Model appendix

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1 Decision problems of firms and households and equilibrium conditions

1.1 Final goods producers

The final good Y_t is a composite made of a continuum of intermediate goods $Y_t(i)$ as in Kimball (1995). The **final good producers** buy intermediate goods on the market, package Y_t, and resell it to consumers, investors and the government in a perfectly competitive market.

The final good producers maximize profits. Their problem is:

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di$$
$$s.t. \left[\int_0^1 G\left(\frac{Y_t(i)}{Y_t}; \lambda_{p,t}\right) di\right] = 1 \quad (\mu_{f,t})$$

where P_t and $P_t(i)$ are the price of the final and intermediate goods respectively, and G is a strictly concave and increasing function characterised by $G(1) = 1$. ϵ_t^p is an exogenous process that reflects shocks to the aggregator function that result in changes in the elasticity of demand and therefore in the markup. We will constrain $\epsilon_t^p \in (0, \infty)$. ϵ_t^p follows the exogenous ARMA process:

$$ln\epsilon_t^p = (1-\rho_p) ln\epsilon^p + \rho_p ln\epsilon_{t-1}^p - \theta_p \eta_{t-1}^p + \eta_t^p, \ \eta_{p,t} \sim N(0, \sigma_p) \tag{1}$$

To simplify notiation, in what follows we leave out this argument.

The FOCs are:

$$(\partial Y_t) \qquad P_t = \frac{\mu_{f,t}}{Y_t} \int_0^1 G'\left(\frac{Y_t(i)}{Y_t}\right) \frac{Y_t(i)}{Y_t} di$$
$$(\partial Y_t(i)) \qquad P_t(i) = \mu_{f,t} G'\left(\frac{Y_t(i)}{Y_t}\right) \frac{1}{Y_t}$$

resulting in

$$Y_t(i) = Y_t G'^{-1}\left[\frac{P_t(i)}{P_t} \int_0^1 G'\left(\frac{Y_t(i)}{Y_t}\right) \frac{Y_t(i)}{Y_t} di\right]$$

As in Kimball (1995), the assumptions on G() imply that the demand for input $Y_t(i)$ is decreasing in its relative price, while the elasticity of demand is a positive function of the relative price (or a negative function of the relative output).

1.2 Intermediate goods producers

Intermediate good producer i uses the following technology:

$$Y_t(i) = \varepsilon_t^a K_t^s(i)^\alpha \left[\gamma^t L_t(i)\right]^{1-\alpha} - \gamma^t \Phi \tag{2}$$

where $K_t^s(i)$ is capital services used in production, $L_t(i)$ is aggregate labour input and Φ is a fixed cost. γ^t respresents the labour-augmenting deterministic growth rate in the economy and ε_t^a is total factor productivity and follows the process:

$$ln\varepsilon_t^a = (1-\rho_z) ln\varepsilon^a + \rho_z ln\varepsilon_{t-1}^a + \eta_t^a,\ \eta_t^a \sim N(0, \sigma_a) \tag{3}$$

The firm's profit is given by:

$$P_t(i)Y_t(i) - W_t L_t(i) - R_t^k K_t(i).$$

where W_t is the aggregate nominal wage rate and R_t^k is the rental rate on capital.

Cost minimization yields the conditions:

$$(\partial L_t(i)): \quad \Theta_t(i)\gamma^{(1-\alpha)t}(1-\alpha)\varepsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t \tag{4}$$

$$(\partial K_t^s(i)): \quad \Theta_t(i)\gamma^{(1-\alpha)t}\alpha\varepsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k \tag{5}$$

where $\Theta_t(i)$ is the Lagrange multiplier associated with the production function and equals marginal cost MC_t.

Combining these FOCs and noting that the capital-labour ratio is equal across firms implies:

$$K_t^s = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}L_t \tag{6}$$

The marginal cost MC_t is the same for all firms and equal to:

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\ \alpha} \gamma^{-(1-\alpha)t} \left(\varepsilon_t^a\right)^{-1}$$

Under Calvo pricing with partial indexation, the optimal price set by the firm that is allowed to re-optimise results from the following optimisation problem:

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{P}_t(i)(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s}\right] Y_{t+s}(i)$$

$$s.t.\ Y_{t+s}(i) = Y_{t+s} G'^{-1}\left(\frac{P_t(i)X_{t,s}}{P_{t+s}}\tau_{t+s}\right)$$

where $\widetilde{P}_t(i)$ is the newly set price, ξ_p is the Calvo probability of being allowed to optimise one's price, π_t is inflation defined as $\pi_t = P_t/P_{t-1}$, $[\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}]$ is the nominal discount factor for firms (which equals the discount factor for the households that are the final owners of the firms), $\tau_t = \int_0^1 G'\left(\frac{Y_t(i)}{Y_t}\right)\frac{Y_t(i)}{Y_t} di$ and

$$X_{t,s} = \left\{ \begin{array}{c} 1 \; for \; s = 0 \\ (\Pi_{l=1}^{s} \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) \; for \; s = 1, ..., \infty \end{array} \right\}$$

The first-order condition is given by:

$$E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} Y_{t+s}(i) \left[X_{t,s} \widetilde{P}_t(i) + \left(\widetilde{P}_t(i) X_{t,s} - MC_{t+s} \right) \frac{1}{G'^{-1}(z_{t+s})} \frac{G'(x_{t+s})}{G''(x_{t+s})} \right] = 0 \tag{7}$$

where $x_t = G'^{-1}(z_t)$ and $z_t = \frac{P_t(i)}{P_t}\tau_t$.

The aggregate price index is in this case given by:

$$P_t = (1-\xi_p) P_t(i) G'^{-1}\left[\frac{P_t(i)\tau_t}{P_t}\right] + \xi_p \pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} G'^{-1}\left[\frac{\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} \tau_t}{P_t}\right] \tag{8}$$

1.3 Households

Household j chooses consumption $C_t(j)$, hours worked $L_t(j)$, bonds $B_t(j)$, investment $I_t(j)$ and capital utilisation $Z_t(j)$, so as to maximise the following objective function:

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma_c} (C_{t+s}(j) - \lambda C_{t+s-1})^{1-\sigma_c} \right] \exp\left(\frac{\sigma_c - 1}{1+\sigma_l} L_{t+s}(j)^{1+\sigma_l} \right)$$

subject to the budget constraint:

$$\begin{aligned} & C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{\varepsilon_t^b R_{t+s} P_{t+s}} - T_{t+s} \qquad (9) \\ \leq \; & \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{W_{t+s}^h(j) L_{t+s}(j)}{P_{t+s}} + \frac{R_{t+s}^k Z_{t+s}(j) K_{t+s-1}(j)}{P_{t+s}} - a(Z_{t+s}(j)) K_{t+s-1}(j) + \frac{Div_{t+s}}{P_{t+s}} \end{aligned}$$

and the capital accumulation equation:

$$K_t(j) = (1-\delta)_{t-1}(j) + \varepsilon_t^i \left[1 - S\left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right] I_t(j)$$

There is external habit formation captured by the parameter λ. The one-period bond is expressed on a discount basis. ε_t^b is an exogenous premium in the return to bonds, which might reflect inefficiencies in the financial sector leading to some premium on the deposit rate versus the risk free rate set by the central bank, or a risk premium that households require to hold the one period bond. ε_t^b follows the stochastic process:

$$ln\varepsilon_t^b = \rho_b \ln \varepsilon_{t-1}^b + \eta_t^b, \eta_t^b \sim N(0, \sigma_b) \tag{10}$$

δ is the depreciation rate, $S(\cdot)$ is the adjustment cost function, with $S(\gamma) = 0$, $S'(\gamma) = 0$, $S''(\cdot) > 0$, and ε_t^i is a stochastic shock to the price of investment relative to consumption goods and follows an exogenous process:

$$ln\varepsilon_t^i \ = \rho_i \ln \varepsilon_{t-1}^i + \eta_t^i, \eta_t^i \sim N(0, \sigma_i) \tag{11}$$

T_{t+s} are lump sum taxes or subsidies and Div_t are the dividends distributed by the labour unions.

Finally, households choose the utilitsation rate of capital. The amount of effective capital that households can rent to the firms is:

$$K_t^s(j) = Z_t(j) K_{t-1}(j) \tag{12}$$

The income from renting capital services is $R_t^k Z_t(j) K_{t-1}(j)$, while the cost of changing capital utilisation is $P_t a(Z_t(j)) K_{t-1}(j)$.

In equilibrium households will make the same choices for consumption, hours worked, bonds, investment and capital utilization. The first-order conditions for consumption, hours worked, bond holdings, investment, capital and capital utilisation can be written as (dropping the j index):

$$(\partial C_t) \qquad \Xi_t = \exp\left(\tfrac{\sigma_c - 1}{1+\sigma_l} L_t(j)^{1+\sigma_l}\right)(C_t - \lambda C_{t-1})^{-\sigma_c}$$

$$(\partial L_t) \qquad \left[\frac{1}{1-\sigma_c}(C_t - hC_{t-1})^{1-\sigma_c}\right] \exp\left(\tfrac{\sigma_c - 1}{1+\sigma_l} L_t^{1+\sigma_l}\right)(\sigma_c - 1) L_t^{\sigma_l} = -\Xi_t \frac{W_t^h}{P_t}$$

$$(\partial B_t) \qquad \Xi_t = \beta \varepsilon_t^b R_t E_t \left[\frac{\Xi_{t+1}}{\pi_{t+1}}\right]$$

$$\begin{aligned} (\partial I_t) \qquad \Xi_t &= \Xi_t^k \varepsilon_t^i \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right) \\ &\quad + \beta E_t \left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2\right] \end{aligned} \tag{13}$$

$$(\partial \bar{K}_t) \qquad \Xi_t^k = \beta E_t \left[\Xi_{t+1}\left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1})\right) + \Xi_{t+1}^k (1-\delta)\right] \tag{14}$$

$$(\partial u_t) \qquad \frac{R_t^k}{P_t} = a'(Z_t) \tag{15}$$

where Ξ_t and Ξ_t^k are the Lagrange multipliers associated with the budget and capital accumulation constraint respectively. Tobin's is $Q_t = \Xi_t^k / \Xi_t$ and equals one in the absence of adjustment costs.

