Appendix

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Solving for the DSGE model for the theoretical economy

0.1 Model Setup

- Setup:
  - Two countries, each inhabitied by identical agents
  - Country 1 produces good a, and country 2 produces good b, with their own technology
  - Labor is internationally immobile
  - Stochastic shocks: productivity shocks and government purchases of goods and services
- Preferences: The utility function is:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_{it}, 1 - n_{it})$$

  where $U(c, 1-n) = [c^{\mu}(1-n)^{1-\mu}]^{\gamma}/\gamma$

- Production: The production functions are:

$$y_{1t} = z_{1t} F(k_{1t}, n_{1t}) = z_{1t} k_{1t}^{\theta} n_{it}^{1-\theta} \tag{1}$$

$$y_{2t} = z_{2t} F(k_{2t}, n_{2t}) = z_{2t} k_{2t}^{\theta} n_{2t}^{1-\theta} \tag{2}$$

  where y_{it} denotes country i's GDP measured in unites of the local good

- Resource constraints in the supply side are:

$$a_{1t} + a_{2t} = y_{1t} \tag{3}$$

$$b_{1t} + b_{2t} = y_{2t} \tag{4}$$

- Goods market-clearing conditions in the demand side are:

$$c_{1t} + x_{1t} + g_{1t} = G(a_{1t}, b_{1t}) = [\omega_1 a_{1t}^{-\rho} + \omega_2 b_{1t}^{-\rho}]^{-1/\rho} \tag{5}$$

$$c_{2t} + x_{2t} + g_{2t} = G(b_{2t}, a_{2t}) = [\omega_1 b_{2t}^{-\rho} + \omega_2 a_{2t}^{-\rho}]^{-1/\rho} \tag{6}$$

- Capital accumulation equations are (assuming the simplest case where J=1):

$$k_{1,t+1} = (1-\delta)k_{1,t} + s_{1,t} \tag{7}$$

$$k_{2,t+1} = (1-\delta)k_{2,t} + s_{2,t} \tag{8}$$

$$x_{1,t} = s_{1,t} \tag{9}$$

$$x_{2,t} = s_{2,t} \tag{10}$$

- Law of motion equations for shocks are:

$$(\mathbf{z_{t+1}} = \mathbf{A}\mathbf{z_t} + \epsilon^{\mathbf{z}}_{\mathbf{t+1}})$$

$$(\mathbf{g_{t+1}} = \mathbf{B}\mathbf{g_t} + \epsilon^{\mathbf{g}}_{\mathbf{t+1}})$$

$$z_{1,t} = 0.906 * z_{1,t-1} + 0.088 * z_{2,t-1} + \epsilon^z_1 \tag{11}$$

$$z_{2,t} = 0.088 * z_{1,t-1} + 0.906 * z_{2,t-1} + \epsilon^z_2 \tag{12}$$

$$g_{1,t} = 0.95 * g_{1,t-1} \tag{13}$$

$$g_{2t} = 0.95 * g_{2,t-1} \tag{14}$$

- Introducing prices of the two goods, we have:

$$c_{1t} + x_{1t} + g_{1t} = q_{1t}a_{1t} + q_{2t}b_{1t} \tag{15}$$

$$c_{2t} + x_{2t} + g_{2t} = q_{1t}a_{2t} + q_{2t}b_{2t} \tag{16}$$

- The terms of trade for country 1 is:

$$p_t = q_{2t}/q_{1t} = \frac{\partial G(a_{1t}, b_{1t})/\partial b_{1t}}{\partial G(a_{1t}, b_{1t})/\partial a_{1t}} = \frac{\omega_2}{\omega_1}(\frac{a_{1t}}{b_{1t}})^{\frac{1}{\sigma}} \tag{17}$$

$$p_t = q_{2t}/q_{1t} = \frac{\partial G(b_{2t}, a_{2t})/\partial b_{2t}}{\partial G(b_{2t}, a_{2t})/\partial a_{2t}} = \frac{\omega_1}{\omega_2}(\frac{a_{2t}}{b_{2t}})^{\frac{1}{\sigma}} \tag{18}$$

$$p_t = q_{2t}/q_{1t} \tag{19}$$

- Net exports of country 1 are:

$$nx_t = (a_{2t} - p_t b_{1t})/y'_{1t} \tag{20}$$

0.2 The Lagrangian Problem

The **social planner problem** is:

$$\max_{c_{1t},c_{2t},n_{1t},n_{2t},k_{1,t+1},k_{2,t+1},x_{1t},x_{2t},a_{1t},b_{1t},a_{2t},b_{2t}} \sum_{t=0}^{\infty} \beta^t \{\frac{1}{2}\frac{[c_{1t}^{\mu}(1-n_{1t}^{1-\mu})]^{\gamma}}{\gamma} + \frac{1}{2}\frac{[c_{2t}^{\mu}(1-n_{2t})^{1-\mu}]^{\gamma}}{\gamma}\}$$

subject to (1)-(10).

