As a first-pass, abstract from the two different consumption goods and energy inputs; here, there is only a homogeneous consumption good and dirty energy input. The reason for this simplification is to highlight the channel of nonseparability between environmental quality and consumption/leisure.[1]

The social planner chooses sequences $\{c_t, h_t, S_t, k_{t+1}, e_t\}$ to maximize the representative household's intertemporal utility

$$\max_{\{c_t,h_t,e_t,k_{t+1},S_t\}} \sum_{t=0}^{\infty} \beta^t \left[\iota c_t^{\frac{\sigma_c-1}{\sigma_c}} + (1-\iota)\left((\chi_s) S_t^{\frac{\sigma_s-1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right)^{\frac{\sigma_s}{\sigma_s-1}\frac{\sigma_c-1}{\sigma_c}} \right]^{\frac{\sigma_c}{\sigma_c-1}}$$

subject to:

(1) the production function

$$Y_t = \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{\sigma_h}{\sigma_h-1}}$$

(2) the law of motion for capital

$$K_{t+1} = (1-\delta)K_t + I_t$$

(3) the labor constraint

$$L_t + H_t = 1$$

(4) the resource constraint

$$C_t + X_t = Y_t$$

and, (5) the environmental constraint

$$S_{t+1} = (1+\gamma)S_t - E_{dt}(\xi + \rho)$$

To simplify the problem computationally, the law of motion for capital and resource constraint can be recombined

$$K_{t+1} = (1-\delta)K_t - C_t + Y_t \equiv (1-\delta)K_t - C_t + \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{\sigma_h}{\sigma_h-1}}$$

(Notice that the labor constraint was written in the objective function already. Further, instead of optimizing over K', simply write out the standard Euler equation.) There are five endogenous variables, so four equilibrium conditions are needed. Denote the Lagrange multipliers on the two constraints as λ and μ.

To ease notation, define the following

$$\Psi_t = \left\{ \iota c_t^{\frac{\sigma_c-1}{\sigma_c}} + (1-\iota)\left[\chi_s S_t^{\frac{\sigma_s-1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right]^{\frac{\sigma_c-1}{\sigma_c}\frac{\sigma_s}{\sigma_s-1}} \right\}^{\frac{1}{\sigma_c-1}}$$

$$\frac{\partial Y}{\partial K} = MP_k = \eta_h \theta \left(\frac{H}{K} \right)^{1-\theta} \left[K_t^{\theta} H_t^{1-\theta} \right]^{-\frac{1}{\sigma_h}} \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{1}{\sigma_h-1}}$$

$$\frac{\partial Y}{\partial H} = MP_h = \eta_h (1-\theta) \left(\frac{K}{H} \right)^{\theta} \left[K_t^{\theta} H_t^{1-\theta} \right]^{-\frac{1}{\sigma_h}} \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{1}{\sigma_h-1}}$$

$$\frac{\partial Y}{\partial E} = MP_e = (1-\eta_h) E_t^{-\frac{1}{\sigma_h}} \left(\eta_h \left[K_t^{\theta} H_t^{1-\theta} \right]^{\frac{\sigma_h-1}{\sigma_h}} + (1-\eta_h) E_t^{\frac{\sigma_h-1}{\sigma_h}} \right)^{\frac{1}{\sigma_h-1}}$$

[1] I also abstract from the AR(1) shocks, technical change, and adjustment costs.

The FOCs are

$$c: \quad \beta^t c_t^{-\frac{1}{\sigma_c}} \Psi_t + \lambda_t = 0$$

$$h: \quad \beta^t (1-\iota)(1-\chi_s)(1-h_t)^{-\frac{1}{\sigma_s}} \left((\chi_s) S_t^{\frac{\sigma_s - 1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right)^{\frac{\sigma_c - \sigma_s}{\sigma_c(\sigma_s - 1)}} \Psi_t + MP_h \lambda_t = 0$$

$$k': \quad \lambda_t[(1-\delta) + MP_k] = 0$$

$$S: \quad \beta^t (1-\iota)\chi_s S_t^{-\frac{1}{\sigma_s}} \left((\chi_s) S_t^{\frac{\sigma_s - 1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right)^{\frac{\sigma_c - \sigma_s}{\sigma_c(\sigma_s - 1)}} - (1+\gamma)\mu_t = 0$$

$$E: \quad -MP_e \lambda_t + (\xi + \rho)\mu_t = 0$$

Solving this by equating Lagrange multipliers yields the following equilibrium conditions. First, the intertemporal Euler equation characterizes the trade off between consumption today and tomorrow consumption

$$c_t^{-\frac{1}{\sigma_c}} \Psi_t = \beta c_{t+1}^{-\frac{1}{\sigma_c}} \Psi_{t+1} (1 + MP_k - \delta)$$

Second, the intratemporal Euler equation characterizes the trade off between consumption and leisure

$$\frac{(1-\iota)(1-\chi_s)(1-h_t)^{-\frac{1}{\sigma_s}} \left((\chi_s) S_t^{\frac{\sigma_s - 1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right)^{\frac{\sigma_c - \sigma_s}{\sigma_c(\sigma_s - 1)}}}{\iota c_t^{-\frac{1}{\sigma_c}}} = MP_h$$

Third, the second intratemporal Euler characterizes the trade off between consumption and environmental quality (and thus necessarily leisure)

$$\frac{(1-\iota)\chi_s S_t^{-\frac{1}{\sigma_s}} \left((\chi_s) S_t^{\frac{\sigma_s - 1}{\sigma_s}} + (1-\chi_s)(1-h_t)^{\frac{\sigma_s-1}{\sigma_s}} \right)^{\frac{\sigma_c - \sigma_s}{\sigma_c(\sigma_s - 1)}}}{(1+\gamma)\iota c_t^{-\frac{1}{\sigma_c}}} = -(\xi + \rho)^{-1} MP_e$$

Finally, the last two necessary conditions are the resource and environmental constraints, respectively

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

$$S_{t+1} = (1+\gamma)S_t - E_t(\xi + \rho)$$