Online Appendix for (not for publication): The Pruned State-Space System for Non-Linear DSGE Models: Theory and Empirical Applications

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Contents

This technical appendix explans in great detail the derivations carried out in relation to our paper. In addition to the material reported in the paper, this technical appendix also provides some additional results - for instance alternative ways of computing second moments (at second and third order) and how to directly implement pruning based on the Dynare notation.

1 The class of DSGE model

We consider the class of DSGE models where the set of equilibrium conditions can be written as

$$E_t \left[\mathbf{f} \left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t \right) \right] = \mathbf{0}. \tag{1}$$

Here, E_t is the conditional expectation given information available at time t. The vector $\mathbf{x}_t$ is the set of state variables (pre-determined variables) and has dimension $n_x \times 1$. The vector $\mathbf{y}_t$ contains the set of control variables (non pre-determined variables) and has dimension $n_y \times 1$. We also let $n \equiv n_x + n_y$.

The state vector is partitioned as $\mathbf{x}_t \equiv \begin{bmatrix} \mathbf{x}_{1,t} \\ \mathbf{x}_{2,t} \end{bmatrix}$, where $\mathbf{x}_{1,t}$ with dimension $n_{x_1} \times 1$ contains the set of endogenous state variables and $\mathbf{x}_{2,t}$ with dimension $n_{x_2} \times 1$ contains the set of exogenous state variables. Note also that $n_{x_1} + n_{x_2} = n_x$.

For the exogenous state variables we assume that

$$\mathbf{x}_{2,t+1} = \mathbf{h} \left(\mathbf{x}_{2,t}, \sigma \right) + \sigma \tilde{\boldsymbol{\eta}} \boldsymbol{\epsilon}_{t+1}, \tag{2}$$

where $\boldsymbol{\epsilon}_{t+1}$ has dimension $n_e \times 1$, and thus, $\tilde{\boldsymbol{\eta}}$ has dimension $n_{x_2} \times n_e$. We assume throughout that $\boldsymbol{\epsilon}_{t+1} \sim \mathcal{IID} \left(\mathbf{0}, \mathbf{I} \right)$, that is the innovations are identical and independent distributed with mean zero and covariance matrix $\mathbf{I}$. Further moment requirements on $\boldsymbol{\epsilon}_{t+1}$ will be imposed later.

The general solution to this class of DSGE model is given by

$$\mathbf{y}_t = \mathbf{g} \left(\mathbf{x}_t, \sigma \right) \tag{3}$$

$$\mathbf{x}_{t+1} = \mathbf{h} \left(\mathbf{x}_t, \sigma \right) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \tag{4}$$

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{0} \\ \tilde{\boldsymbol{\eta}} \end{bmatrix} \tag{5}$$

where the functions $\mathbf{g} \left(\cdot, \cdot \right)$ and $\mathbf{h} \left(\cdot, \cdot \right)$ are unknown. We will therefore approximate these functions up to any desired order. This is done around the non-stochastic steady state, i.e. $\mathbf{x}_t = \mathbf{x}_{ss}$ and $\sigma = 0$. Formally, the expression for non-stochastic steady state is given as the solution of $\left(\mathbf{y}_{ss}, \mathbf{x}_{ss} \right)$ to

$$\mathbf{f} \left(\mathbf{y}_{ss}, \mathbf{y}_{ss}, \mathbf{x}_{ss}, \mathbf{x}_{ss} \right) = \mathbf{0}. \tag{6}$$

Note also that $\mathbf{x}_{ss} = \mathbf{h} \left(\mathbf{x}_{ss}, 0 \right)$ and $\mathbf{y}_{ss} = \mathbf{g} \left(\mathbf{x}_{ss}, 0 \right)$.

2 The pruning scheme:

2.1 Second order approximation

We start by partitioning the state vector using the approximated expression

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s,$$

where $\mathbf{x}_t^f$ denotes the first order terms and $\mathbf{x}_t^s$ denotes the second order terms.

A second-order approximation of the state equation reads (for $j = 1, 2, ..., n_x$)

$x_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t + \frac{1}{2}\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$
$\Updownarrow$

$$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
$\Updownarrow$

$$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}^f_t + \mathbf{h_x}(j,:)\mathbf{x}^s_t + \frac{1}{2}\left(\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:) + \left(\mathbf{x}^s_t\right)'\mathbf{h_{xx}}(j,:,:)\right)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
$\Updownarrow$

$$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}^f_t + \mathbf{h_x}(j,:)\mathbf{x}^s_t + \frac{1}{2}\left(\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^f_t + \left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^s_t + \left(\mathbf{x}^s_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^f_t + \left(\mathbf{x}^s_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^s_t\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
$\Updownarrow$

$$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}^f_t + \mathbf{h_x}(j,:)\mathbf{x}^s_t + \frac{1}{2}\left(\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^f_t + 2\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^s_t + \left(\mathbf{x}^s_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^s_t\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$
due to the symmetry of $\mathbf{h_{xx}}(j,:,:)$.

A law of motion for the first order terms is thus

$$x^f_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}^f_t + \sigma\boldsymbol{\eta}(j,:)\boldsymbol{\epsilon}_{t+1}$$

A law of motion for the second order terms is thus

$$x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}^s_t + \frac{1}{2}\left(\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\mathbf{x}^f_t\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$$

Inserting the decomposition of the state variables into the control variables we get (for $i = 1, 2, ...n_y$)
$y^s_t(i,1) = \mathbf{g_x}(i,:)\mathbf{x}_t + \frac{1}{2}\mathbf{x}_t'\mathbf{g_{xx}}(i,:,:)\mathbf{x}_t + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$
$\Updownarrow$

$$y^s_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right)'\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$
$\Updownarrow$

$$y^s_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}\left(\left(\mathbf{x}^f_t\right)'\mathbf{g_{xx}}(i,:,:) + \left(\mathbf{x}^s_t\right)'\mathbf{g_{xx}}(i,:,:)\right)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$
$\Updownarrow$

$$y^s_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}\left(\left(\mathbf{x}^f_t\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}^f_t + 2\left(\mathbf{x}^f_t\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}^s_t + \left(\mathbf{x}^s_t\right)'\mathbf{g_{xx}}(i,:,:)\mathbf{x}^s_t\right) + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$$

due to the symmetry of $\mathbf{g_{xx}}(i,:,:)$

We want to preserve terms up to second order, hence the pruned approximation is

$$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f + \frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$$

because $\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^s$ is a third order term and $\left(\mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^s$ is a fourth order term

2.2 Third order approximation

We decompose the state vector using the approximated exprresion

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd},$$

where the new term $\mathbf{x}_t^{rd}$ denotes the third order term.

A third order approximation of the state equation reads (for $j = 1, 2, ..., n_x$)

$$\begin{aligned}
&x_{t+1}(j,1) = \mathbf{h_x}(j,:)\,\mathbf{x}_t + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\
&\quad + \tfrac{1}{2}\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\
&\quad + \tfrac{1}{6}\mathbf{x}_t'\begin{bmatrix} \mathbf{x}_t'\mathbf{h_{xxx}}(j,1,:,:)\mathbf{x}_t \\ \ldots \\ \mathbf{x}_t'\mathbf{h_{xxx}}(j,n_x,:,:)\mathbf{x}_t \end{bmatrix} + \tfrac{3}{6}\mathbf{h_{\sigma\sigma x}}(j,:)\,\sigma^2\mathbf{x}_t + \tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 \\
&\Updownarrow \\
&x_{t+1}^f(j,1) + x_{t+1}^s(j,1) + x_{t+1}^{rd}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\
&\quad + \tfrac{1}{2}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\
&\quad + \tfrac{1}{6}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\begin{bmatrix} \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,1,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\ \ldots \\ \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,n_x,:,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \end{bmatrix} \\
&\quad + \tfrac{3}{6}\mathbf{h_{\sigma\sigma x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 \\
&\Updownarrow \\
&x_{t+1}^f(j,1) + x_{t+1}^s(j,1) + x_{t+1}^{rd}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\
&\quad + \tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:) + \left(\mathbf{x}_t^{rd}\right)'\mathbf{h_{xx}}(j,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\
&\quad + \tfrac{1}{6}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)' \times \\
&\qquad \begin{bmatrix} \left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xxx}}(j,1,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xxx}}(j,1,:,:) + \left(\mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,1,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\ \ldots \\ \left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xxx}}(j,n_x,:,:) + \left(\mathbf{x}_t^s\right)'\mathbf{h_{xxx}}(j,n_x,:,:) + \left(\mathbf{x}_t^{rd}\right)'\mathbf{h_{xxx}}(j,n_x,:,:)\right)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \end{bmatrix} \\
&\quad + \tfrac{3}{6}\mathbf{h_{\sigma\sigma x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 \\
&\Updownarrow \\
&x_{t+1}^f(j,1) + x_{t+1}^s(j,1) + x_{t+1}^{rd}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \\
&\quad + \tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^f + \left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^s + \left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)
\end{aligned}$$

$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$
$$+\tfrac{1}{6}\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)'\times$$
$$\begin{bmatrix}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\ \dots \\ \left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\end{bmatrix}$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

$\Updownarrow$

$$x_{t+1}^f(j,1)+x_{t+1}^s(j,1)+x_{t+1}^{rd}(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1}$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$
$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

due to symmetry in $\mathbf{h}_{\mathbf{xx}}(j,:,:)$

$\Updownarrow$

$$x_{t+1}^f(j,1)+x_{t+1}^s(j,1)+x_{t+1}^{rd}(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1}$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$
$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$$
$$\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$$
$$\times\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$$
$$\times\left(\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

$\Updownarrow$

$$x_{t+1}^f(j,1)+x_{t+1}^s(j,1)+x_{t+1}^{rd}(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1}$$

$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$$
$$\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+2\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)$$
$$\times\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^s+2\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

due to symmetries in $\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)$

A law of motion for $x_t^f(j,1)$ is then (as before)

$$x_{t+1}^f(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^f+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1}$$

because we only keep first order terms

A law of motion for $x_t^s(j,1)$ is then (as before)

$$x_{t+1}^s(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^s+\frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^f\right)+\frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$$

because we only keep second order terms.

