau_{kt+1}\right) r_{t+1} + (1-\delta)\right] = \left(c_t^k\right)^{\theta(1-\sigma)-1} \tag{3.14}$$

$$\frac{(1-\theta)}{\theta} \frac{c_t^w}{1-h_t^w} = \left(1-\tau_{ht}\right) w_t \tag{3.15}$$

$$\alpha A \left(k_t^f\right)^{\alpha-1} \left(h_t^f\right)^{1-\alpha} = r_t \tag{3.16}$$

$$(1-\alpha) A \left(k_t^f\right)^{\alpha} \left(h_t^f\right)^{-\alpha} = w_t \tag{3.17}$$

$$c_t^k + k_{t+1}^k - (1-\delta) k_t^k = \left(1-\tau_{kt}\right) r_t k_t^k \tag{3.18}$$

$$c_t^w = \left(1-\tau_{ht}\right) w_t h_t^w \tag{3.19}$$

$$g_t = n^k \tau_{kt} r_t k_t^k + \left(1-n^k\right) \tau_{ht} w_t h_t^w \tag{3.20}$$

$$k_t^k = k_t^f \tag{3.21}$$

$$h_t^w = \frac{n^k}{(1-n^k)} h_t^f \tag{3.22}$$