1.4 Intermediate labour union sector

Households supply their homogenous labour to an intermediate labour union which differentiates the labour services and sets wages subject to a Calvo schemeiates about the wage with the intermediate labour packers.

Before going into the household's and the union's decision on labour supply and wage setting, more details on the labor market are needed. Labor used by the intermediate goods producers L_t is a composite:

$$L_t = \left[\int_0^1 L_t(l)^{\frac{1}{1+\lambda_{w,t}}} dl \right]^{1+\lambda_{w,t}}. \tag{16}$$

There are **labor packers** who buy the labor from the unions, package L_t, and resell it to the intermediate goods producers. Labor packers maximize profits in a perfectly competitive environment. From the FOCs of the labor packers one obtains:

$$L_t(l) = \left(\frac{W_t(l)}{W_t} \right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t \tag{17}$$

Combining this condition with the zero profit condition one obtains an expression for the wage cost for the intermediate goods producers:

$$W_t = \left[\int_0^1 W_t(l)^{\frac{1}{\lambda_{w,t}}} dl \right]^{\lambda_{w,t}} \tag{18}$$

We assume that $\lambda_{w,t}$ follows the exogenous ARMA process:

$$ln\lambda_{w,t} = (1 - \rho_{w)} ln\lambda_w + \rho_w ln\lambda_{w,t-1} - \theta_w \epsilon_{w,t-1} + \epsilon_{w,t}, \epsilon_{w,t} \sim .. \tag{19}$$

Labor packers buy the labour from the unions. The unions are an intermediate between the households and the labor packers. The unions allocate and differentiate the labour services from the households and have market power: they can choose the wage subject to the labour demand equation 17. The household's budget constraint now also contains the dividends of the union distributed to the households:

$$\begin{aligned} & C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{b_t^2 R_{t+s} P_{t+s}} + A_{t+s}(j) - T_{t+s} \\ \leq \quad & \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{W_{t+s}^h(j) L_{t+s}(j)}{P_{t+s}} + \frac{Div_{t+s}}{P_{t+s}} + \frac{R_{t+s}^k u_{t+s}(j) \overline{K}_{t+s-1}(j)}{P_{t+s}} - a(u_{t+s}(j)) \bar{K}_{t+s-1}(j) \end{aligned}$$

The household labour supply decision is the same for all households and is given by the following FOC:

$$\frac{W_t^h}{P_t} = -\frac{\left[\frac{1}{1-\sigma_c}(C_t - hC_{t-1})^{1-\sigma_c}\right] \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_t^{1+\nu_l}\right) (\sigma_c - 1) L_t^{\nu_l}}{\Xi_t}$$

The real wage desired by the households here reflects the marginal rate of substitution between leisure and consumption. The marginal disutility of labour is equal accross households and equal to:

$$U'_{l,t} =$$

Labour unions take this marginal rate of substitution as the cost of the labour services in their negotiations with the labour packers. The markup above the marginal disutility is distributed to the households. However, the union is also subject to nominal rigidities á la Calvo. Specifically, unions can readjust wages with probability $1-\zeta_w$ in each period. For those that cannot adjust wages, $W_t(l)$ will increase at the determistic growth rate γ and weighted average of the steady state inflation π_* and of last period's inflation (π_{t-1}). For those that can adjust, the problem is to choose a wage $\tilde{W}_t(l)$ that maximizes the wage income in all states of nature where the union is stuck with that wage in the future:

$$\max_{\widetilde{W}_t(l)} E_t \sum_{s=0}^{\infty} \zeta_w^s [\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}] \left[W_{t+s}(l) - W_{t+s}^h\right] L_{t+s}(l)$$

$$\begin{aligned} \text{and } L_{t+s}(l) &= \left(\frac{W_{t+s}(l)}{W_{t+s}}\right)^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}} L_{t+s} \\ \text{with } W_{t+s}(l) &= \widetilde{W}_t(l)(\Pi_{l=1}^s \gamma \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w}) \text{ for } s = 1, ..., \infty \end{aligned}$$

The first order condition becomes:

$$\begin{aligned} (\partial W_t) \qquad 0 &= E_t \sum_{s=0}^{\infty} \zeta_w^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \Bigg[(W_{t+s}(l) - W_{t+s}^h) \left(\frac{X_{t,s}\widetilde{W}_t(l)}{W_{t+s}}\right)^{-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}-1} \\ & \quad (-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}})(\frac{X_{t,s}}{W_{t+s}}) L_{t+s} - X_{t,s} L_{t+s}(l) \Bigg] \end{aligned}$$

where

$$X_{t,s} = \left\{ \begin{array}{c} 1 \; for \; s = 0 \\ (\Pi_{l=1}^s \gamma \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w}) \; for \; s = 1, ..., \infty \end{array} \right\}$$

Simplifying by substituting for the individual labour and multiplying with the optimal wage

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[(X_{t,s}\widetilde{W}_t(l) - W_{t+s}^h)(-\frac{1+\lambda_{w,t+s}}{\lambda_{w,t+s}}) L_{t+s}(l) - X_{t,s}\widetilde{W}_t(l) L_{t+s}(l) \right] = 0$$

or

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} L_{t+s}(l) \frac{1}{\lambda_{w,t+s}} \left[(1+\lambda_{w,t+s}) W_{t+s}^h - X_{t,s}\widetilde{W}_t(l) \right] = 0 \tag{20}$$

The aggregate wage expression is

$$W_t = [(1-\zeta_w)\widetilde{W}_t^{\frac{1}{\lambda_{w,t}}} + \zeta_w(\gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} W_{t-1})^{\frac{1}{\lambda_{w,t}}}]^{\lambda_{w,t}}. \tag{21}$$

1.5 Government Policies

The central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^*}\right)^{\psi_2}\right]^{1-\rho_R} \left(\frac{Y_t/Y_{t-1}}{Y_t^*/Y_{t-1}^*}\right)^{\psi_3} r_t \tag{22}$$

where R^* is the steady state nominal rate (gross rate) and Y_t^* is the natural output. The parameter ρ_R determines the degree of interest rate smoothing. The monetary policy shock r_t is determined as

$$lnr_t = \rho_r lnr_{t-1} + \epsilon_{r,t} \tag{23}$$

The central bank supplies the money demanded by the household to support the desired nominal interest rate.

The government budget constraint is of the form

$$P_t G_t + B_{t-1} = T_t + \frac{B_t}{R_t} \tag{24}$$

where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint. Government spending expressed relative to the steady state output path $g_t = G_t/(Y\gamma^t)$ follows the process:

$$lng_t = (1-\rho_g)lng + \rho_g lng_{t-1} + \rho_{ga} lnZ_t - \rho_{ga} lnZ_{t-1} + \epsilon_{g,t}, \epsilon_{g,t} \sim ... \tag{25}$$

where we allow for a reaction of government spending to respond on the productivity process. Or a slightly different specification:

$$lng_t - \rho_{ga} lnZ_t = (1-\rho_g)lng + \rho_g(lng_{t-1} - \rho_{ga} lnZ_{t-1}) + \epsilon_{g,t}, \epsilon_{g,t} \sim ...$$

1.6 The natural output level

The natural output level is defined as the output in the flexible price and wage economy. The question is which shocks need to be taken into account. More in particular the question is whether the markup shocks in prices and wages must be taken into account in the calculation of the natural output level. If the markup shock are not taken into account this will lead to a trade-off problem between output-gap stabilisation and inflation stabilisation. Persistent markup shocks might result in persistent conflicts between the two objectives and therefore in persistent deviations of inflation from the inflation target.

1.7 Resource constraints

To obtain the market clearing condition for the final goods market first integrate the HH budget constraint across households, and combine it with the government budget constraint:

$$P_tC_t + P_tI_t + P_tG_t \leq \Pi_t + \int W_t(j)L_t(j)dj + R_t^k \int K_t(j)dj - P_ta(u_t) \int \bar{K}_{t-1}(j)dj.$$

or in case of the labour unions:

$$P_tC_t + P_tI_t + P_tG_t \leq \Pi_t + \int W_t^h(j)L_t(j)dj + Div_t + R_t^k \int K_t(j)dj - P_ta(u_t) \int \bar{K}_{t-1}(j)dj.$$

Next, realize that

$$\Pi_t = \int \Pi_t(i)di \;\; = \;\; \int P_t(i)Y_t(i)di - W_tL_t - R_t^kK_t,$$

where $L_t = \int L(i)_t di$ is total labor supplied by the labor packers (and demanded by the firms), and $K_t = \int K_t(i)di = \int K_t(j)dj$. Now replace the defintion of Π_t into the HH budget constraint, realize that by the labor and goods' packers' zero profit condition $W_tL_t = \int W_t(j)L_t(j)dj$, or $= \int W_t^h(j)L_t(j)dj + Div_t$ and $P_tY_t = \int P(i)_tY(i)_t di$ and obtain:

$$P_tC_t + P_tI_t + P_tG_t + P_ta(u_t)\bar{K}_{t-1} = P_tY_t,$$

or

$$C_t + I_t + G_t + a(u_t)\bar{K}_{t-1} = Y_t \tag{26}$$

where Y_t is defined by eq(1) .

In the data we do observe $\overline{Y}_t$ and $\overline{L}_t$ instead of Y_t and L_t where

$$\overline{Y}_t = \int Y_t(i)di$$

Starting from eq (), the relationship between output and the aggregate inputs, labor and capital, is:

$$\begin{aligned}
\overline{Y}_t &= \int \left(\frac{P_t(i)}{P_t}\right)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} Y_t di \\
&= Y_t\left(P_t\right)^{\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} \int P_t(i)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} di \\
&= Y_t\left(P_t\right)^{\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} \int P_t(i)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} di \\
&= Y_t\left(P_t\right)^{\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} \overline{P}_t^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}}
\end{aligned}$$

where $\overline{P}_t = \left[\int P_t(i)^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} di\right]^{-\frac{\lambda_{p,t}}{1+\lambda_{p,t}}}$
and

$$\begin{aligned}
\overline{L}_t &= \int L_t(j)dj \\
&= \int \left(\frac{W_t(j)}{W_t}\right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t dj \\
&= L_t \left(W_t\right)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \overline{W}_t^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}}
\end{aligned}$$

where $\overline{W}_t = \left[\int W_t(j)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} dj\right]^{-\frac{\lambda_{w,t}}{1+\lambda_{w,t}}}$
In the first order approximation the barred concepts will equal the unbarred.