A law of motion for $x_{t+1}^{rd}(j,1)$ is then

$$\begin{aligned}x_{t+1}^{rd}(j,1) &= \mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^{rd}+\frac{2}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\,\mathbf{x}_t^s+\frac{1}{6}\sum_{\gamma=1}^{n_x}x_t^f(\gamma,1)\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,\gamma,:,:)\mathbf{x}_t^f \\ &\quad+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t^f+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3\end{aligned}$$

Note that σ^2 is in perturbation a variable and $\sigma^2\mathbf{x}_t^f$ is therefore a third order effect.

Inserting the decomposition of the state variables into the control variables we get (for $i=1,2,...n_y$)

$$y_t^{rd}(i,1)=\mathbf{g}_{\mathbf{x}}(i,:)\,\mathbf{x}_t+\tfrac{1}{2}\mathbf{x}_t'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$$
$$+\tfrac{1}{6}\mathbf{x}_t'\begin{bmatrix}\mathbf{x}_t'\mathbf{g}_{\mathbf{xxx}}(i,1,:,:)\mathbf{x}_t\\ \ldots \\ \mathbf{x}_t'\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:)\mathbf{x}_t\end{bmatrix}+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3$$

$\Updownarrow$

$$y_t^{rd}(i,1)=\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{2}\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$$
$$+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3$$

$\Updownarrow$

$$y_t^{rd}(i,1)=\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\right)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)$$
$$+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2$$

$$
\begin{aligned}
&+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\\
&\qquad\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\right)\\
&\qquad\times\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
y_t^{rd}(i,1)=&\;\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2\\
&+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\\
&\qquad\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\right)\\
&\qquad\times\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}
$$

$\Updownarrow$

$$
\begin{aligned}
y_t^{rd}(i,1)=&\;\mathbf{g}_{\mathbf{x}}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^f\right)\\
&+\tfrac{1}{2}\left(\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+2\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^s+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xx}}(i,:,:)\,\mathbf{x}_t^{rd}\right)\\
&+\tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2\\
&+\tfrac{1}{6}\textstyle\sum_{\gamma=1}^{n_x}\left(x_t^f(\gamma,1)+x_t^s(\gamma,1)+x_t^{rd}(\gamma,1)\right)\\
&\qquad\times\left(\left(\mathbf{x}_t^f\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^s\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)+\left(\mathbf{x}_t^{rd}\right)'\mathbf{g}_{\mathbf{xxx}}(i,\gamma,:,:)\right)\\
&\qquad\times\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)+\tfrac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}
$$

We want to preserve terms up to third order, hence the pruned approximation is

$$\begin{aligned}
y_t^{rd}(i,1) &= \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\right)\\
&+\frac{1}{2}\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f+2\left(\mathbf{x}_t^s\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t^f\right)\\
&+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2\\
&+\frac{1}{6}\sum_{\gamma=1}^{n_x}x_t^f(\gamma,1)\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,\gamma,:,:)\mathbf{x}_t^f\right)\\
&+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t^f\\
&+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3
\end{aligned}$$

$i=1,2,...n_y$

2.3 Summary: NO pruning up to third order

The approximation of the state variables ($\mathbf{x}_t$) is here

$$\begin{aligned}
x_{t+1}(j,1) &= \mathbf{h_x}(j,:)\,\mathbf{x}_t+\frac{1}{2}\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t && (7)\\
&+\frac{1}{6}\mathbf{x}_t'\begin{bmatrix}\mathbf{x}_t'\mathbf{h_{xxx}}(j,1,:,:)\mathbf{x}_t\\ ...\\ \mathbf{x}_t'\mathbf{h_{xxx}}(j,n_x,:,:)\mathbf{x}_t\end{bmatrix}\\
&+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t+\frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2\\
&+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3+\sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} && (8)
\end{aligned}$$

for $j=1,2,...,n_x$.

The approximation of the control variables ($\mathbf{y}_t$) is

$$\begin{aligned}
y_t(i,1) &= \mathbf{g_x}(i,:)\,\mathbf{x}_t && (9)\\
&+\frac{1}{2}\mathbf{x}_t'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_t\\
&+\frac{1}{6}\mathbf{x}_t'\begin{bmatrix}\mathbf{x}_t'\mathbf{g_{xxx}}(i,1,:,:)\mathbf{x}_t\\ ...\\ \mathbf{x}_t'\mathbf{g_{xxx}}(i,n_x,:,:)\mathbf{x}_t\end{bmatrix}\\
&+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2\\
&+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3 && (10)
\end{aligned}$$

$i=1,2,...n_y$

2.4 Summary: pruning up to third order

The approximation of the state variables is

$$\mathbf{x}_{t+1}^{f} = \mathbf{h_x x}_t^f + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \tag{11}$$

$$x_{t+1}^{s}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}_t^s + \frac{1}{2}\left(\mathbf{x}_t^f\right)' \mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f\right) + \frac{1}{2} h_{\sigma\sigma}(j,1)\sigma^2 \tag{12}$$

$$\begin{aligned} x_{t+1}^{rd}(j,1) &= \mathbf{h_x}(j,:)\mathbf{x}_t^{rd} + \frac{2}{2}\left(\mathbf{x}_t^f\right)' \mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^s\right) \\ &+ \frac{1}{6}\left(\mathbf{x}_t^f\right)' \begin{bmatrix} \left(\mathbf{x}_t^f\right)' \mathbf{h_{xxx}}(j,1,:,:)\left(\mathbf{x}_t^f\right) \\ \dots \\ \left(\mathbf{x}_t^f\right)' \mathbf{h_{xxx}}(j,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix} \\ &+ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\sigma^2 \mathbf{x}_t^f + \frac{1}{6} h_{\sigma\sigma\sigma}(j,1)\sigma^3 \end{aligned} \tag{13}$$

$$\mathbf{x}_{t+1} = \mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s + \mathbf{x}_{t+1}^{rd} \tag{14}$$

for $j = 1, 2, ..., n_x$.

The approximation of the control variables ($\mathbf{y}_t$) is

$$\begin{aligned} y_t^{rd}(i,1) &= \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \\ &+ \frac{1}{2}\left(\mathbf{x}_t^f\right)' \mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_t^f + 2\mathbf{x}_t^s\right) \\ &+ \frac{1}{6}\left(\mathbf{x}_t^f\right)' \begin{bmatrix} \left(\mathbf{x}_t^f\right)' \mathbf{g_{xxx}}(i,1,:,:)\left(\mathbf{x}_t^f\right) \\ \dots \\ \left(\mathbf{x}_t^f\right)' \mathbf{g_{xxx}}(i,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix} \\ &+ \frac{1}{2} g_{\sigma\sigma}(i,1)\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\sigma^2 \mathbf{x}_t^f \\ &+ \frac{1}{6} g_{\sigma\sigma\sigma}(i,1)\sigma^3 \end{aligned} \tag{15}$$

for $i = 1, ..., n_y$.

2.5 Increasing efficiency for the simulation in FORTRAN

When simulating the pruned state space system, the efficency can be improved by re-expressing some of the sums in the matrices and by using some of the symmetry in the second and third order terms (due to Young's theorem). This is useful in FORTRAN because we can reduce the number of summations. However, in MATLAB, this trick does not work as it induces more loops.

