In the benchmark model, I consider only a single, composite consumption good. Unlike the social planner's problem, the representative household does not take into account their externality, leading to a suboptimal decentralized solution since environmental quality is too low (i.e. air pollution is too high).

Definition 1. *For a given series of taxes τ^t, production shocks z^t, and an initial condition K_0, the competitive equilibrium is a sequence of prices ($\{r_t^*\}$, $\{w_t^*\}$, and $\{f_t^*\}$) and quantities ($\{K_t^*\}$, $\{X_t^*\}$, $\{H_t^*\}$, $\{E_t^*\}$, $\{C_t^*\}$, $\{Y_t^*\}$, $\{\psi_t^*\}$, $\{T_t^*\}$) such that*

1. *(Households): taking prices, taxes, environmental quality and transfers as given, $\{c_t^*\}$, $\{h_t^*\}$, and $\{k_{t+1}^*\}$ are the solutions to*

$$\max_{c_t,h_t,k_{t+1}} \sum_{t=0}^{\infty} \beta^t \left[\iota c_t^{\frac{\sigma_c-1}{\sigma_c}} + (1-\iota)\left((\chi_s) S_t^{\frac{\sigma_s-1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right)^{\frac{\sigma_s}{\sigma_s-1}\frac{\sigma_c-1}{\sigma_c}} \right]^{\frac{\sigma_c}{\sigma_c-1}}$$

subject to

$$c_t(1+\tau_{ct}) + x_t(1+\tau_{xt}) = w_t h_t (1-\tau_{ht})(1+T_t) + (1-\tau_{kt})(r_t-\delta)k_t + \delta k_t + \psi_t$$

$$H_t + L_t = 1; \quad K_t, C_t, H_t \geq 0$$

2. *(Firms): taking prices as given, $\{Y_t^*\}$, $\{H_t^*\}$, $\{E_t^*\}$, and $\{K_t^*\}$ are the solution to*

$$\max_{K_t,L_t,E_t} Y_t - r_t K_t - w_t H_t - f_t E_t (1+\tau_{dt})$$

subject to

$$Y_t = A_t \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{\sigma_h}{\sigma_h-1}}$$

where $S_t = 1/(\xi E_t)$.

3. *(Government): the government budget constraint is balanced each period*

$$\psi_t + T_t = \tau_{ht} w_t H_t + \tau_{dt} \xi E_t + \tau_{ct} C_t + \tau_{xt} X_t + r_t \tau_{kt} K_t$$

4. *Markets clear*

$$Y_t^* = c_t^* + x_t^*$$

$$K_t^* = k_t^*; \quad H_t^* = h_t^*$$

Because there are three unknowns, three equilibrium conditions are needed. First, the intertemporal Euler condition

$$C_t^{-\frac{1}{\sigma_c}} \Psi_t = \beta C_{t+1}^{-\frac{1}{\sigma_c}} \Psi_{t+1} (1 - \delta + (1-\tau_{kt}) r_t)$$

$$\text{where } r_t = A_t \eta_h \theta \left(\frac{H_t}{K_t} \right)^{1-\theta} \left[K_t^{\theta} H_t^{1-\theta} \right]^{-\frac{1}{\sigma_h}} \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{1}{\sigma_h-1}}$$

$$\text{and } \Psi_t = \left\{ \iota c_t^{\frac{\sigma_c - 1}{\sigma_c}} + (1-\iota)\left[\chi_s \left(\frac{1}{\xi E_t}\right)^{\frac{\sigma_s - 1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s - 1}{\sigma_s}} \right]^{\frac{\sigma_c - 1}{\sigma_c}\frac{\sigma_s}{\sigma_s - 1}} \right\}^{\frac{1}{\sigma_c - 1}}$$

Second, the intratemporal Euler condition between labor and consumption

$$\frac{(1-\iota)(1-\chi_s)(1-H_t)^{-\frac{1}{\sigma_s}} \left((\chi_s)\left(\frac{1}{\xi E_t}\right)^{\frac{\sigma_s - 1}{\sigma_s}} + (1-\chi_s)(1-H_t)^{\frac{\sigma_s - 1}{\sigma_s}} \right)^{\frac{\sigma_c - \sigma_s}{\sigma_c(\sigma_s - 1)}}}{\iota C_t^{-\frac{1}{\sigma_c}}} = \frac{(1-\tau_{ht})(1+T_t)w_t}{(1+\tau_{ct})}$$

$$\text{where } w_t = A_t \eta_h (1-\theta)\left(\frac{K_t}{H_t}\right)^{\theta} \left[K_t^{\theta} H_t^{1-\theta}\right]^{-\frac{1}{\sigma_h}} \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta}\right]^{\frac{\sigma_h - 1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h - 1}{\sigma_h}} \right)^{\frac{1}{\sigma_h - 1}}$$

Third, Hotelling's rule equates the marginal net product of energy with the rental rate of return

$$\frac{f_{t+1}}{f_t} = r_t$$

$$\text{where } f_t = (1+\tau_{dt})^{-1} A_t (1-\eta_h) E_t^{-\frac{1}{\sigma_h}} \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta}\right]^{\frac{\sigma_h - 1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h - 1}{\sigma_h}} \right)^{\frac{1}{\sigma_h - 1}}$$

Fourth, the aggregate resource constraint

$$\begin{array}{rcc} K_{t+1} & = & (1-\delta)K_t - E_t \\ & - & \underbrace{(1+\tau_{ct})^{-1}\left[w_t H_t (1-\tau_{ht})(1+T_t) - X_t(1+\tau_{xt}) + (1-\tau_{kt})(r_t - \delta)K_t + \delta K_t + \psi_t\right]}_{C_t} \\ & + & \underbrace{\left(\eta_h \left[K_t^{\theta} H_t^{1-\theta}\right]^{\frac{\sigma_h - 1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h - 1}{\sigma_h}} \right)^{\frac{\sigma_h}{\sigma_h - 1}}}_{Y_t} \end{array}$$

Linearizing

$$\begin{array}{rcl} r_t & = & A + \eta + \theta + (1-\theta)\exp\left(\frac{\tilde{H}}{\tilde{K}}\right)\left(-\frac{1}{\sigma_h}\right)\left[\theta \exp(\tilde{K}) + (1-\theta)\exp(\tilde{H})\right] \\ & \times & \left(\frac{1}{\sigma_h - 1}\right)\left[\eta \frac{\sigma_h - 1}{\sigma_h}\left(\theta \exp(\tilde{K}) + (1-\theta)\exp(\tilde{H})\right) + (1-\eta)\frac{\sigma_h - 1}{\sigma_h}\exp(\tilde{E})\right] \\ w_t & = & A + \eta + (1-\theta) + \theta \exp\left(\frac{\tilde{K}}{\tilde{H}}\right)\left(-\frac{1}{\sigma_h}\right)\left[\theta \exp(\tilde{K}) + (1-\theta)\exp(\tilde{H})\right] \\ & \times & \left(\frac{1}{\sigma_h - 1}\right)\left[\eta \frac{\sigma_h - 1}{\sigma_h}\left(\theta \exp(\tilde{K}) + (1-\theta)\exp(\tilde{H})\right) + (1-\eta)\frac{\sigma_h - 1}{\sigma_h}\exp(\tilde{E})\right] \end{array}$$