1.8 Exogenous Processes

There are seven exogenoues processes in the model:

- Technology process:

$$lnZ_t = (1 - \rho_z)lnZ + \rho_z lnZ_{t-1} + \epsilon_{z,t}$$

- Investment relative price process:

$$ln\mu_t = (1 - \rho_\mu)ln\mu + \rho_\mu ln\mu_{t-1} + \epsilon_{\mu,t}$$

- Intertemporal preference shifter (financial risk premium process):

$$lnb_t = (1 - \rho_{b)} \ln b + \rho_b \ln b_{t-1} + \epsilon_{b,t}$$

- Government spending process:

$$lng_t = (1 - \rho_g)lng + \rho_g lng_{t-1} + \rho_{ga} lnZ_t - \rho_{ga} lnZ_{t-1} + \epsilon$$

$$lng_t - \rho_{ga} lnZ_t = (1 - \rho_g)lng + \rho_g(lng_{t-1} - \rho_{ga} lnZ_{t-1}) + \epsilon_{g,t}$$

- Monetary Policy Shock:

$$lnr_t = \rho_r lnr_{t-1} + \epsilon_{r,t}$$

- Price Mark-up shock:

$$ln\lambda_{p,t} = (1 - \rho_{p)} ln\lambda_p + \rho_p ln\lambda_{p,t-1} - \theta_p \epsilon_{p,t-1} + \epsilon_{p,t}$$

- Wage Mark-up shock: $ln\lambda_{w,t} = (1 - \rho_{w)} ln\lambda_w + \rho_w ln\lambda_{w,t-1} - \theta_w \epsilon_{w,t-1} + \epsilon_{w,t}$

  and where the innovations ϵ are distributed as i.i.d. Normal innovations:

$$\epsilon_{i,t} \sim N(0, \sigma_i)$$

2 Detrending and steady state

2.1 Intermediate goods producers

The model can be detrended with the deterministic trend γ and nominal variables can be replaced by their real counterparts. Lower case variables represent detrended real variables which can be considered as stationary processes that have a well defined steady state: for instance

$$k_t = K_t/\gamma^t, w_t = W_t/(P_t\gamma^t), r_t^k = R_t^k/P_t, \xi_t = \Xi_t\gamma^{\sigma_c t}$$

The aggregate production function 2 becomes

$$y_t(i) = Z_t k_t(i)^\alpha (L_t(i))^{1-\alpha} - \Phi \tag{27}$$

Equation 6 becomes:

$$k_t = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t^k} L_t \tag{28}$$

and at st.st.:

$$k_* = \frac{\alpha}{1-\alpha}\frac{w_*}{r_*^k} L_*$$

The maginal cost expression **??** becomes:

$$mc_t = \frac{MC_t}{P_t} = \frac{w_t^{1-\alpha} r_t^{k\ \alpha}}{\alpha^\alpha (1-\alpha)^{(1-\alpha)} Z_t} \tag{29}$$

Option 1: Expression **??** becomes

$$\begin{aligned}
&\frac{1}{\lambda_{p,t}}\left(\widetilde{P}_t - (1+\lambda_{p,t})MC_t\right) Y_t(i) + \\
&E_t \sum_{s=1}^{\infty} \zeta_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \frac{1}{\lambda_{p,t+s}} \left(\widetilde{P}_t(i)(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - (1+\lambda_{p,t+s}) MC_{t+s}\right) Y_{t+s}(i) \\
= &\quad 0
\end{aligned}$$

becomes:

$$\begin{aligned}
&\frac{1}{\lambda_{p,t}}\left(\widetilde{p}_t - (1+\lambda_{p,t})mc_t\right) y_t(i) + \\
&E_t \sum_{s=1}^{\infty} \zeta_p^s \beta^s \gamma^{(1-\sigma_c)s} \frac{\xi_{t+s}}{\xi_t} \frac{1}{\lambda_{p,t+s}} \left(\widetilde{p}_t(i) \frac{(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})}{(\Pi_{l=1}^s \pi_{t+l})} - (1+\lambda_{p,t+s}) mc_{t+s}\right) y_{t+s}(i) \\
= &\quad 0
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{\lambda_{p,t}}\left(\widetilde{p}_t - (1+\lambda_{p,t})mc_t\right) y_t(i) + \\
&E_t \sum_{s=1}^{\infty} \zeta_p^s \overline{\beta}^s \gamma^s \frac{\xi_{t+s}}{\xi_t} \frac{1}{\lambda_{p,t+s}} \left(\widetilde{p}_t(i) \frac{(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})}{(\Pi_{l=1}^s \pi_{t+l})} - (1+\lambda_{p,t+s}) mc_{t+s}\right) y_{t+s}(i) \\
= &\quad 0
\end{aligned} \tag{30}$$

where $\widetilde{p}_t = \widetilde{P}_t / P_t$, $\overline{\beta}^s = \beta^s \gamma^{-\sigma_c s}$ and $\xi_{t+s} = \Xi_{t+s} \gamma^{\sigma_c (t+s)}$ is the real discount factor. Note that in case $\sigma_c = 1$ the expression simplifies to the standard problem with β as the discount factor. This implies that in steady state:

$$\tilde{p}_* = (1+\lambda_p)\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} w_*^{1-\alpha} r_*^{k\ \alpha} Z_*^{-1}$$

Expression **??** becomes:

$$1 = \left[(1-\zeta_p)\tilde{p}_t^{\frac{1}{\lambda_{p,t}}} + \zeta_p (\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} \pi_t^{-1})^{\frac{1}{\lambda_{p,t}}}\right]^{\lambda_{p,t}}. \tag{31}$$

which means that:

$$\tilde{p}_* = 1$$

Option 2: Equation (7)

$$E_t \sum_{s=0}^{\infty} \zeta_p^s [\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}] Y_{t+s}(i) \left[\left(1 + \frac{1}{G'^{-1}(z_{t+s})} \frac{G'(x_{t+s})}{G''(x_{t+s})}\right) \widetilde{P}_t(i) X_{t,s} - \left(\frac{1}{G'^{-1}(z_{t+s})} \frac{G'(x_{t+s})}{G''(x_{t+s})}\right) MC_{t+s} \right] = 0$$

$$E_t \sum_{s=0}^{\infty} \zeta_p^s \overline{\beta}^s \gamma^s \frac{\xi_{t+s}}{\xi_t} y_{t+s}(i) \left[\left(1 + \frac{1}{G'^{-1}(z_{t+s})} \frac{G'(x_{t+s})}{G''(x_{t+s})}\right) \frac{\widetilde{p}_t(i) X_{t,s}}{X_{t+s}^p} - \left(\frac{1}{G'^{-1}(z_{t+s})} \frac{G'(x_{t+s})}{G''(x_{t+s})}\right) mc_{t+s} \right] = 0 \tag{32}$$

where

$$X_{t+s}^p = 0 \text{ for } s = 0 \text{ or else } (\Pi_{l=0}^s \pi_{t+l})$$

In steady state this gives (using $G'^{-1}(G'(1)) = x = 1$)

$$\begin{aligned} 1 + [1 - mc] \frac{1}{G'^{-1}(G'(1))} \frac{G'(1)}{G''(1)} &= 0 \\ 1 + \frac{G''(1)}{G'(1)} &= mc \end{aligned}$$

The implied markup is equal to

$$mu(1) = \frac{1}{1 + \frac{G''(1)}{G'(1)}} = \frac{\frac{G'(1)}{G''(1)}}{\frac{G'(1)}{G''(1)} + 1} = \frac{\varepsilon(1)}{\varepsilon(1) - 1} = (1 + \lambda_p)$$

which corresponds with the elasticity of demand $\varepsilon(1) = -\frac{G'(1)}{G''(1)}$

The aggregate price index in the case of the Calvo model becomes (8):

$$1 = (1 - \zeta_p) p_t(i) G'^{-1} [p_t(i)\tau] + \zeta_p \pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} \pi_t^{-1} G'^{-1} \left[\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} \pi_t^{-1} \tau\right] \tag{33}$$

In steady-state[1]

$$\begin{aligned} 1 &= (1 - \zeta_p) p_*(i) G'^{-1} [p_*(i)\tau] + \zeta_p G'^{-1} [\tau] \\ 1 &= p_*(i) G'^{-1} [z_*] \\ 1 &= p_*(i) \end{aligned}$$

[1] Recall that $z_t = p_t(i)\tau_t$, hence $z_* = p_*(i)\tau$ and $G'^{-1}(z_*) = x_* = 1$, $\tau = G'(1) \Leftrightarrow G'^{-1}(\tau) = 1$.