First, to re-express some of the summations implied by the matrix notation, recall the following rules for the vec and kronecker operators:

1. $vec(\mathbf{A} + \mathbf{B}) = vec(\mathbf{A}) + vec(\mathbf{B})$
2. $\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n_x}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n_x}\mathbf{B} \\ \dots & \dots & \dots & \dots \\ a_{n_x1}\mathbf{B} & a_{n_x2}\mathbf{B} & \dots & a_{n_xn_x}\mathbf{B} \end{bmatrix}$

3. $vec\left(\mathbf{ABC}\right) = \left(\mathbf{C}' \otimes \mathbf{A}\right) vec\left(\mathbf{B}\right)$ hence $\mathbf{x}_t'\mathbf{A}\mathbf{x}_t = vec\left(\mathbf{x}_t'\mathbf{A}\mathbf{x}_t\right) = \left(\mathbf{x}_t' \otimes \mathbf{x}_t'\right) vec\left(\mathbf{A}\right)$

4. $\left(\mathbf{A} \otimes \mathbf{B}\right)' = \left(\mathbf{A}' \otimes \mathbf{B}'\right)$ and hence $vec\left(\mathbf{A} \otimes \mathbf{B}\right)' = vec\left(\mathbf{A}' \otimes \mathbf{B}'\right)$

5. $\left(\mathbf{A} \otimes \mathbf{B}\right)\left(\mathbf{C} \otimes \mathbf{D}\right) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

6. $\left(\mathbf{A} + \mathbf{B}\right) \otimes \left(\mathbf{C} + \mathbf{D}\right) = \mathbf{A} \otimes \mathbf{C} + \mathbf{A} \otimes \mathbf{D} + \mathbf{B} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{D}$ if $\mathbf{A} + \mathbf{B}$ and $\mathbf{C} + \mathbf{D}$ are defined

7. $\left[\mathbf{x}_t' \otimes \mathbf{x}_t'\right] = vec\left(\left[\mathbf{x}_t\mathbf{x}_t'\right]\right)' \iff \left[\mathbf{x}_t \otimes \mathbf{x}_t\right] = vec\left(\left[\mathbf{x}_t\mathbf{x}_t'\right]\right)$

where $\mathbf{x}_t$ has dimension $n_x \times 1$ and $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ have dimension $n_x \times n_x$. Hence, we may also write the terms of the form $\mathbf{x}_t'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t$ in the following way

$$\begin{aligned}
\mathbf{x}_t'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t &= \left(\mathbf{x}_t' \otimes \mathbf{x}_t'\right) vec\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right) \\
&= vec\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)'\left(\mathbf{x}_t \otimes \mathbf{x}_t\right) \\
&= vec\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vec\left(\left[\mathbf{x}_t\mathbf{x}_t'\right]\right)
\end{aligned}$$

To exploit the symmetry in the second and third order terms, we use the *vech*-operator which stacks all elements of a matrix on or below the diagonal. For instance, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $vech\left(A\right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{22} \\ a_{32} \\ a_{33} \end{bmatrix}$.

It then holds that

$$vec\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vec\left(\left[\mathbf{x}_t\mathbf{x}_t'\right]\right) = vech\left(2\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right) - diag\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)\right)' vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \tag{16}$$

Here, $diag\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)$ is an $n_x \times n_x$ with zeros except at the diagonal where the matrix has the diagonal elements of $\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)$ for $i = 1, ..., n_x$. To realize the validity of the expression in (16), consider

$$\begin{aligned}
& vec\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vec\left(\left[\mathbf{x}_t\mathbf{x}_t'\right]\right) \\
&= \mathbf{x}_t'\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\mathbf{x}_t \\
&= \textstyle\sum_{h=1}^{n_x}\sum_{k=1}^{n_x}\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)x_t\left(h\right)x_t\left(k\right) \\
&= \textstyle\sum_{h=1}^{n_x}\mathbf{h}_{\mathbf{xx}}\left(j,h,h\right)x_t\left(h\right)^2 + 2\sum_{h=1}^{n_x}\sum_{k=h+1}^{n_x}\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)x_t\left(h\right)x_t\left(k\right) \\
&= \mathbf{x}_t' diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\mathbf{x}_t + 2\textstyle\sum_{h=1}^{n_x}\sum_{k=h+1}^{n_x}\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)x_t\left(h\right)x_t\left(k\right) \\
&= \mathbf{x}_t' diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\mathbf{x}_t + 2\left[vech\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vech\left(\mathbf{x}_t\mathbf{x}_t'\right) - \mathbf{x}_t' diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\mathbf{x}_t\right] \\
&= 2vech\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vech\left(\mathbf{x}_t\mathbf{x}_t'\right) - \mathbf{x}_t' diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\mathbf{x}_t \\
&= 2vech\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vech\left(\mathbf{x}_t\mathbf{x}_t'\right) - vec\left(diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\right)' vec\left(\mathbf{x}_t\mathbf{x}_t'\right) \\
&= 2vech\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right)' vech\left(\mathbf{x}_t\mathbf{x}_t'\right) - vech\left(diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\right)' vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \\
&= \left[2vech\left(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right)\right) - vech\left(diag(\mathbf{h}_{\mathbf{xx}}\left(j,:,:\right))\right)\right]' vech\left(\mathbf{x}_t\mathbf{x}_t'\right)
\end{aligned}$$

WITHOUT PRUNING:

For the state variables in (7), we have

$$\begin{aligned} x_{t+1}(j,1) &= \mathbf{h_x}(j,:)\,\mathbf{x}_t + \hat{\mathbf{H}}_{\mathbf{xx}}(j,:)\,vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \\ &+\mathbf{x}_t' \begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{xxx}}(j,1,:)\,vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \\ ... \\ \hat{\mathbf{H}}_{\mathbf{xxx}}(j,n_x,:)\,vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \end{bmatrix} \\ &+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t + \frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \\ &+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3 + \sigma\boldsymbol{\eta}(j,:)\,\boldsymbol{\epsilon}_{t+1} \end{aligned} \tag{17}$$

for $j = 1, 2, ..., n_x$ where we define

$$\hat{\mathbf{H}}_{\mathbf{xx}}(1:n_x, 1:n_x(n_x+1)/2) = \frac{1}{2}\begin{bmatrix} vech\left(2\mathbf{h}_{\mathbf{xx}}(1,:,:) - diag\left(\mathbf{h}_{\mathbf{xx}}(1,:,:)\right)\right)' \\ ... \\ vech\left(2\mathbf{h}_{\mathbf{xx}}(n_x,:,:) - diag\left(\mathbf{h}_{\mathbf{xx}}(n_x,:,:)\right)\right)' \end{bmatrix} \tag{18}$$

$$\hat{\mathbf{H}}_{\mathbf{xxx}}(j,1:n_x, 1:n_x(n_x+1)/2) = \frac{1}{6}\begin{bmatrix} vech\left(2\mathbf{h}_{\mathbf{xxx}}(j,1,:,:) - diag\left(\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)\right)\right)' \\ ... \\ vech\left(2\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:) - diag\left(\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)\right)\right)' \end{bmatrix} \tag{19}$$

for $j = 1, 2, ..., n_x$. The advantage of this formulation compared to one which use all the symmetry in the third order terms is simply that we only need to compute $vech\left(\mathbf{x}_t\mathbf{x}_t'\right)$ once.

For the control variables in (9), we have

$$\begin{aligned} y_t(i,1) &= \mathbf{g_x}(i,:)\,\mathbf{x}_t \\ &+\hat{\mathbf{G}}_{\mathbf{xx}}(i,:)\,vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \\ &+\mathbf{x}_t' \begin{bmatrix} \hat{\mathbf{G}}_{\mathbf{xxx}}(i,1,:)\,vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \\ ... \\ \hat{\mathbf{G}}_{\mathbf{xxx}}(i,n_x,:)\,vech\left(\mathbf{x}_t\mathbf{x}_t'\right) \end{bmatrix} \\ &+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}_t \\ &+\frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3 \end{aligned} \tag{20}$$

for $i = 1, 2, ..., n_y$ where we define

$$\hat{\mathbf{G}}_{\mathbf{xx}}(1:n_y, 1:n_x(n_x+1)/2) = \frac{1}{2}\begin{bmatrix} vech\left(2\mathbf{g}_{\mathbf{xx}}(1,:,:) - diag\left(\mathbf{g}_{\mathbf{xx}}(1,:,:)\right)\right)' \\ ... \\ vech\left(2\mathbf{g}_{\mathbf{xx}}(n_y,:,:) - diag\left(\mathbf{g}_{\mathbf{xx}}(n_y,:,:)\right)\right)' \end{bmatrix} \tag{21}$$

$$\hat{\mathbf{G}}_{\mathbf{xxx}}(i,1:n_x, 1:n_x(n_x+1)/2) = \frac{1}{6}\begin{bmatrix} vech\left(2\mathbf{g}_{\mathbf{xxx}}(i,1,:,:) - diag\left(\mathbf{g}_{\mathbf{xxx}}(i,1,:,:)\right)\right)' \\ ... \\ vech\left(2\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) - diag\left(\mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:)\right)\right)' \end{bmatrix} \tag{22}$$

for $i = 1, 2, ..., n_y$.