So the Lagrangian becomes:

$$\begin{aligned} L = \sum_{t=0}^{\infty} \beta^t \{ & \frac{1}{2}\frac{[c_{1t}^{\mu}(1-n_{1t}^{1-\mu})]^{\gamma}}{\gamma} + \frac{1}{2}\frac{[c_{2t}^{\mu}(1-n_{2t})^{1-\mu}]^{\gamma}}{\gamma} \\ & + \lambda_{1t}(z_{1t}k_{1t}^{\theta}n_{it}^{1-\theta} - a_{1t} - a_{2t})\} + \lambda_{2t}(z_{2t}k_{2t}^{\theta}n_{2t}^{1-\theta} - b_{1t} - b_{2t})((1)-(4)) \\ & + \phi_{1t}([\omega_1 a_{1t}^{-\rho} + \omega_2 b_{1t}^{-\rho}]^{-1/\rho} - c_{1t} - x_{1t} - g_{1t}) \\ & + \phi_{2t}([\omega_1 b_{2t}^{-\rho} + \omega_2 a_{2t}^{-\rho}]^{-1/\rho} - c_{2t} - x_{2t} - g_{2t})((5)\&(6)) \\ & + \mu_{1t}(k_{1,t+1} - (1-\delta)k_{1t} - x_{1t}) + \mu_{2t}(k_{2,t+1} - (1-\delta)k_{2t} - x_{2t})(7)-(10) \end{aligned}$$

FOC for country 1:

1. c_{1t}: $\frac{1}{2}\mu\frac{[c_{1t}^{\mu}(1-n_{1t})^{1-\mu}]^{\gamma}}{c_{1t}} = \phi_{1t}$
2. n_{1t}: $\frac{1}{2}(1-\mu)\frac{[c_{1t}^{\mu}(1-n_{1t})^{1-\mu}]^{\gamma}}{1-n_{1t}} = (1-\theta)\lambda_{1t}\frac{y_{1t}}{n_{1t}}$
3. $k_{1,t+1}$: $\mu_{1t} = \beta[(1-\delta)\mu_{1,t+1} - \lambda_{1,t+1}\theta\frac{y_{1,t+1}}{k_{1,t+1}}]$
4. x_{1t}: $\phi_{1t} = -\mu_{it}$

5. a_{1t}: $\lambda_{1t} = \phi_{1t}[\omega_1 a_{1t}^{-\rho} + \omega_2 b_{1t}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_1 a_{1t}^{-\rho-1}$

6. a_{2t}: $\lambda_{1t} = \phi_{2t}[\omega_1 b_{2t}^{-\rho} + \omega_2 a_{2t}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_2 a_{2t}^{-\rho-1}$

Combining 1,3,4,5, I could derive a **Euler equation** as follows:

$$\frac{[c_{1t}^{\mu}(1-n_{1t})^{1-\mu}]^{\gamma}}{c_{1t}} = \beta\frac{[c_{1,t+1}^{\mu}(1-n_{1,t+1})^{1-\mu}]^{\gamma}}{c_{1,t+1}}[\theta\frac{y_{1,t+1}}{k_{1,t+1}}[\omega_1 a_{1,t+1}^{-\rho} + \omega_2 b_{1,t+1}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_1 a_{1,t+1}^{-\rho-1} + 1 - \delta] \tag{21}$$

Combining 1,2,3,5, I could derive a **intra-temporal substitution equation** as follows:

$$(1-\mu)c_{1t} = \mu(1-\theta)(1-n_{1t})\frac{y_{1t}}{n_{1t}}[\omega_1 a_{1t}^{-\rho} + \omega_2 b_{1t}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_1 a_{1t}^{-\rho-1} \tag{22}$$

FOC for country 2:

1. c_{2t}: $\frac{1}{2}\mu\frac{[c_{2t}^{\mu}(1-n_{2t})^{1-\mu}]^{\gamma}}{c_{2t}} = \phi_{2t}$

2. n_{2t}: $\frac{1}{2}(1-\mu)\frac{[c_{2t}^{\mu}(1-n_{2t})^{1-\mu}]^{\gamma}}{1-n_{2t}} = (1-\theta)\lambda_{2t}\frac{y_{2t}}{n_{2t}}$

3. $k_{2,t+1}$: $\mu_{2t} = \beta[(1-\delta)\mu_{2,t+1} - \lambda_{2,t+1}\theta\frac{y_{2,t+1}}{k_{2,t+1}}]$

4. x_{2t}: $\phi_{2t} = -\mu_{2t}$

5. b_{1t}: $\lambda_{2t} = \phi_{1t}[\omega_1 a_{1t}^{-\rho} + \omega_2 b_{1t}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_2 b_{1t}^{-\rho-1}$

6. b_{2t}: $\lambda_{2t} = \phi_{2t}[\omega_1 b_{2t}^{-\rho} + \omega_2 a_{2t}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_1 b_{2t}^{-\rho-1}$

Similarly, combining 1,3,4,6, I could derive a **Euler equation** as follows:

$$\frac{[c_{2t}^{\mu}(1-n_{2t})^{1-\mu}]^{\gamma}}{c_{2t}} = \beta\frac{[c_{2,t+1}^{\mu}(1-n_{2,t+1})^{1-\mu}]^{\gamma}}{c_{2,t+1}}[\theta\frac{y_{2,t+1}}{k_{2,t+1}}[\omega_1 b_{2,t+1}^{-\rho} + \omega_2 a_{2,t+1}^{-\rho}]^{-\frac{1}{\rho}-1}\omega_1 b_{2,t+1}^{-\rho-1} + 1 - \delta] \tag{23}$$

Combining 1,2,6, I could derive the **intra-temporal substitution equation** as follows:

$$(1-\mu)c_{2t} = \mu(1-\theta)(1-n_{2t})\frac{y_{2t}}{n_{2t}}(\omega_1 b_{2t}^{-\rho} + \omega_2 a_{2t}^{-\rho})^{-\frac{1}{\rho}-1}\omega_1 b_{2t}^{-\rho-1} \tag{24}$$