Recall that aggregate profits are equal to:

$$\Pi_t = P_t Y_t - W_t L_t - R_t^k K_t.$$

In terms of detrended variables we then have, using 2 :

$$\begin{aligned}
\frac{\Pi_t}{P_t \gamma^t} &= y_t - w_t L_t - r_t^k k_t \\
&= Z_t k_t^{\alpha} L_t^{1-\alpha} - \Phi - w_t L_t - \frac{\alpha}{1-\alpha} w_t L_t \\
&= \left[Z_t k_t^{\alpha} L_t^{-\alpha} - \frac{1}{1-\alpha} w_t \right] L_t - \Phi \\
&= \left[\alpha^{\alpha} (1-\alpha)^{-\alpha} Z_t w_t^{\alpha} (r_t^k)^{-\alpha} - \frac{w_t}{1-\alpha} \right] L_t - \Phi \\
&= \left[\alpha^{\alpha} (1-\alpha)^{1-\alpha} Z_t w_t^{\alpha-1} (r_t^k)^{-\alpha} - 1 \right] \frac{w_t L_t}{1-\alpha} - \Phi \\
&= \left(\frac{1}{mc_t} - 1 \right) \frac{w_t L_t}{1-\alpha} - \Phi
\end{aligned}$$

At steady state we can use $1 = p_*(i) = (1+\lambda_p) mc_*$ to get st. st. profits:

$$\begin{aligned}
\Pi_* &= (1 + \lambda_p - 1) \frac{w_* L_*}{1-\alpha} - \Phi \\
&= \lambda_p \frac{w_* L_*}{1-\alpha} - \Phi
\end{aligned}$$

2.2 Households

Expression **??** and **??** become:

$$\xi_t = b_t^1 \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_t^{1+\nu_l} \right) (c_t - (h/\gamma) c_{t-1})^{-\sigma_c} \tag{34}$$

$$\xi_t = \overline{\beta} b_t^2 R_t E_t [\xi_{t+1} \pi_{t+1}^{-1}] \tag{35}$$

respectively with where $\xi_t = \Xi_t \gamma^{\sigma_c t}$ and $\overline{\beta} = (\beta / \gamma^{\sigma_c})$. At steady state:

$$\xi_* = b_*^1 \exp\left(\frac{\sigma_c - 1}{1+\nu_l} {L_*}^{1+\nu_l} \right) c_*^{-\sigma_c} (1 - (h/\gamma))^{-\sigma_c}$$

$$R_* = \overline{\beta}^{-1} \pi_*$$

Equation 12 and **??** become:

$$\begin{aligned}
k_t &= u_t \bar{k}_{t-1} / \gamma, & (36) \\
\bar{k}_t &= [(1-\delta)/\gamma] \, \bar{k}_{t-1} + \mu_t \left(1 - S(\gamma \frac{i_t}{i_{t-1}}) \right) i_t & (37)
\end{aligned}$$

which deliver the steady state relationships:

$$\begin{aligned} k_* &= u_* \bar{k}_* / \gamma = \bar{k}_* / \gamma \\ i_* &= (1/\mu_*)\left(1 - (1-\delta)/\gamma\right) \bar{k}_* \\ &= \left(1 - (1-\delta)/\gamma\right) \bar{k}_* \end{aligned}$$

under the assumption that $S(\gamma) = 0$, $u_* = 1$, and $\mu_* = 1$.

The FOC with respect to investment, capital, and capital utilization are:

$$\begin{aligned} (\partial I_t) \qquad 1 &= Q_t^k \mu_t \left[1 - S\left(\frac{i_t \gamma}{i_{t-1}}\right) - S'\left(\frac{i_t \gamma}{i_{t-1}}\right) \frac{i_t \gamma}{i_{t-1}}\right] \\ &\quad + (\beta/\gamma^{\sigma_c}) E_t \left\{ \frac{\Xi_{t+1} \gamma^{\sigma_c t+1}}{\Xi_t \gamma^{\sigma_c t}} \left[Q_{t+1}^k \mu_{t+1} S'\left(\frac{i_{t+1}\gamma}{i_t}\right) \left(\frac{i_{t+1}\gamma}{i_t}\right)^2 \right] \right\} \\ (\partial \bar{K}_t) \qquad Q_t &= (\beta/\gamma^{\sigma_c}) E_t \left\{ \frac{\Xi_{t+1} \gamma^{\sigma_c t+1}}{\Xi_t \gamma^{\sigma_c t}} \left[(r_{t+1}^k u_{t+1} - a(u_{t+1})) + Q_{t+1}(1-\delta) \right] \right\} \\ (\partial u_t) \qquad r_t^k &= a'(u_t) \end{aligned}$$

with $\Xi_t^k / \Xi_t = Q_t$ so that

$$\begin{aligned} (\partial I_t) \qquad 1 &= Q_t^k \mu_t \left(1 - S\left(\frac{i_t \gamma}{i_{t-1}}\right) - S'\left(\frac{i_t \gamma}{i_{t-1}}\right) \frac{i_t \gamma}{i_{t-1}}\right) \\ &\quad + (\overline{\beta}) E_t \left\{ \frac{\xi_{t+1}}{\xi_t} \left[Q_{t+1}^k \mu_{t+1} S'\left(\frac{i_{t+1}\gamma}{i_t}\right) \left(\frac{i_{t+1}\gamma}{i_t}\right)^2 \right] \right\} && (38) \\ (\partial \bar{K}_t) \qquad Q_t &= (\overline{\beta}) E_t \left\{ \frac{\xi_{t+1}}{\xi_t} \left[(r_{t+1}^k u_{t+1} - a(u_{t+1})) + Q_{t+1}(1-\delta) \right] \right\} && (39) \\ (\partial u_t) \qquad r_t^k &= a'(u_t) && (40) \end{aligned}$$

so that in steady state where $Q_* = 1$, $u_* = 1$, and $a(u_*) = 0$:

$$1 = \overline{\beta}[(r_*^k + (1-\delta)]$$

In the variant for the utilisation cost, the accumulated utilisation costs becomes

$$ak_t = \frac{a(u_{t+s}(j))}{\gamma^{1-s}} \overline{k}_{t+s-1}(j) + \frac{(1-\delta)}{\gamma} ak_{t-1}$$

Option 1: Expressed in terms of detrended variables, equations **??** and 21 become:

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \beta^s \Xi_{t+s} L_{t+s}(j) \frac{1}{\lambda_{w,t+s}} \left[\frac{U'_{l,t+s}(j)}{\Xi_{t+s}} (1 + \lambda_{w,t+s}) + \frac{X_{t,s}}{P_{t+s}} \widetilde{W}_t(j) \right] = 0$$

where

$$\begin{aligned} \frac{U'_{l,t}}{\Xi_t} &= \frac{b_t^1 \left[\frac{1}{1-\sigma_c} (C_t - hC_{t-1})^{1-\sigma_c} + \ell_t \right] \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_t^{1+\nu_l} \right) (\sigma_c - 1) L_t^{\nu_l}}{b_t^1 \exp\left(\frac{\sigma_c - 1}{1+\nu_l} L_t(j)^{1+\nu_l} \right) (C_t - hC_{t-1})^{-\sigma_c}} \\ &= \frac{\left[\frac{1}{1-\sigma_c} (C_t - hC_{t-1})^{1-\sigma_c} + \ell_t \right] (\sigma_c - 1) L_t^{\nu_l}}{(C_t - hC_{t-1})^{-\sigma_c}} \\ &= -\left[(C_t - hC_{t-1}) \right] L_t^{\nu_l} \end{aligned}$$

for $\ell_t = 0$.

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \beta^s \gamma^{t+s} (\Xi_{t+s}) L_{t+s}(j) \frac{1}{\lambda_{w,t+s}} \left[- \left[(c_t - (h/\gamma) c_{t-1}) \right] L_t^{\nu_l} (1 + \lambda_{w,t+s}) + \frac{(\Pi_{l=0}^s \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w})}{(\Pi_{l=0}^s \pi_{t+l-1})} \widetilde{w}_t(j) \right] = 0$$

$$E_t \sum_{s=0}^{\infty} \zeta_w^s (\beta^s / \gamma^{\sigma_s s}) \gamma^s (\gamma^{\sigma_s t+s} \Xi_{t+s}) L_{t+s}(j) \frac{1}{\lambda_{w,t+s}} \left[- \left[(c_t - (h/\gamma) c_{t-1}) \right] L_t^{\nu_l} (1 + \lambda_{w,t+s}) \right.$$
$$\left. + \frac{(\Pi_{l=0}^s \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w})}{(\Pi_{l=0}^s \pi_{t+l-1})} \widetilde{w}_t(j) \right] = 0$$

$$E_t \sum_{s=0}^{\infty} \zeta_w^s (\overline{\beta}^s) \gamma^s (\xi_{t+s}) L_{t+s}(j) \frac{1}{\lambda_{w,t+s}} \left[- \left[(c_t - (h/\gamma) c_{t-1}) \right] L_t^{\nu_l} (1 + \lambda_{w,t+s}) + \frac{(\Pi_{l=0}^s \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w})}{(\Pi_{l=0}^s \pi_{t+l-1})} \widetilde{w}_t(j) \right] = 0 \tag{41}$$

or option 2: from (20)

$$E_t \sum_{s=0}^{\infty} \zeta_w^s [\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}] L_{t+s}(l) \frac{1}{\lambda_{w,t+s}} \left[(1 + \lambda_{w,t+s}) W_{t+s}^h - X_{t,s} \widetilde{W}_t(l) \right] = 0$$

which becomes

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \beta^s \gamma^{s(1-\sigma_c)} \frac{\xi_{t+s}}{\xi_t} L_{t+s}(l) \frac{1}{\lambda_{w,t+s}} \left[(1 + \lambda_{w,t+s}) w_{t+s}^h - \frac{(\Pi_{l=0}^s \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w})}{(\Pi_{l=0}^s \pi_{t+l-1})} \widetilde{w}_t(l) \right] = 0$$

$$E_t \sum_{s=0}^{\infty} \zeta_w^s \overline{\beta}^s \gamma^s \frac{\xi_{t+s}}{\xi_t} L_{t+s}(l) \frac{1}{\lambda_{w,t+s}} \left[(1 + \lambda_{w,t+s}) w_{t+s}^h - \frac{(\Pi_{l=0}^s \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w})}{(\Pi_{l=0}^s \pi_{t+l-1})} \widetilde{w}_t(l) \right] = 0 \tag{42}$$