WITH PRUNING:

For the state variables in (12) and (13), we have

$$x^s_{t+1}(j,1) = \mathbf{h_x}\mathbf{x}^s_t + \hat{\mathbf{H}}_{\mathbf{xx}}(j,:)\,vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\right)'\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2 \tag{23}$$

$$x^{rd}_{t+1}(j,1) = \mathbf{h_x}\mathbf{x}^{rd}_t + \hat{\mathbf{H}}_{\mathbf{xx}}(j,:)\left(vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^s_t\right)'\right) + vech\left(\left(\mathbf{x}^s_t\right)\left(\mathbf{x}^f_t\right)'\right)\right) \tag{24}$$

$$+\left(\mathbf{x}^f_t\right)'\begin{bmatrix} \hat{\mathbf{H}}_{\mathbf{xxx}}(j,1,:)\,vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\right)'\right) \\ ... \\ \hat{\mathbf{H}}_{\mathbf{xxx}}(j,n_x,:)\,vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\right)'\right)\end{bmatrix}$$

$$+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}^f_t + \frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

For the control variables in (15), we have

$$y^{rd}_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t + \mathbf{x}^{rd}_t\right) \tag{25}$$

$$+\hat{\mathbf{G}}_{\mathbf{xx}}(i,:)\left[vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\right)'\right) + vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^s_t\right)'\right) + vech\left(\left(\mathbf{x}^s_t\right)\left(\mathbf{x}^f_t\right)'\right)\right]$$

$$+\left(\mathbf{x}^f_t\right)'\begin{bmatrix} \hat{\mathbf{G}}_{\mathbf{xxx}}(i,1,:)\,vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\right)'\right) \\ ... \\ \hat{\mathbf{G}}_{\mathbf{xxx}}(i,n_x,:)\,vech\left(\left(\mathbf{x}^f_t\right)\left(\mathbf{x}^f_t\right)'\right)\end{bmatrix}$$

$$+\frac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\,\sigma^2\mathbf{x}^f_t + \frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\,\sigma^3$$

2.6 Increasing efficiency for the simulation in MATLAB

In MATLAB the most important thing is to avoid for-loops. We therefore provide a representation based on the kronecker product which does not require any loops. Even without using the symmetry in the non-linear terms, this greatly increases the execution speed in MATLAB. Note first that

$\mathbf{x}_t'\mathbf{h_{xx}}(j,:,:)\mathbf{x}_t = reshape(\mathbf{h_{xx}}, n_x, n_x^2)\,(\mathbf{x}_t \otimes \mathbf{x}_t)$

where

$$reshape(\mathbf{h_{xx}}, n_x, n_x^2) = \begin{bmatrix} \mathbf{h_{xx}}(1,1:n_x,1)' & \mathbf{h_{xx}}(1,1:n_x,2)' & ... & \mathbf{h_{xx}}(1,1:n_x,n_x)' \\ \mathbf{h_{xx}}(2,1:n_x,1)' & \mathbf{h_{xx}}(2,1:n_x,2)' & ... & \mathbf{h_{xx}}(1,1:n_x,n_x)' \\ ... & ... & ... & ... \\ \mathbf{h_{xx}}(n_x,1:n_x,1)' & \mathbf{h_{xx}}(n_x,1:n_x,2)' & ... & \mathbf{h_{xx}}(n_x,1:n_x,n_x)' \end{bmatrix}$$

And for the third order terms:

$$\mathbf{x}_t'\begin{bmatrix} \mathbf{x}_t'\mathbf{h_{xxx}}(j,1,:,:)\mathbf{x}_t \\ ... \\ \mathbf{x}_t'\mathbf{h_{xxx}}(j,n_x,:,:)\mathbf{x}_t \end{bmatrix} = \sum_{j_1=1}^{n_x} x_t(j_1,1)\,\mathbf{x}_t'\mathbf{h_{xxx}}(j,j_1,:,:)\mathbf{x}_t$$

$$= \sum_{j_1=1}^{n_x}\sum_{j_2=1}^{n_x}\sum_{j_3=1}^{n_x} x_t(j_1,1)\,x_t(j_2,1)\,x_t(j_3,1)\,\mathbf{h_{xxx}}(j,j_1,j_2,j_3)$$

$$= reshape(\mathbf{h_{xxx}}, n_x, n_x^3)\,(\mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t)$$

WITHOUT PRUNING:

For the state variables in (7), we have

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{h_x}\mathbf{x}_t + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t\right) + \tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t\right) \\ &+ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\end{aligned} \tag{26}$$

where we define

$$\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{h}_{\mathbf{xx}}, n_x, n_x^2\right) \tag{27}$$

$$\tilde{\mathbf{H}}_{\mathbf{xxx}} \equiv \frac{1}{6} reshape\left(\mathbf{h}_{\mathbf{xxx}}, n_x, n_x^3\right) \tag{28}$$

For the control variables in (9), we have

$$\mathbf{y}_t = \mathbf{g_x}\mathbf{x}_t + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}}\left(\mathbf{x}_t \otimes \mathbf{x}_t \otimes \mathbf{x}_t\right) \tag{29}$$

$$+ \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3 \tag{30}$$

where we define

$$\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{xx}}, n_y, n_x^2\right) \tag{31}$$

$$\tilde{\mathbf{G}}_{\mathbf{xxx}} \equiv \frac{1}{6} reshape\left(\mathbf{g}_{\mathbf{xxx}}, n_y, n_x^3\right) \tag{32}$$

WITH PRUNING:

For the state variables in (12) and (13), we have

$$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \tag{33}$$

$$\begin{aligned}\mathbf{x}_{t+1}^{rd} &= \mathbf{h_x}\mathbf{x}_t^{rd} + 2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\ &+ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\end{aligned} \tag{34}$$

For the control variables in (15), we have

$$\begin{aligned}\mathbf{y}_t^{rd} &= \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + 2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\ &+ \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3\end{aligned} \tag{35}$$

3 Stastical properties: Second-order approximation

3.1 Covariance-stationary

Proposition 1:

The pruned second-order approximation for $\mathbf{x}_t^f, \mathbf{x}_t^s$, and $\mathbf{y}_t^s$ is covariance-stationary if

1. the DSGE model has a unique stable equilibrium, i.e. all eigenvalue of $\mathbf{h_x}$ have modulus less than one
2. $\boldsymbol{\epsilon}_{t+1}$ has finite fourth moment

Proof

Note first that

$x_{t+1}^s(j,1) = \mathbf{h_x}(j,:)\,\mathbf{x}_t^s + \frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f\right) + \frac{1}{2}h_{\sigma\sigma}(j,1)\,\sigma^2$

$\Updownarrow$

$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}} vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$

where $\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{h_{xx}}, n_x, n_x^2\right)$

$\Updownarrow$

$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$

because $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) = vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right)$

We now form the extended state vector

$$\mathbf{z}_t \equiv \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$

We know the law of motion for $\mathbf{x}_t^f$ and $\mathbf{x}_t^s$, so we only need to find the law of motion for $\mathbf{x}_t^f \otimes \mathbf{x}_t^f$. Hence consider

$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$

$= \mathbf{h_x}\mathbf{x}_t^f \otimes \mathbf{h_x}\mathbf{x}_t^f + \mathbf{h_x}\mathbf{x}_t^f \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$

using $(\mathbf{A}+\mathbf{B}) \otimes (\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}$

$= (\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$
$+ (\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD}$

$= (\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$
$+ (\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}) - vec(\mathbf{I}_{n_e})\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})$

Note that $E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\right] = vec(\mathbf{I}_{n_e})$. Thus

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{h_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n_x\times 1} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta})\,vec(\mathbf{I}_{n_e}) \end{bmatrix}$$

$$+\begin{bmatrix} \sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}) & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \end{bmatrix}$$

$\Updownarrow$

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{c} + \mathbf{B}\boldsymbol{\xi}_{t+1} \tag{36}$$

where $Cov\left(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_{t-s}\right) = \mathbf{0}$ for $s = 1, 2, 3, ...$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time

The absolute value of the eigenvalues in $\mathbf{h_x}$ are all strictly less then one by assumption. Accordingly, all eigenvalues of $\mathbf{A}$ are also strictly less than one. To see this note first that

$p(\lambda) = \left|\mathbf{A} - \lambda\mathbf{I}_{2n_x+n_x^2}\right|$

$$= \left|\begin{bmatrix} \mathbf{h_x} - \lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x} - \lambda\mathbf{I}_{n_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}_{n_x^2} \end{bmatrix}\right|$$

$$= \begin{vmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{vmatrix}$$

where we let

$\mathbf{B}_{11} \equiv \begin{bmatrix} \mathbf{h_x} - \lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x} - \lambda\mathbf{I}_{n_x} \end{bmatrix}$ which is $2n_x \times 2n_x$

$\mathbf{B}_{12} \equiv \begin{bmatrix} \mathbf{0}_{n_x\times n_x^2} \\ \tilde{\mathbf{H}}_{\mathbf{xx}} \end{bmatrix}$ which is $2n_x \times n_x^2$

$\mathbf{B}_{21} \equiv \begin{bmatrix} \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} \end{bmatrix}$ which is $n_x^2 \times 2n_x$