With

$$w_{t+s}^h = -\frac{U'_{l,t}}{\gamma^t \Xi_t} = \left[(c_t - (h/\gamma) c_{t-1}) \right] L_t^{\nu_l}$$

in steady state

$$\begin{aligned} (1 + \lambda_{w*}) w_*^h &= \widetilde{w_*} \\ \text{with } w_*^h &= -\frac{U'_{l,*}}{\Xi_*} = \left[(c_* - (h/\gamma) c_*) \right] L_*^{\nu_l} \\ &= (1 - h/\gamma) c_* L_*^{\nu_l} \end{aligned}$$

and where

$$W_t = \left[(1 - \zeta_w) \widetilde{W}_t^{\frac{1}{\lambda_{w,t}}} + \zeta_w (\gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} W_{t-1})^{\frac{1}{\lambda_{w,t}}} \right]^{\lambda_{w,t}}$$

becomes

$$w_t = \left[(1 - \zeta_w) \widetilde{w}_t^{\frac{1}{\lambda_{w,t}}} + \zeta_w (\gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} \pi_t^{-1} \gamma^{-1} w_{t-1})^{\frac{1}{\lambda_{w,t}}} \right]^{\lambda_{w,t}} \tag{43}$$

which imply at steady state:

$$w_* = \tilde{w}_* = (1 + \lambda_w) w_*^h$$

2.3 Resource constraints

The resource constraint(s) become:[2]

$$C_t + I_t + G_t + a(u_t)\bar{K}_{t-1} = Y_t$$

$$c_t + i_t + y_* g_t + a(u_t)\bar{k}_{t-1}/\gamma = y_t. \tag{44}$$

and

$$\dot{y}_t = Z_t k_t^\alpha L_t^{1-\alpha} - \Phi.$$

$$Y_t = \left(\frac{\dot{P}_t}{P_t}\right)^{\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} \dot{Y}_t$$

becomes

$$y_t = (\dot{p}_t)^{\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} \dot{y}_t$$

where

$$\begin{aligned}
\dot{p}_t &= \frac{\dot{(P)}_t}{P_t} \\
&= [(1-\zeta_p)(\frac{\tilde{P}_t}{P_t})^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} + \zeta_p(\pi_* \frac{\dot{(P)}_{t-1}}{P_t})^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}}]^{-\frac{\lambda_{p,t}}{1+\lambda_{p,t}}} \\
&= [(1-\zeta_p)\tilde{p}_t^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}} + \zeta_p(\pi_* \dot{(p)}_{t-1}\pi_t^{-1})^{-\frac{1+\lambda_{p,t}}{\lambda_{p,t}}}]^{-\frac{\lambda_{p,t}}{1+\lambda_{p,t}}}
\end{aligned}$$

While

$$L_t = \left(\frac{\dot{W}_t}{W_t}\right)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{(L)}_t$$

becomes

$$L_t = \left(\dot{(w)}_t\right)^{\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} \dot{(L)}_t$$

where

At steady state we have:

$$c_* + i_* + g_* y_* = y_*.$$

and

$$y_* = Z_* k_*^\alpha L_*^{1-\alpha} - \Phi.$$

and

$$\dot{y}_* = y_*, \ \dot{L}_* = L_*.$$

2.4 Government Policies

The Taylor rule becomes:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{y_t}{y_t^*}\right)^{\psi_2}\right]^{1-\rho_R} \left(\frac{y_t/y_{t-1}}{y_t^*/y_{t-1}^*}\right)^{\psi_3} r_t \tag{45}$$

[2] Using $g_t = \frac{G_t}{Y_* \gamma^t} \Leftrightarrow \frac{G_t}{\gamma^t} = Y_* g_t$.

3 Steady state

Combining the steady state expressions:

From (28)

$$k_* = \frac{\alpha}{1-\alpha}\frac{w_*}{r_*^k}L_*.$$

From (30)

$$\tilde{p}_* = 1 = (1+\lambda_p)\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_*^{1-\alpha}(r_*^k)^{\alpha}Z_*^{-1}$$

From (35)

$$R_* = \overline{\beta}^{-1}\pi_*$$

From (36) and (37):

$$\begin{aligned} \bar{k}_* &= \gamma k_*, \\ i_* &= (1-(1-\delta)/\gamma)\,\bar{k}_*. \end{aligned}$$

From (42):

$$w_* = \tilde{w}_* = (1+\lambda_w)w_*^h$$

with

$$w_*^h = -\frac{U'_{l,*}}{\Xi_*} = [(c_* - (h/\gamma)c_*)]\,L_*^{\nu_l} = (1-h/\gamma)c_*L_*^{\nu_l}$$

From (34):

$$\Xi_* = b_*^1 \exp\left(\frac{\sigma_c - 1}{1+\nu_l}L_*{}^{1+\nu_l}\right)c_*^{-\sigma_c}(1-(h/\gamma))^{-\sigma_c}$$

From (39) and (40)

$$\begin{aligned} 1 &= (\overline{\beta})[(r_*^k) + (1-\delta)] \\ r_*^k &= \overline{\beta}^{-1} - (1-\delta) \\ &= \beta^{-1}\gamma^{\sigma_c} - (1-\delta) \\ \text{and} \qquad r_*^k &= a'(u_*) \end{aligned}$$

From (27):

$$y_* = Z_* k_*^{\alpha} L_*^{1-\alpha} - \Phi.$$

From (44):

$$\frac{c_*}{y_*} + \frac{i_*}{y_*} + g_* = 1$$

The ratio

$$\begin{aligned}\frac{w_*^h L_*}{c_*} &= \frac{1}{1+\lambda_w}\frac{w_* L_*}{c_*}\\ &= \frac{1}{1+\lambda_w}\frac{1-\alpha}{\alpha}\frac{r_*^k k_*}{c_*}\\ &= \frac{1}{1+\lambda_w}\frac{1-\alpha}{\alpha} r_*^k \frac{k_*}{y_*}\frac{y_*}{c_*}\end{aligned}$$

with

$$\frac{k_*}{y_*} = \frac{y_* + \Phi}{y_*}(\frac{L_*}{k_*})^{\alpha-1}$$

and

$$\begin{aligned}\frac{c_*}{y_*} &= (1-g-\frac{i_*}{y_*})\\ &= (1-g-\frac{i_*}{k_*}\frac{k_*}{y_*})\end{aligned}$$

4 Log-linearized model

Eq. (29) becomes:

$$\widehat{mc}_t = (1-\alpha)\ \widehat{w}_t + \alpha\ \widehat{r}_t^k - \widehat{Z}_t \tag{46}$$

Eq. (31) becomes:

$$\widehat{\widetilde{p}}_t = \frac{\zeta_p}{1-\zeta_p}(\ \widehat{\pi}_t - \ \iota_p \widehat{\pi}_{t-1}) \tag{47}$$

- **Option1**:

  Eq. (30) becomes:

$$\begin{aligned}& d(\widetilde{p}_t) - (1+\lambda_p) d(mc_t) - mc d(\lambda_{p,t}) + \\ & E_t \sum_{s=1}^{\infty} \zeta_p^s \overline{\beta}^s \gamma^s \left(d(\widetilde{p}_t) + d(\frac{(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})}{(\Pi_{l=1}^s \pi_{t+l})}) - (1+\lambda_p) d(mc_{t+s}) - mc d(\lambda_{p,t+s}) \right) \\ = & \ 0\end{aligned}$$

  minus the same expression for time t+1 multiplied with $\zeta_p \overline{\beta}\gamma$

$$\begin{aligned}& \frac{1}{(1-\zeta_p\overline{\beta}\gamma)} d(\widetilde{p}_t) - (1+\lambda_p) d(mc_t) - mc d(\lambda_{p,t}) - \frac{\zeta_p\overline{\beta}\gamma}{(1-\zeta_p\overline{\beta}\gamma)}(d(\widetilde{p}_{t+1})) \\ & + \frac{\zeta_p\overline{\beta}\gamma}{(1-\zeta_p\overline{\beta}\gamma)} \left(\iota_p \frac{\pi_*^{\iota_p-1} \pi_*^{1-\iota_p}}{\pi_*} d(\pi_t) - \frac{(\pi_*^{\iota_p}\pi_*^{1-\iota_p})}{(\pi_*)^2} d(\pi_{t+1}) \right) \\ = & \ 0\end{aligned}$$

  in terms of deviations from steady state this becomes:

$$\begin{aligned}& \frac{1}{(1-\zeta_p\overline{\beta}\gamma)} \widehat{\widetilde{p}}_t \ - (1+\lambda_p) mc(\widehat{mc}_t) - mc\lambda_p \widehat{\lambda}_{p,t} - \frac{\zeta_p\overline{\beta}\gamma}{(1-\zeta_p\overline{\beta}\gamma)} \widehat{\widetilde{p}}_{t+1} \\ & + \frac{\zeta_p\overline{\beta}\gamma}{(1-\zeta_p\overline{\beta}\gamma)} (\iota_p \widehat{\pi}_t - \widehat{\pi}_{t+1}) \\ = & \ 0\end{aligned}$$

using 47 to substitute the optimal price expression

$$\begin{aligned}
&\frac{1}{(1-\zeta_p\overline{\beta}\gamma)}\frac{\zeta_p}{1-\zeta_p}(\ \widehat{\pi}_t-\ \iota_p\widehat{\pi}_{t-1})\ -(\widehat{mc}_t)-mc\lambda_p\widehat{\lambda}_{p,t}-\frac{\zeta_p\overline{\beta}\gamma}{(1-\zeta_p\overline{\beta}\gamma)}\frac{\zeta_p}{1-\zeta_p}(\ \widehat{\pi}_{t+1}-\ \iota_p\widehat{\pi}_t) \\
&+\frac{\zeta_p\overline{\beta}\gamma}{(1-\zeta_p\overline{\beta}\gamma)}\frac{(1-\zeta_p)}{(1-\zeta_p)}(\iota_p\widehat{\pi}_t-\widehat{\pi}_{t+1}) \\
=\ & 0
\end{aligned}$$

regrouping

$$\begin{aligned}
&(\ \widehat{\pi}_t-\ \iota_p\widehat{\pi}_{t-1})\ -\zeta_p\overline{\beta}\gamma(\ \widehat{\pi}_{t+1}-\ \iota_p\widehat{\pi}_t)+(\overline{\beta}\gamma-\overline{\beta}\gamma\zeta_p)(\iota_p\widehat{\pi}_t-\widehat{\pi}_{t+1}) \\
=\ &\frac{(1-\zeta_p\overline{\beta}\gamma)(1-\zeta_p)}{\zeta_p}(\widehat{mc}_t+mc\lambda_p\widehat{\lambda}_{p,t})
\end{aligned}$$

$$\begin{aligned}
&(1+\overline{\beta}\gamma\iota_p)\ \widehat{\pi}_t-\ \iota_p\widehat{\pi}_{t-1}\ -\overline{\beta}\gamma\widehat{\pi}_{t+1} \\
=\ &\frac{(1-\zeta_p\overline{\beta}\gamma)(1-\zeta_p)}{\zeta_p}(\widehat{mc}_t+mc\lambda_p\widehat{\lambda}_{p,t})
\end{aligned}$$

Finally we obtain:

$$\widehat{\pi}_t=\frac{(1-\zeta_p\overline{\beta}\gamma)(1-\zeta_p)}{(1+\iota_p\overline{\beta}\gamma)\zeta_p}\left[\widehat{m}c_t+\frac{\lambda_p}{1+\lambda_p}\widehat{\lambda}_{p,t}\right]+\frac{\iota_p}{1+\iota_p\overline{\beta}\gamma}\widehat{\pi}_{t-1}+\frac{\overline{\beta}\gamma}{1+\iota_p\overline{\beta}\gamma}E_t[\widehat{\pi}_{t+1}] \qquad (48)$$

if $\sigma_c=1$, the expression $\overline{\beta}\gamma=\beta$ and the traditional expression is obtained again (see also DSSW).

Option 2:

For the flexible price case the derivation is relative simple. Remember the presence of the markup shock in the aggregtor function results in:[3]

$$\begin{aligned}
d(\frac{G'^{-1}(z_t)G''(x_t)}{\tau})-d(\lambda_{p,t})+d(p_t(i))-d(mc_t) &= 0 \\
\frac{G''(x_t)}{\tau}d(G'^{-1}(z_t))+\frac{G'^{-1}(z_t)}{\tau}d(G''(x_t))-d(\lambda_{p,t})+d(p_t(i))-d(mc_t) &= 0 \\
\frac{G''(x_t)}{\tau}\frac{1}{G''(x_t)}d(z_t)+\frac{G'^{-1}(z_t)}{\tau}G'''d(x_t)-d(\lambda_{p,t})+d(p_t(i))-d(mc_t) &= 0 \\
\frac{G''(x)}{\tau}\frac{1}{G''(x)}\tau d(p_t)+\frac{G'^{-1}(z)}{G''(x)}G'''d(p_t)-d(\lambda_{p,t})+d(p_t(i))-d(mc_t) &= 0 \\
d(p_t)+\frac{G'^{-1}(z)}{G''(x)}G'''d(p_t)-d(\lambda_{p,t})+d(p_t(i))-d(mc_t) &= 0 \\
(2+\frac{1}{G''(x)}G''')\widehat{p}_t(i)-mc\widehat{\lambda_{p,t}}-mc\widehat{mc}_t &= 0
\end{aligned}$$

[3]The shock has been scaled with $-\frac{G'''(x)G'^{-1}(z)}{\tau}=-\frac{G'''(1)}{G'(1)}$.

where the deviations in the markup are exppressed in percentage of the marginal cost

$$\widehat{p}_t(i) = \frac{\left[1 + \frac{G''(1)}{G'(1)}\right]}{\left[2 + \frac{G'''}{G''(1)}\right]}(\widehat{mc}_t + \widehat{\lambda_{p,t}})$$

Linearization the aggregate price expression (33) (in stst: $G'^{-1}[z] = x = 1$ and $\tau = G'(1)$ and $\pi = 0$)

$$\begin{aligned}
0 &= (1-\zeta_p)\left(\widehat{\widetilde{p}}_t(i) + \frac{G'(1)}{G''(1)}\widehat{\widetilde{p}}_t(i)\right) \\
&\quad + \zeta_p\left(\iota_p\widehat{\pi}_{t-1} - \widehat{\pi}_t + \iota_p\frac{G'(1)}{G''(1)}\widehat{\pi}_{t-1} - \frac{G'(1)}{G''(1)}\widehat{\pi}_t\right) \\
0 &= (1-\zeta_p)\widehat{\widetilde{p}}_t(i) + \zeta_p(\iota_p\widehat{\pi}_{t-1} - \widehat{\pi}_t) \\
\widehat{\widetilde{p}}_t(i) &= \frac{\zeta_p}{1-\zeta_p}(\widehat{\pi}_t - \iota_p\widehat{\pi}_{t-1})
\end{aligned}$$

Linearize the FOC for the Calvo wase with indexation (32):

$$E_t\sum_{s=0}^{\infty}\zeta_p^s\overline{\beta}^s\gamma^s\frac{\xi_{t+s}}{\xi_t}y_{t+s}(i)\left(\frac{1}{G'^{-1}(z_{t+s})}\frac{G'(x_{t+s})}{G''(x_{t+s})}\right)\left[(1+G'^{-1}(z_{t+s})\frac{G''(x_{t+s})}{G'(x_{t+s})})\frac{\widetilde{p}_t(i)X_{t,s}}{X^p_{t+s}} - mc_{t+s}\right] = 0 \tag{49}$$

$$E_t\sum_{s=0}^{\infty}\zeta_p^s\overline{\beta}^s\gamma^s\frac{\xi_{t+s}}{\xi_t}y_{t+s}(i)\left(\frac{1}{G'^{-1}(z_{t+s})}\frac{G'(x_{t+s})}{G''(x_{t+s})}\right)\left[\frac{\widetilde{p}_t(i)X_{t,s}}{X^p_{t+s}} + \frac{G'^{-1}(z_{t+s})G''(x_{t+s};\lambda_{p,t+s})}{\tau} - mc_{t+s}\right] = 0 \tag{50}$$

$$\begin{aligned}
&E_t\sum_{s=0}^{\infty}\zeta_p^s\overline{\beta}^s\gamma^s\Big[d(\widetilde{p}_t(i)) + d(\frac{X_{t,s}}{X^p_{t+s}}) - d(mc_{t+s}) - d(\lambda_{p,t+s}) + \\
&(\frac{G'''}{\tau}d(x_{t+s}) + \frac{G''(x)}{\tau}\frac{1}{G''(x)}d(z_{t+s}))\Big] \\
=&\ 0
\end{aligned} \tag{51}$$

$$\begin{aligned}
&E_t\sum_{s=0}^{\infty}\zeta_p^s\overline{\beta}^s\gamma^s\Big[d(\widetilde{p}_t(i)) + d(\frac{X_{t,s}}{X^p_{t+s}}) - d(mc_{t+s}) - d(\lambda_{p,t+s}) \\
&+\frac{G'''}{\tau}\frac{\tau}{G''}d(p_{t+s}) + \frac{G''(x)}{\tau}\frac{\tau}{G''}d(p_{t+s}))\Big] \\
=&\ 0
\end{aligned} \tag{52}$$

$$E_t \sum_{s=0}^{\infty} \zeta_p^s \overline{\beta}^s \gamma^s \left[d(\widetilde{p}_t(i)) + d(\frac{X_{t,s}}{X_{t+s}^p}) - d(\lambda_{p,t+s}) - d(mc_{t+s}) + \frac{G'''}{G''} d(p_{t+s}) + d(p_{t+s})) \right] = 0$$

$$E_t \sum_{s=0}^{\infty} \zeta_p^s \overline{\beta}^s \gamma^s \left[(2 + \frac{G'''}{G''}) d(\widetilde{p}_t(i)) + (2 + \frac{G'''}{G''}) d(\frac{X_{t,s}}{X_{t+s}^p}) - d(mc_{t+s}) - d(\lambda_{p,t+s}) \right] = 0$$

where

$$\begin{aligned} d(\frac{X_{t,s}}{X_{t+s}^p}) &= d(\frac{(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})}{(\Pi_{l=1}^s \pi_{t+l})}) \\ &= \iota_p \frac{\pi_*^{\iota_p - 1} \pi_*^{1-\iota_p}}{\pi_*} d(\pi_t) + \iota_p \frac{\pi_*^{\iota_p - 1} \pi_*^{1-\iota_p}}{\pi_*} d(\pi_{t+1}) + ... \\ &\quad - \frac{(\pi_*^{\iota_p} \pi_*^{1-\iota_p})}{(\pi_*)^2} d(\pi_{t+1}) - \frac{(\pi_*^{\iota_p} \pi_*^{1-\iota_p})}{(\pi_*)^2} d(\pi_{t+2}) - ... \end{aligned}$$

minus the same equation for t+1 multiplied with $\zeta_p \overline{\beta} \gamma$

$$\begin{aligned} 0 &= (2 + \frac{G'''}{G''}) \frac{1}{1 - \zeta_p \overline{\beta} \gamma} d(\widetilde{p}_t(i)) - (2 + \frac{G'''}{G''}) \frac{\zeta_p \overline{\beta} \gamma}{1 - \zeta_p \overline{\beta} \gamma} d(\widetilde{p}_{t+1}(i)) - d(mc_t) - d(\lambda_{p,t)} \\ &\quad + \frac{\zeta_p \overline{\beta} \gamma}{(1 - \zeta_p \overline{\beta} \gamma)} (2 + \frac{G'''}{G''}) \left(\iota_p \frac{\pi_*^{\iota_p - 1} \pi_*^{1-\iota_p}}{\pi_*} d(\pi_t) - \frac{(\pi_*^{\iota_p} \pi_*^{1-\iota_p})}{(\pi_*)^2} d(\pi_{t+1}) \right) \end{aligned}$$