$\mathbf{B}_{22} \equiv \mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}_{n_x^2}$ which is $n_x^2 \times n_x^2$

$$= |\mathbf{B}_{11}|\,|\mathbf{B}_{22}|$$

using $\begin{vmatrix} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{vmatrix} = |\mathbf{U}|\,|\mathbf{Y}|$ where $\mathbf{U}$ is $m \times m$ and $\mathbf{Y}$ is $n \times n$

$$= \left|\begin{bmatrix} \mathbf{h_x} - \lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x} - \lambda\mathbf{I}_{n_x} \end{bmatrix}\right| \left|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}_{n_x^2}\right|$$

$$= \left|\mathbf{h_x} - \lambda\mathbf{I}_{n_x}\right| \left|\mathbf{h_x} - \lambda\mathbf{I}_{n_x}\right| \left|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}_{n_x^2}\right|$$

Hence, the eigenvalue λ solves the problem

$p(\lambda) = 0$

$\Updownarrow$

$\left|\mathbf{h_x} - \lambda\mathbf{I}_{n_x}\right| \left|\mathbf{h_x} - \lambda\mathbf{I}_{n_x}\right| \left|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}_{n_x^2}\right| = 0$

$\Updownarrow$

$\left|\mathbf{h_x} - \lambda\mathbf{I}_{n_x}\right| = 0$ or $\left|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}_{n_x^2}\right| = 0$

The absolute value of all eigenvalues to the first problem are strictly less than one. That is $|\lambda_i| < 1$ $i = 1, 2, ..., n_x$. This is also the case for the second problem because the eigenvalues to $\mathbf{h_x}\otimes\mathbf{h_x}$ are $\lambda_i\lambda_j$ for $i = 1, 2, ..., n_x$ and $j = 1, 2, ..., n_x$

Thus, the system in (36) is covariance stationary if $\boldsymbol{\xi}_{t+1}$ has finite first and second moment. It follows directly that $E\left[\boldsymbol{\xi}_{t+1}\right] = \mathbf{0}$ and $\boldsymbol{\xi}_{t+1}$ has finite second moments if $\boldsymbol{\epsilon}_{t+1}$ has a finite fourth moment. The latter holds by assumption.

For the control variables we have

$y_t^s(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\left(\mathbf{x}_t^f\right)' \mathbf{g_{xx}}(i,:,:)\mathbf{x}_t^f + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$
$\Updownarrow$

$\mathbf{y}_t^s = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
where $\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{g_{xx}}, n_y, n_x^2\right)$
$\Updownarrow$

$$\mathbf{y}_t^s = \left[\begin{array}{ccc} \mathbf{g_x} & \mathbf{g_x} & \tilde{\mathbf{G}}_{\mathbf{xx}} \end{array}\right] \left[\begin{array}{c} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{array}\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$
$\Updownarrow$

$\mathbf{y}_t^s = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
where $\mathbf{D} \equiv \left[\begin{array}{ccc} \mathbf{g_x} & \mathbf{g_x} & \tilde{\mathbf{G}}_{\mathbf{xx}} \end{array}\right]$
That is $\mathbf{y}_t$ is linear function of $\mathbf{z}_t$ and $\mathbf{y}_t$ is therefore also covariance-stationary.

Q.E.D.

3.2 Method 1: Formulas for the first and second moments

This section computes first and second moments using the representation of the second-order system stated above. This method is fairly direct but has the computational disadvantage of requiring a lot of memory because we work directly with the big $\mathbf{B}$ matrix.

The system

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$
$$\mathbf{y}_t^s = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

The mean values are

$$E[\mathbf{z}_t] = \left(\mathbf{I}_{2n_x+n_x^2} - \mathbf{A}\right)^{-1}\mathbf{c}.$$
$$E[\mathbf{y}_t] = \mathbf{D}E[\mathbf{z}_t] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

For the variances we first have that

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$$

$$= E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$= E\left[\mathbf{c}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$

$$= \mathbf{c}\mathbf{c}' + \mathbf{c}E[\mathbf{z}_t']\mathbf{A}'$$
$$+\mathbf{A}E[\mathbf{z}_t]\mathbf{c}' + \mathbf{A}E[\mathbf{z}_t\mathbf{z}_t']\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

We then note that

$$E\left[\mathbf{z}_t\boldsymbol{\xi}'_{t+1}\right] = E\left[\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{array}\right]\left[\begin{array}{cccc}\boldsymbol{\epsilon}'_{t+1} & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]\right]$$

$$= E\left[\begin{array}{cccc}\mathbf{x}_t^f\boldsymbol{\epsilon}'_{t+1} & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \mathbf{x}_t^s\boldsymbol{\epsilon}'_{t+1} & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}'_{t+1} & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\end{array}\right]$$

$$= \left[\begin{array}{cccc}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]$$

Thus

$$E\left[\mathbf{z}_{t+1}\mathbf{z}'_{t+1}\right] = \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}'_t\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}'_t\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}'_{t+1}\right]\mathbf{B}'$$

$$= \mathbf{c}E\left[\mathbf{z}'_t\right]\mathbf{A}' + \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}'_t\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}'_{t+1}\right]\mathbf{B}'$$

Note also that

$$E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)'$$

$$= \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)E\left[\mathbf{z}'_t\right]\mathbf{A}'$$

$$= \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}'_t\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}'_t\right]\mathbf{A}'$$

So

$$E\left[\mathbf{z}_{t+1}\mathbf{z}'_{t+1}\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \mathbf{c}E\left[\mathbf{z}'_t\right]\mathbf{A}' + \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}'_t\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}'_{t+1}\right]\mathbf{B}' - \left(\mathbf{c}+\mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' - \mathbf{c}E\left[\mathbf{z}'_t\right]\mathbf{A}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}'_t\right]\mathbf{A}'$$

$$= \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}'_t\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}'_{t+1}\right]\mathbf{B}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}'_t\right]\mathbf{A}'$$

$$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}'_t\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}'_t\right]\right)\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}'_{t+1}\right]\mathbf{B}'$$

$\Updownarrow$

$$Var\left(\mathbf{z}_{t+1}\right) = \mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}' + \mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = vec\left(\mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}'\right) + vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{A}\otimes\mathbf{A}\right)vec\left(Var\left(\mathbf{z}_t\right)\right) + vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)\left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A}\otimes\mathbf{A}\right)\right) = vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A}\otimes\mathbf{A}\right)\right)^{-1}vec\left(\mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'\right)$$

Hence we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$.

$$Var\left(\boldsymbol{\xi}_{t+1}\right) = E\left[\begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \end{bmatrix}\begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \end{bmatrix}'\right]$$

$$= E\left[\begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \end{bmatrix} \quad \boldsymbol{\epsilon}_{t+1}' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' \quad \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' \quad \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\begin{bmatrix} \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}_{t+1}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I}_{n_e} & E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] & \mathbf{0} & \mathbf{0} \\ E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right] & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] \\ \mathbf{0} & \mathbf{0} & E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] \end{bmatrix}$$

All elements in this matrix can be computed (and coded) directly as shown below. The variance of the control variables is then given by

$$Var\left[\mathbf{y}_t^s\right] = \mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

3.2.1 Computing the variance of the innovations

1) for $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

```
E_eps_eps2 = zeros(ne, (ne)^2)
for phi1 = 1 : ne
    index2 = 0
    for phi2 = 1 : ne
```

$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ (phi1 = phi2 = phi3)$
$E\_eps\_eps2(phi1, index2) = m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end

Note also that $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right] = \left(E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\right)'$

2) $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right]$
Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right]$

$$= E\left[\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$
$$+E\left[-vec\left(\mathbf{I}_{n_e}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$

$$= E\left[\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\right]$$
$$+E\left[-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' + vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right]$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] - vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'$$
$$-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)' + vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] - vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'$$

Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{\epsilon_{t+1}\left(\phi_4,1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are
$E\_eps2\_eps2 = zeros(n_e^2, n_e^2)$
$index1 = 0$
$for\ phi1 = 1 : n_e$
$for\ phi2 = 1 : n_e$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi3 = 1 : n_e$
$for\ phi4 = 1 : n_e$
$index2 = index2 + 1$
% second moments
$if\ (phi1 == phi2\ \&\&\ phi3 == phi4\ \&\&\ phi1\tilde{} = phi4)$
$E\_eps2\_eps2(index1, index2) = 1$
$elseif\ (phi1 == phi3\ \&\&\ phi2 == phi4\ \&\&\ phi1\tilde{} = phi2)$
$E\_eps2\_eps2(index1, index2) = 1$
$elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{} = phi2)$
$E\_eps2\_eps2(index1, index2) = 1$

$\%$ fourth moments
$elseif(phi1 == phi2 \;\&\&\; phi1 == phi3 \;\&\&\; phi1 == phi4)$
$E\_eps2\_eps2(index1, index2) = m^4 \left(\epsilon_{t+1} \left(phi1\right)\right)$
end
end
end
end
end

3) $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$
Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right]$$

Hence the quasi MATLAB codes are
$E\_epsxf\_epsxf = zeros(n_e n_x, n_x n_e)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ gama1 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 = phi2$
$E\_epsxf\_epsxf\left(index1, index2\right) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

4) $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$
Here
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\left\{x_t^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Hence the quasi MATLAB codes are
$E\_epsxf\_xfeps = zeros(n_e n_x, n_e n_x)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ gama1 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$

$for\ gama2 = 1 : nx$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 = phi2$
$E\_epsxf\_xfeps\,(index1, index2) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end

5) $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$
Here
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] = \left[E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]\right]'$

so $E\_xfeps\_epsxf = E\_epsxf\_xfeps'$

6) $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$
Here
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1} \otimes \left(\mathbf{x}_t^f\right)'\right)\right]$

$$= E\left[\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are
$E\_xfeps\_epsxf = zeros(n_x n_e, n_x n_e)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 = phi2$
$E\_xfeps\_epsxf\,(index1, index2) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

3.3 Method 2: Formulas for the first and second moments

This section computes first and second moments using a slightly different representation of the second-order system than stated above. (Basically, this was the first representation we considered for computing these moments). The advantage of this method is that it compared to Method 1 is less memory intensive because some of the matrix multiplications are done by hand.

We start by deriving an alternative representation of the pruned state space system (the old representation). Hence consider

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h_x x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)$$

$$= \mathbf{h_x x}_t^f \otimes \mathbf{h_x x}_t^f + \mathbf{h_x x}_t^f \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

using $(\mathbf{A}+\mathbf{B}) \otimes (\mathbf{C}+\mathbf{D}) = \mathbf{A}\otimes\mathbf{C} + \mathbf{A}\otimes\mathbf{D} + \mathbf{B}\otimes\mathbf{C} + \mathbf{B}\otimes\mathbf{D}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = \mathbf{AC}\otimes\mathbf{BD}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x})\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{v}(t+1)$$

where

$$\mathbf{v}(t+1) = (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

Note that $E[\mathbf{v}(t+1)] = (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e})$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time and therefore also independent of $\mathbf{x}_t^f$. Moreover, $E\left[\mathbf{x}_t^f\right] = 0$ and $E[\boldsymbol{\epsilon}_{t+1}] = 0$.

Thus

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{h_x} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0}_{n_x^2 \times n_x} & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{h_x} \otimes \mathbf{h_x} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{0}_{n_x \times 1} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta})\, vec(\mathbf{I}_{n_e}) \end{bmatrix} + \begin{bmatrix} \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x \times 1} \\ \mathbf{v}(t+1) - (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\, vec(\mathbf{I}_{n_e}) \end{bmatrix}$$

$\Updownarrow$

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \tilde{\boldsymbol{\xi}}_{t+1} \tag{37}$$

where $Cov\left(\tilde{\boldsymbol{\xi}}_{t+1}, \tilde{\boldsymbol{\xi}}_{t-s}\right) = \mathbf{0}$ for $s = 1, 2, 3, ...$ because $\boldsymbol{\epsilon}_{t+1}$ is independent across time. The expression for the controls are as above, i.e.

$$\mathbf{y}_t^s = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

The mean values are

$$E[\mathbf{z}_t] = \left(\mathbf{I}_{2n_x + n_x^2} - \mathbf{A}\right)^{-1}\mathbf{c}$$

$$E[\mathbf{y}_t] = \mathbf{D}E[\mathbf{z}_t] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

and the covariance matrix is

$$Var(\mathbf{z}_{t+1}) = \mathbf{A}Var(\mathbf{z}_t)\mathbf{A}' + Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)$$

$\Updownarrow$

$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = vec\left(\mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}'\right) + vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$
$$\Updownarrow$$
$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{A}\otimes\mathbf{A}\right)vec\left(Var\left(\mathbf{z}_t\right)\right) + vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$
$$\Updownarrow$$
$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right)\left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A}\otimes\mathbf{A}\right)\right) = vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$
$$\Updownarrow$$
$$vec\left(Var\left(\mathbf{z}_{t+1}\right)\right) = \left(\mathbf{I}_{(2n_x+n_x^2)^2} - \left(\mathbf{A}\otimes\mathbf{A}\right)\right)^{-1} vec\left(Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)\right)$$

The variance of the control variables is then given by

$$Var\left[\mathbf{y}_t^s\right] = \mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

Hence we only need to compute $Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)$.

$$Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right) = E\left(\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x\times 1} \\ \mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\end{bmatrix}\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x\times 1} \\ \mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\end{bmatrix}'\right)$$

$$= E\left(\begin{bmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ \mathbf{0}_{n_x\times 1} \\ \mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\end{bmatrix}\begin{bmatrix}\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{1\times n_x} & \mathbf{v}'\left(t+1\right) - vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\end{bmatrix}\right)$$

$$= E\begin{pmatrix}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{n_x\times n_x} & \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{v}'\left(t+1\right) - vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right) \\ \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \left(\mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{n_x\times n_x} & Var\left[\boldsymbol{\xi}_{t+1}\right]_{33}\end{pmatrix}$$

where

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33} \equiv \left(\mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{v}'\left(t+1\right) - vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)$$

Recall that

$$\mathbf{v}\left(t+1\right) = \left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

3.3.1 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13}$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13} \equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{v}'\left(t+1\right) - vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(\left(\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right)'$$
$$-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)')]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'))]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

$$-\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)']$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)']$$

because $E\left[\boldsymbol{\epsilon}_{t+1}\right] = \mathbf{0}$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\left(\sigma\boldsymbol{\eta}'\otimes\boldsymbol{\sigma\eta}'\right)]$$

because $\boldsymbol{\epsilon}_{t+1}$ is independent of $\mathbf{x}_t^f$ and $E\left[\mathbf{x}_t^f\right] = 0$. Hence, for shocks with a symmetry distribution $Var\left[\boldsymbol{\xi}_{t+1}\right]_{13} = \mathbf{0}$.

For the implementation, consider:
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right]$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

```
E_eps_eps2 = zeros(ne, (ne)^2)
for phi1 = 1 : ne
    index2 = 0
    for phi2 = 1 : ne
        for phi3 = 1 : ne
            index2 = index2 + 1
            if (phi1 = phi2 = phi3)
                E_eps_eps2(phi1, index2) = m^3 (ϵ_{t+1} (phi1))
            end
        end
    end
end
```

3.3.2 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33} \equiv E[\left(\mathbf{v}\left(t+1\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{v}'\left(t+1\right) - vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\left(\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\right)$$
$$\left(\left(\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right)' - vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\left(\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - vec\left(\mathbf{I}_{n_e}\right)\right)\right)$$
$$\left(\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - vec\left(\mathbf{I}_{n_e}\right)\right)\right)']$$

$$= E[\left(\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - vec\left(\mathbf{I}_{n_e}\right)\right)\right)$$
$$\left(\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_\mathbf{x}\right)' + \left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' - vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$= E[\left(\mathbf{h}_\mathbf{x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\times$$

$$
\begin{aligned}
&\quad\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' + \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right) \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \times \\
&\quad\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' + \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right) \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right) \times \\
&\quad\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' + \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)]
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})']
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - vec\left(\mathbf{I}_{n_e}\right)\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' - vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})']
\end{aligned}
$$

$$
\begin{aligned}
&= E[(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})' \\
&+ (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})' \\
&+ (\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})\left(\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right) - \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) vec\left(\mathbf{I}_{n_e}\right)'\right)(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h}_\mathbf{x})\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)' \otimes \mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)(\mathbf{h}_\mathbf{x} \otimes \sigma\boldsymbol{\eta})'
\end{aligned}
$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)-\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\Big]$$

$$=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)-\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)-\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

note that $\boldsymbol{\epsilon}_{t+1}$ is independent of $\mathbf{x}_t^f$ and $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$ is know and $E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right]=\mathbf{I}_{n_e}$

$$=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

terms with three or one $\boldsymbol{\epsilon}_{t+1}$ are zero because $\mathbf{x}_t^f$ is independent of $\boldsymbol{\epsilon}_{t+1}$ and $E\left[\mathbf{x}_t^f\right]=0$

$$=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\times$$

$$E\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)vec\left(\mathbf{I}_{n_e}\right)'-vec\left(\mathbf{I}_{n_e}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)+vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

$$\begin{aligned}
&=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\times\\
&\left(E\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'+vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'
\end{aligned}$$

$$\begin{aligned}
&=\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left(\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\otimes\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\right)\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'\\
&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(E\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)-vec\left(\mathbf{I}_{n_e}\right)vec\left(\mathbf{I}_{n_e}\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'
\end{aligned}$$

Next consider (with dimensions $n_x n_e \times n_e n_x$)

$$E\left[\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right]$$

$$\begin{aligned}
&=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\right)\right]\\
&=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\right)\right)'\right]\\
&=E\left[\left\{x_t^f\left(\gamma_1,1\right)\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\mathbf{x}_t^f\right\}_{\phi_2=1}^{n_e}\right)'\right]\\
&=E\left[\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]
\end{aligned}$$