$$\begin{aligned} 0 &= (2 + \frac{G'''}{G''}) \frac{1}{1 - \zeta_p \overline{\beta} \gamma} \widehat{\widetilde{p}}_t(i) - (2 + \frac{G'''}{G''}) \frac{\zeta_p \overline{\beta} \gamma}{1 - \zeta_p \overline{\beta} \gamma} \widehat{\widetilde{p}_{t+1}}(i) - (1 + \frac{G''}{G'}) (\widehat{mc}_t + \widehat{\lambda_{p,t}}) \\ &\quad + \frac{\zeta_p \overline{\beta} \gamma}{(1 - \zeta_p \overline{\beta} \gamma)} (2 + \frac{G'''}{G''}) \left(\iota_p \frac{\pi_*^{\iota_p - 1} \pi_*^{1-\iota_p}}{\pi_*} \pi_* \widehat{\pi}_t - \frac{(\pi_*^{\iota_p} \pi_*^{1-\iota_p})}{(\pi_*)^2} \pi_* \widehat{\pi}_{t+1} \right) \end{aligned}$$

using the expression for $\widehat{\widetilde{p}}_t(i)$

$$\begin{aligned} 0 &= \frac{1}{1 - \zeta_p \overline{\beta} \gamma} (\frac{\zeta_p}{1 - \zeta_p} (\ \widehat{\pi}_t - \ \iota_p \widehat{\pi}_{t-1})) - \frac{\zeta_p \overline{\beta} \gamma}{1 - \zeta_p \overline{\beta} \gamma} \frac{\zeta_p}{1 - \zeta_p} (\ \widehat{\pi}_{t+1} - \ \iota_p \widehat{\pi}_t) \\ &\quad - \frac{(1 + \frac{G''}{G'})}{(2 + \frac{G'''}{G''})} (\widehat{mc}_t + \widehat{\lambda_{p,t}}) + \frac{\zeta_p \overline{\beta} \gamma}{(1 - \zeta_p \overline{\beta} \gamma)} \left(\iota_p \frac{\pi_*^{\iota_p - 1} \pi_*^{1-\iota_p}}{\pi_*} \pi_* \widehat{\pi}_t - \frac{(\pi_*^{\iota_p} \pi_*^{1-\iota_p})}{(\pi_*)^2} \pi_* \widehat{\pi}_{t+1} \right) \end{aligned}$$

$$\begin{aligned} 0 &= ((\ \widehat{\pi}_t - \ \iota_p \widehat{\pi}_{t-1})) - \frac{\zeta_p \overline{\beta} \gamma}{1} (\ \widehat{\pi}_{t+1} - \ \iota_p \widehat{\pi}_t) \\ &\quad + \frac{\zeta_p \overline{\beta} \gamma (1 - \zeta_p)}{\zeta_p} (\iota_p \widehat{\pi}_t - \widehat{\pi}_{t+1}) \\ &\quad - \frac{(1 + \frac{G''}{G'})}{(2 + \frac{G'''}{G''})} \frac{(1 - \zeta_p \overline{\beta} \gamma)(1 - \zeta_p)}{\zeta_p} (\widehat{mc}_t + \widehat{\lambda_{p,t}}) \end{aligned}$$

$$
\begin{aligned}
0 &= \widehat{\pi}_t - \iota_p\widehat{\pi}_{t-1} - \zeta_p\overline{\beta}\gamma\ \widehat{\pi}_{t+1} + \zeta_p\overline{\beta}\gamma\ \iota_p\widehat{\pi}_t \\
&+\overline{\beta}\gamma\left(\iota_p\widehat{\pi}_t - \widehat{\pi}_{t+1}\right) - \overline{\beta}\gamma\zeta_p\left(\iota_p\widehat{\pi}_t - \widehat{\pi}_{t+1}\right) \\
&-\frac{(1+\frac{G''}{G'})}{(2+\frac{G'''}{G''})}\frac{(1-\zeta_p\overline{\beta}\gamma)(1-\zeta_p)}{\zeta_p}(\widehat{mc}_t + \widehat{\lambda_{p,t}})
\end{aligned}
$$

$$
\begin{aligned}
(1+\overline{\beta}\gamma\iota_p)\ \widehat{\pi}_t &= \iota_p\widehat{\pi}_{t-1} + \overline{\beta}\gamma\widehat{\pi}_{t+1} + A\frac{(1-\zeta_p\overline{\beta}\gamma)(1-\zeta_p)}{\zeta_p}(\widehat{mc}_t + \widehat{\lambda_{p,t}}) \qquad (53) \\
\text{with } A &= \frac{(1+\frac{G''}{G'})}{(2+\frac{G'''}{G''})}
\end{aligned}
$$

If we consider the special case in which the aggregator function G reduces to the Dixit-Stiglitz aggregator, we can see that option 1 and option 2 results in the same NKPC: in that case $G(x) = (x)^{1/(1+\lambda)}$ and $A = 1$. The following relation between A and the curvature holds:

$$
\begin{aligned}
A &= \frac{(1+\frac{G''}{G'})}{(2+\frac{G'''}{G''})} = \frac{1}{\lambda\epsilon+1} = \frac{1}{\frac{\epsilon}{\varepsilon-1}+1} \\
\epsilon &= \frac{d(\varepsilon)}{d(x)} = \frac{d(\frac{G''}{xG'})}{d(x)} = 1+\varepsilon+\varepsilon\frac{G'''}{G''}
\end{aligned}
$$

Eq. 28 becomes:

$$
\widehat{k}_t \ = \ \widehat{w}_t \ - \ \widehat{r}_t^k \ + \ \widehat{L}_t. \qquad (54)
$$

Eq. 34 becomes:

$$
(1-h/\gamma)\widehat{\xi}_t = (1-h/\gamma)\widehat{b^1}_t \ - \ \sigma_c\widehat{c}_t + \sigma_c(h/\gamma)\widehat{c}_{t-1} \ + (1-h/\gamma)(\sigma_c-1)L^{1+\nu_l}\widehat{L}_t
$$

using $w_*^h = \frac{U'_{l,*}}{\Xi_*} = \left[(c_t - (h/\gamma)c_{t-1})\right] L_t^{\nu_l}$:

$$
(1-h/\gamma)\widehat{\xi}_t = (1-h/\gamma)\widehat{b^1}_t \ - \ \sigma_c\widehat{c}_t + \sigma_c(h/\gamma)\widehat{c}_{t-1} \ + (\sigma_c-1)(w_*^hL/c_*)\widehat{L}_t \qquad (55)
$$

$[\frac{w_*^hL}{c_*} = \frac{1}{1+\lambda_w}\frac{1-\alpha}{\alpha}r_*^k\frac{k_*}{y_*}\frac{y_*}{c_*}]$

Eq. 35 becomes:

$$
\widehat{\xi}_t = \widehat{b}_t^2 + \widehat{R}_t \ + \ E_t[\widehat{\xi}_{t+1}] \ - \ E_t[\widehat{\pi}_{t+1}]. \qquad (56)
$$

Eq. 36 becomes:

$$
\widehat{k}_t \ = \ \widehat{u}_t \ + \ \widehat{\overline{k}}_{t-1}.
$$

Eq. 37 becomes:

$$
\widehat{\overline{k}}_t = \ (1-\frac{i_*}{\overline{k}_*})\ \widehat{\overline{k}}_{t-1} + \frac{i_*}{\overline{k}_*}\mu_t \ + \ \frac{i_*}{\overline{k}_*}\widehat{i}_t. \qquad (57)
$$

Eq. 38 becomes:

$$Q_t^k \mu_t \left[1 - S\left(\frac{i_t\gamma}{i_{t-1}}\right) - S'\left(\frac{i_t\gamma}{i_{t-1}}\right)\frac{i_t\gamma}{i_{t-1}}\right]$$
$$+\overline{\beta}E_t\left\{\frac{\xi_{t+1}}{\xi_t}[Q_{t+1}^k\mu_{t+1}S'\left(\frac{i_{t+1}\gamma}{i_t}\right)\left(\frac{i_{t+1}\gamma}{i_t}\right)^2]\right\} = 1$$

becomes:

$$\widehat{Q}_t^k \ + \widehat{\mu}_t \ - \gamma^2 S''\widehat{i}_t + \gamma^2 S''\widehat{i}_{t-1} + (\overline{\beta}\gamma)\gamma^2 S''\widehat{i}_{t+1} - (\overline{\beta}\gamma)\gamma^2 S''\widehat{i}_t \ = 0$$

$$\widehat{i}_t = \frac{1}{(1+\overline{\beta}\gamma)}(\widehat{i}_{t-1} + (\overline{\beta}\gamma)\widehat{i}_{t+1} + \frac{1}{\gamma^2 S''}\widehat{Q}_t^k \ + \frac{1}{\gamma^2 S''}\widehat{\mu}_t \tag{58}$$

Eq. 39 becomes:

$$\widehat{Q}_t^k = -\xi_t + E_t[\xi_{t+1}] \ + \ \frac{r_*^k}{r_*^k + (1-\delta)}E_t[r_{t+1}^k] + \frac{(1-\delta)}{r_*^k+(1-\delta)}E_t[Q_{t+1}^k]. \tag{59}$$

where the steady state expression $\overline{\beta}(r_*^k + (1-\tau)) = 1$ is used.
Eq. 40 becomes:

$$r_*^k\widehat{r}_t^k = a''\widehat{u}_t. \tag{60}$$

or in case of accumulated costs (assuming $s = 0$) expressed relative to output:

$$\begin{aligned} ak_t &= \frac{a'(u_*)\overline{k}_*}{\gamma}\widehat{u}_t + \frac{1-\delta}{\gamma}ak_{t-1} \\ &= r_*^k k\widehat{u}_t + \frac{1-\delta}{\gamma}ak_{t-1} \end{aligned}$$

Option 2 for wage setting:
Eq. 42 becomes:

$$\begin{aligned} & (1+\lambda_w)w^h\widehat{w}_t^h + \lambda_w w^h\widehat{\lambda_{w,t}} - \frac{\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)}\widehat{\widetilde{w}}_t + \frac{\zeta_w\overline{\beta}\gamma\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)}\widehat{\widetilde{w}}_{t+1} \\ & -\frac{\zeta_w\overline{\beta}\gamma\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)}\iota_w\widehat{\pi}_t + \frac{\zeta_w\overline{\beta}\gamma\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)}\widehat{\pi}_{t+1} \\ = \ & 0 \end{aligned}$$