Thus the quasi Matlab codes are

```
E_xfeps_epsxf = zeros(n_x n_e, n_x n_e)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2
                    E_xfeps_epsxf (index1, index2) = E_xf_xf(gama1, gama2)
```

end
end
end
end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

Note also that

$$E\left[\left(\mathbf{x}_t^f \boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f \boldsymbol{\epsilon}_{t+1}'\right)' \otimes \left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)'\right)'\right]$$

$$= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\right)' \otimes \mathbf{x}_t^f \boldsymbol{\epsilon}_{t+1}'\right]$$

$\Updownarrow$

$(E\_xfeps\_epsxf)' = E\_epsxf\_xfeps$

Finally consider the matrix (with dimension $n_e^2 \times n_e^2$)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)\right]$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1) \\ \epsilon_{t+1}(2,1) \\ ... \\ \epsilon_{t+1}(n_e,1) \end{bmatrix} \begin{bmatrix} \epsilon_{t+1}'(1,1) & \epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}'(1,n_e) \end{bmatrix}$$
$$\otimes \begin{bmatrix} \epsilon_{t+1}(1,1) \\ \epsilon_{t+1}(2,1) \\ ... \\ \epsilon_{t+1}(n_e,1) \end{bmatrix} \begin{bmatrix} \epsilon_{t+1}'(1,1) & \epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}'(1,n_e) \end{bmatrix}]$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,1) & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,n_e) \\ \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,1) & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,n_e) \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1) & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,2) & ... & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}'(1,n_e) \end{bmatrix}$$
$$\otimes \begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,1) & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,n_e) \\ \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,1) & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,n_e) \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1) & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,2) & ... & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}'(1,n_e) \end{bmatrix}]$$

$$= E[\begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,1)\mathbf{A}_{\epsilon\epsilon} & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,2)\mathbf{A}_{\epsilon\epsilon} & ... & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,n_e)\mathbf{A}_{\epsilon\epsilon} \\ \epsilon_{t+1}(2,1)\epsilon_{t+1}(1,1)\mathbf{A}_{\epsilon\epsilon} & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,2)\mathbf{A}_{\epsilon\epsilon} & ... & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,n_e)\mathbf{A}_{\epsilon\epsilon} \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1)\mathbf{A}_{\epsilon\epsilon} & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}'(1,2)\mathbf{A}_{\epsilon\epsilon} & ... & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}'(1,n_e)\mathbf{A}_{\epsilon\epsilon} \end{bmatrix}$$

where $\mathbf{A}_{\epsilon\epsilon} \equiv \begin{bmatrix} \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,1) & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}(1,1)\epsilon_{t+1}'(1,n_e) \\ \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,1) & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,2) & ... & \epsilon_{t+1}(2,1)\epsilon_{t+1}'(1,n_e) \\ ... & ... & ... & ... \\ \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,1) & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}(1,2) & ... & \epsilon_{t+1}(n_e,1)\epsilon_{t+1}'(1,n_e) \end{bmatrix}$

$$= E\left[\begin{array}{cccc} \epsilon_{t+1}(1,1)\,\epsilon'_{t+1}(1,1)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \epsilon_{t+1}(1,1)\,\epsilon'_{t+1}(1,2)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \ldots & \epsilon_{t+1}(1,1)\,\epsilon'_{t+1}(1,n_e)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} \\ \epsilon_{t+1}(2,1)\,\epsilon'_{t+1}(1,1)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \epsilon_{t+1}(2,1)\,\epsilon'_{t+1}(1,2)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \ldots & \epsilon_{t+1}(2,1)\,\epsilon'_{t+1}(1,n_e)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} \\ \ldots & \ldots & \ldots & \ldots \\ \epsilon_{t+1}(n_e,1)\,\epsilon'_{t+1}(1,1)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \epsilon_{t+1}(n_e,1)\,\epsilon'_{t+1}(1,2)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} & \ldots & \epsilon_{t+1}(n_e,1)\,\epsilon'_{t+1}(1,n_e)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e} \end{array}\right]$$

$$= E\left[\begin{array}{c} \epsilon_{t+1}(1,1)\left\{\epsilon'_{t+1}(1,\phi_3)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e}\right\}_{\phi_3=1}^{n_e} \\ \epsilon_{t+1}(2,1)\left\{\epsilon'_{t+1}(1,\phi_3)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e}\right\}_{\phi_3=1}^{n_e} \\ \ldots \\ \epsilon_{t+1}(n_e,1)\left\{\epsilon'_{t+1}(1,\phi_3)\left\{A_{\epsilon\epsilon}(\phi_1,\phi_2)\right\}_{\phi_1=1,\phi_2=1}^{n_e,n_e}\right\}_{\phi_3=1}^{n_e} \end{array}\right]$$

Hence the quasi MATLAB codes are

```
E_eps2_eps2 = zeros(n_e^2, n_e^2)
index1 = 0
for phi4 = 1 : n_e
    for phi1 = 1 : n_e
        index1 = index1 + 1
        index2 = 0
        for phi3 = 1 : n_e
            for phi2 = 1 : n_e
                index2 = index2 + 1
                % second moments
                if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                    E_eps2_eps2(index1, index2) = 1
                elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                    E_eps2_eps2(index1, index2) = 1
                elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                    E_eps2_eps2(index1, index2) = 1
                % fourth moments
                elseif(phi1 == phi2 && phi1 == phi3 && phi1 == phi4)
                    E_eps2_eps2(index1, index2) = m^4 (eps_{t+1} (phi1))
                end
            end
        end
    end
end
```

3.4 Method 3: Simple formulas for first and second moments

This section computes first and second moments at second order using a more direct approach. The advantage of this method is that we do not in a second-order approximation re-compute some of the moments already know from a first-order approximation. A direct implication is that we in Method 3 only need to invert smaller matrices than in Method 1 and 2.

3.4.1 First moments

Note first that

$$E[\mathbf{x}_t] = E\left[\mathbf{x}_t^f\right] + E[\mathbf{x}_t^s]$$

For the first order effects, we have due to stationary of the linear model

$E\left[\mathbf{x}_t^f\right] = \mathbf{h_x} E\left[\mathbf{x}_{t-1}^f\right] + \sigma\boldsymbol{\eta} E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$\mathbf{I}E\left[\mathbf{x}_t^f\right] - \mathbf{h_x} E\left[\mathbf{x}_{t-1}^f\right] = E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$(\mathbf{I} - \mathbf{h_x}) E\left[\mathbf{x}_t^f\right] = E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$E\left[\mathbf{x}_t^f\right] = (\mathbf{I} - \mathbf{h_x})^{-1} E[\boldsymbol{\epsilon}_t]$

$\Updownarrow$

$E\left[\mathbf{x}_t^f\right] = \mathbf{0}$

since $E[\boldsymbol{\epsilon}_t] = \mathbf{0}$

For the second order effects we have

$E\left[x_{t+1}^s(j,1)\right] = \mathbf{h_x}(j,:) E[\mathbf{x}_t^s] + \frac{1}{2} E\left[\left(\mathbf{x}_t^f\right)' \mathbf{h_{xx}}(j,:,:) \left(\mathbf{x}_t^f\right)\right] + \frac{1}{2} h_{\sigma\sigma}(j,1)\sigma^2$

$\Updownarrow$

$(\mathbf{I} - \mathbf{h_x}) E[\mathbf{x}_t^s] = \tilde{\mathbf{H}}_{\mathbf{xx}} E\left[vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right)\right] + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$

where $\tilde{\mathbf{H}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{h_{xx}}, n_x, n_x^2\right)$

To compute $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$, consider

$Var\left(\mathbf{x}_t^f\right) = \mathbf{h_x} Var\left(\mathbf{x}_t^f\right)\mathbf{h}_\mathbf{x}' + \sigma^2\boldsymbol{\eta}\boldsymbol{\eta}'$

$\Downarrow$

$vec\left(Var\left(\mathbf{x}_t^f\right)\right) = vec\left(\mathbf{h_x} Var\left(\mathbf{x}_t^f\right)\mathbf{h}_\mathbf{x}'\right) + vec\left(\sigma^2\boldsymbol{\eta}\boldsymbol{\eta}'\right)$

$\Updownarrow$

$vec\left(Var\left(\mathbf{x}_t^f\right)\right) = (\mathbf{h_x} \otimes \mathbf{h_x}) vec\left(Var\left(\mathbf{x}_t^f\right)\right) + vec\left(\sigma^2\boldsymbol{\eta}\boldsymbol{\eta}'\right)$

$\Updownarrow$

$\left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right) vec\left(Var\left(\mathbf{x}_t^f\right)\right) = vec\left(\sigma^2\boldsymbol{\eta}\boldsymbol{\eta}'\right)$

$\Updownarrow$

$vec\left(Var\left(\mathbf{x}_t^f\right)\right) = \left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} vec\left(\sigma^2\boldsymbol{\eta}\boldsymbol{\eta}'\right)$

Notice there that

$Var\left(\mathbf{x}_t^f\right) = E\left[\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)\left(\mathbf{x}_t^f - E\left[\mathbf{x}_t^f\right]\right)'\right] = E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$
since $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$
$\Updownarrow$
$vec\left(E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right) = vec\left(Var\left(\mathbf{x}_t^f\right)\right)$