Using Eq. 43:

$$\frac{1}{1-\zeta_w}\widehat{w}_t \ - \frac{\zeta_w}{1-\zeta_w} \ \widehat{w}_{t-1} \ + \frac{\zeta_w}{1-\zeta_w} \ \widehat{\pi}_t - \frac{\zeta_w\iota_w}{1-\zeta_w} \ \widehat{\pi}_{t-1} = \widehat{\widetilde{w}}_t.$$

and

$$\widehat{w}_t^h = \frac{1}{1-h/\gamma}\widehat{c}_t - \frac{h/\gamma}{1-h/\gamma}\widehat{c}_{t-1} \ + \nu_l\widehat{L}_t$$

results in

$$
\begin{aligned}
& (1+\lambda_w) w^h \widehat{w}_t^h + \lambda_w w^h \widehat{\lambda_{w,t}} - \widetilde{w}\widehat{w}_t + \widetilde{w}\widehat{w}_t \\
& -\frac{\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)} \frac{1}{(1-\zeta_w)} \left[\widehat{w}_t \ - \zeta_w \ \widehat{w}_{t-1} \ + \zeta_w \ \widehat{\pi}_t - \zeta_w \iota_w \ \widehat{\pi}_{t-1}\right] \\
& + \frac{\zeta_w\overline{\beta}\gamma\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)} \frac{1}{(1-\zeta_w)} \left[\widehat{w}_{t+1} \ - \zeta_w \ \widehat{w}_t \ + \zeta_w \ \widehat{\pi}_{t+1} - \zeta_w \iota_w \ \widehat{\pi}_t\right] \\
& - \frac{\zeta_w\overline{\beta}\gamma\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)} \frac{(1-\zeta_w)}{(1-\zeta_w)} \iota_w \widehat{\pi}_t + \frac{\zeta_w\overline{\beta}\gamma\widetilde{w}}{(1-\zeta_w\overline{\beta}\gamma)} \frac{(1-\zeta_w)}{(1-\zeta_w)} \widehat{\pi}_{t+1} \\
= \ & 0
\end{aligned}
$$

$$
\begin{aligned}
& (1-\zeta_w\overline{\beta}\gamma)(1-\zeta_w) \left[\widehat{w}_t^h + \frac{\lambda_w}{(1+\lambda_w)} \widehat{\lambda_{w,t}} - \widehat{w}_t\right] \\
& + (1-\zeta_w\overline{\beta}\gamma - \zeta_w + \zeta_w\overline{\beta}\gamma\zeta_w)\widehat{w}_t \\
& - \left[\widehat{w}_t \ - \zeta_w \ \widehat{w}_{t-1} \ + \zeta_w \ \widehat{\pi}_t - \zeta_w \iota_w \ \widehat{\pi}_{t-1}\right] \\
& + \zeta_w\overline{\beta}\gamma \left[\widehat{w}_{t+1} \ - \zeta_w \ \widehat{w}_t \ + \zeta_w \ \widehat{\pi}_{t+1} - \zeta_w \iota_w \ \widehat{\pi}_t\right] \\
& - \zeta_w\overline{\beta}\gamma\iota_w(1-\zeta_w)\widehat{\pi}_t + \zeta_w\overline{\beta}\gamma(1-\zeta_w)\widehat{\pi}_{t+1} \\
= \ & 0
\end{aligned}
$$

$$
\begin{aligned}
& (1+\overline{\beta}\gamma)\widehat{w}_t \ - \widehat{w}_{t-1} - \overline{\beta}\gamma\widehat{w}_{t+1} \\
= \ & \frac{(1-\zeta_w\overline{\beta}\gamma)(1-\zeta_w)}{\zeta_w} \left[\widehat{w}_t^h + \frac{\lambda_w}{(1+\lambda_w)} \widehat{\lambda_{w,t}} - \widehat{w}_t \ \right] \\
& - (1+\overline{\beta}\gamma\iota_w)\widehat{\pi}_t + \iota_w\widehat{\pi}_{t-1} + \overline{\beta}\gamma\widehat{\pi}_{t+1}
\end{aligned} \tag{61}
$$

Eq. (44) becomes:

$$
\widehat{y}_t \ = \ \widehat{g}_t \ + \ \frac{c_*}{y_*}\widehat{c}_t \ + \ \frac{i_*}{y_*}\widehat{i}_t \ + \ \frac{r_*^k k_*}{y_*}\widehat{u}_t. \tag{62}
$$

Eq. (27) becomes (remember $\widehat{y}_t = \widehat{\dot{y}}_t$):

$$
\widehat{y}_t \ = \ \alpha\frac{y_*+\Phi}{y_*}\widehat{k}_t \ + \ (1-\alpha)\frac{y_*+\Phi}{y_*}\widehat{L}_t + \frac{y_*+\Phi}{y_*}\widehat{Z}_t \tag{63}
$$

Eq. (45) becomes:

$$
\begin{aligned}
\widehat{R}_t \quad = \quad & \rho_R\widehat{R}_{t-1} \ + \ (1-\rho_R)(\psi_1\widehat{\pi}_t \ + \ \psi_2(\widehat{y}_t - \widehat{y}_t^{flex})) \\
& + \psi_3(\widehat{y}_t - \widehat{y}_{t-1} - (\widehat{y_t^{flex}} - \widehat{y}_{t-1}^{flex})) \ + r_t
\end{aligned} \tag{64}
$$

5 Estimated equations and rescaling residuals

5.1 For the sticky economy

Eq. (46):

$$\widehat{mc}_t = (1-\alpha)\ \widehat{w}_t + \alpha\ \widehat{r}_t^k - \widehat{Z}_t$$

Eq (53)

$$(1+\overline{\beta}\gamma\iota_p)\ \widehat{\pi}_t = \iota_p\widehat{\pi}_{t-1} + \overline{\beta}\gamma E_t\left[\widehat{\pi}_{t+1}\right] + A\frac{(1-\zeta_p\overline{\beta}\gamma)(1-\zeta_p)}{\zeta_p}(\widehat{mc}_t) + \widehat{\lambda_{p,t}}$$

Eq (61)

$$\begin{aligned} & (1+\overline{\beta}\gamma)\widehat{w}_t\ - \widehat{w}_{t-1} - \overline{\beta}\gamma E_t\left[\widehat{w}_{t+1}\right] \\ = & \frac{(1-\zeta_w\overline{\beta}\gamma)(1-\zeta_w)}{\zeta_w}\left[\frac{1}{1-h/\gamma}\widehat{c}_t - \frac{h/\gamma}{1-h/\gamma}\widehat{c}_{t-1}\ + \nu_l\widehat{L}_t - \widehat{w}_t\ \right] \\ & -(1+\overline{\beta}\gamma\iota_w)\widehat{\pi}_t + \iota_w\widehat{\pi}_{t-1} + \overline{\beta}\gamma E_t\left[\widehat{\pi}_{t+1}\right] + \widehat{\lambda_{w,t}} \end{aligned}$$

Eq (56)

$$\begin{aligned} & \widehat{c}_t \\ = & \frac{1}{(1+(h/\gamma))}E_t\left[\widehat{c}_{t+1}\right]\ \ + \frac{(h/\gamma)}{(1+(h/\gamma))}\widehat{c}_{t-1} \\ & -\frac{(1-h/\gamma)}{\sigma_c(1+(h/\gamma))}(\widehat{b}_t^2 + \widehat{R}_t - E_t[\widehat{\pi}_{t+1}]) \\ & -\frac{(\sigma_c-1)(w_*^hL/c_*)}{\sigma_c(1+(h/\gamma))}(E_t\left[\widehat{L}_{t+1}\right] - \widehat{L}_t). \end{aligned}$$

Eq (54)

$$\widehat{k}_t\ =\ \widehat{w}_t\ -\ \widehat{r}_t^k\ +\ \widehat{L}_t.$$

Eq (60)

$$r_*^k\widehat{r}_t^k = a''\widehat{u}_t.$$

Eq (57)

$$\widehat{\overline{k}}_t = \ (1-\frac{i_*}{\overline{k}_*})\ \widehat{\overline{k}}_{t-1} + \frac{i_*}{\overline{k}_*}\mu_t\ +\ \frac{i_*}{\overline{k}_*}\widehat{i}_t.$$

Eq (58)

$$\widehat{i}_t = \frac{1}{(1+\overline{\beta}\gamma)}(\widehat{i}_{t-1} + (\overline{\beta}\gamma)E_t\left[\widehat{i}_{t+1}\right] + \frac{1}{\gamma^2S''}\widehat{Q}_t^k\ + \frac{1}{\gamma^2S''}\widehat{\mu}_t$$

Eq (59)

$$\widehat{Q}_t^k = -\widehat{b}_t^2 - (\widehat{R}_t - E_t[\widehat{\pi}_{t+1}]) \ + \ \frac{r_*^k}{r_*^k + (1-\delta)} E_t[r_{t+1}^k] + \frac{(1-\delta)}{r_*^k + (1-\delta)} E_t[Q_{t+1}^k].$$

Eq. (64):

$$\begin{aligned} \widehat{R}_t \quad &= \quad \rho_R \widehat{R}_{t-1} \ + \ (1-\rho_R)(\psi_1 \widehat{\pi}_t \ + \ \psi_2(\widehat{y}_t - \widehat{y}_t^{flex})) \\ &\quad + \psi_3(\widehat{y}_t - \widehat{y}_{t-1} - (\widehat{y_t^{flex}} - \widehat{y}_{t-1}^{flex})) \ + r_t \end{aligned}$$

Eq. (63):

$$\widehat{y}_t \ = \ \alpha \frac{y_* + \Phi}{y_*} \widehat{k}_t \ + \ (1-\alpha) \frac{y_* + \Phi}{y_*} \widehat{L}_t + \frac{y_* + \Phi}{y_*} \widehat{Z}_t$$

Eq. (62):

$$\widehat{y}_t \ = \ \widehat{g}_t \ + \ \frac{c_*}{y_*} \widehat{c}_t \ + \ \frac{i_*}{y_*} \widehat{i}_t \ + \ \frac{r_*^k k_*}{y_*} \widehat{u}_t.$$

5.2 For the flexible economy