Hence,
$E\left[\mathbf{x}_t^s\right] = \left(\mathbf{I} - \mathbf{h}_{\mathbf{x}}\right)^{-1}\left(\tilde{\mathbf{H}}_{\mathbf{xx}} E\left[vec\left(\left[\left(\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\right)\right] + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$

The mean value of the control variables is given by
$E\left[\mathbf{y}_t^s\right] = \mathbf{g}_{\mathbf{x}}\left(E\left[\mathbf{x}_t^f\right] + E\left[\mathbf{x}_t^s\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
$\Updownarrow$
$E\left[\mathbf{y}_t^s\right] = \mathbf{g}_{\mathbf{x}} E\left[\mathbf{x}_t^s\right] + \tilde{\mathbf{G}}_{\mathbf{xx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$

3.4.2 Second moments

We need to compute $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]$, $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ and $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^s\right)'\right]$ as this will allow us to find $Var\left(\mathbf{z}_t\right)$. This is because

$$Var\left(\mathbf{z}_t\right) = E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)'\right]$$

$$= E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t' - E\left[\mathbf{z}_t'\right]\right)\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t' - \mathbf{z}_t E\left[\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]\mathbf{z}_t' + E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]$$

and

$$E\left[\mathbf{z}_t\mathbf{z}_t'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \end{bmatrix}\right]$$

$$= E\begin{bmatrix} \mathbf{x}_t^f\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \\ \mathbf{x}_t^s\mathbf{x}_t^f & \mathbf{x}_t^s\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{x}_t^f & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \end{bmatrix}$$

Finding $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$
From above:
$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{v}\left(t+1\right)$
where
$\mathbf{v}\left(t+1\right) = \left(\mathbf{h}_{\mathbf{x}} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$
and $E\left[\mathbf{v}\left(t+1\right)\right] = \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)$. Note that $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$ and $\mathbf{v}\left(t+1\right)'$ are uncorrelated

So

$$E\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'$$

$$=E\{(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\}\{(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\}'$$

$$=E\{(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{v}\left(t+1\right)\}\{\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)'\}$$

$$=E\left[(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)'\right)\right]$$
$$+E\left[\mathbf{v}\left(t+1\right)\left(\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)'\right)\right]$$

$$=E\left[(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+(\mathbf{h_x}\otimes\mathbf{h_x})\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\mathbf{v}\left(t+1\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'(\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{v}\left(t+1\right)\mathbf{v}\left(t+1\right)'\right]$$

$$=(\mathbf{h_x}\otimes\mathbf{h_x})\,E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})\,E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\right]E\left[\mathbf{v}\left(t+1\right)'\right]+E\left[\mathbf{v}\left(t+1\right)\right]E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x})'+E\left[\mathbf{v}\left(t+1\right)\mathbf{v}\left(t+1\right)'\right]$$

Letting
$$\mathbf{c}\equiv(\mathbf{h_x}\otimes\mathbf{h_x})\,E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\right]E\left[\mathbf{v}\left(t+1\right)'\right]+E\left[\mathbf{v}\left(t+1\right)\right]E\left[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right](\mathbf{h_x}\otimes\mathbf{h_x})'+E\left[\mathbf{v}\left(t+1\right)\mathbf{v}\left(t+1\right)'\right]$$
we therefore have (due to stationarity)
$$E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']=(\mathbf{h_x}\otimes\mathbf{h_x})\,E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'](\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{c}$$
$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)=vec\left((\mathbf{h_x}\otimes\mathbf{h_x})\,E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'](\mathbf{h_x}\otimes\mathbf{h_x})'\right)+vec\left(\mathbf{c}\right)$$
$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)=((\mathbf{h_x}\otimes\mathbf{h_x})\otimes(\mathbf{h_x}\otimes\mathbf{h_x}))\,vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)+vec\left(\mathbf{c}\right)$$
because $vec\left(\mathbf{ABC}\right)=\left(\mathbf{C}'\otimes\mathbf{A}\right)vec\left(\mathbf{B}\right)$
$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)\left(\mathbf{I}_{n_x^4}-(\mathbf{h_x}\otimes\mathbf{h_x})\otimes(\mathbf{h_x}\otimes\mathbf{h_x})\right)=vec\left(\mathbf{c}\right)$$
$\Updownarrow$

$$vec\left(E[\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)']\right)=vec\left(\mathbf{c}\right)\left(\mathbf{I}_{n_x^4}-(\mathbf{h_x}\otimes\mathbf{h_x})\otimes(\mathbf{h_x}\otimes\mathbf{h_x})\right)^{-1}$$

Finding $E\left[\mathbf{x}_{t}^{s}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]=E\left[\left(\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\mathbf{v}\left(t+1\right)\right)'\right]$$

$$=E\left[\left(\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{v}\left(t+1\right)'\right)\right]$$

$$=E\left[\mathbf{h_x}\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{h_x}\mathbf{x}_t^s\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\mathbf{v}\left(t+1\right)'\right]$$

$$=\mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{h_x}E\left[\mathbf{x}_t^s\right]E\left[\mathbf{v}\left(t+1\right)'\right]$$
$$+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]E\left[\mathbf{v}\left(t+1\right)'\right]$$
$$+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\mathbf{v}\left(t+1\right)'\right]$$

Letting

$$\mathbf{c}\equiv\mathbf{h_x}E\left[\mathbf{x}_t^s\right]E\left[\mathbf{v}\left(t+1\right)'\right]+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]E\left[\mathbf{v}\left(t+1\right)'\right]$$
$$+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2E\left[\mathbf{v}\left(t+1\right)'\right]$$

we therefore have (due to stationarity)

$$E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]=\mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'+\mathbf{c}$$

$$\Updownarrow$$

$$vec\left(E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]\right)=vec\left(\mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)'\right)+vec\left(\mathbf{c}\right)$$

$$\Updownarrow$$

$$vec\left(E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)'\right]\right)=\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)vec\left(E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)+vec\left(\mathbf{c}\right)$$

$$\Updownarrow$$

$$vec\left(E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\mathbf{I}_{n_x^3}-\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\right)=vec\left(\mathbf{c}\right)$$

$$\Updownarrow$$

$$vec\left(E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)=vec\left(\mathbf{c}\right)\left(\mathbf{I}_{n_x^3}-\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\right)^{-1}$$

Finding $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^{s}\left(\mathbf{x}_{t+1}^{s}\right)'\right]=E\left[\left(\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right]$$

$$=E\left[\left(\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_{\mathbf{x}}'+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}'+\frac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$

$$= E\left[\mathbf{h_x x}_t^s\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_\mathbf{xx}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$
$$+E\left[\tilde{\mathbf{H}}_\mathbf{xx}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_\mathbf{xx}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$
$$+E\left[\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_\mathbf{xx}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$

$$= E\left[\mathbf{h_x x}_t^s\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \mathbf{h_x x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_\mathbf{xx}' + \mathbf{h_x x}_t^s\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right]$$
$$+E\left[\tilde{\mathbf{H}}_\mathbf{xx}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \tilde{\mathbf{H}}_\mathbf{xx}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_\mathbf{xx}' + \tilde{\mathbf{H}}_\mathbf{xx}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right]$$
$$+E\left[\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\mathbf{x}_t^s\right)'\mathbf{h}_\mathbf{x}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_\mathbf{xx}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right]$$

$$= \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_\mathbf{xx}' + \mathbf{h_x}E\left[\mathbf{x}_t^s\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tilde{\mathbf{H}}_\mathbf{xx}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tilde{\mathbf{H}}_\mathbf{xx}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_\mathbf{xx}' + \tilde{\mathbf{H}}_\mathbf{xx}E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 E\left[\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_\mathbf{xx}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$

Letting

$$\mathbf{c} \equiv \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_\mathbf{xx}' + \mathbf{h_x}E\left[\mathbf{x}_t^s\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tilde{\mathbf{H}}_\mathbf{xx}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tilde{\mathbf{H}}_\mathbf{xx}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_\mathbf{xx}' + \tilde{\mathbf{H}}_\mathbf{xx}E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right]\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$
$$+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 E\left[\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_\mathbf{xx}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2$$

we therefore have (due to stationarity)

$$E\left[\mathbf{x}_{t+1}^s\left(\mathbf{x}_{t+1}^s\right)'\right] = \mathbf{h_x}E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right]\mathbf{h}_\mathbf{x}' + \mathbf{c}$$
$$\Updownarrow$$
$$E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^s\right)'\right] = vec\left(c\right)\left(\mathbf{I}_{n_x^2} - \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\right)^{-1}$$

Finding $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$

$$E\left[\mathbf{x}_{t+1}^f\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right)'\right]$$

$$= E\left[\left(\mathbf{h_x x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{v}\left(t+1\right)\right)'\right]$$

$$= E\left[\left(\mathbf{h_x x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)' + \mathbf{v}\left(t+1\right)'\right)\right]$$

$$= E\left[\mathbf{h_x x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)' + \mathbf{h_x x}_t^f\mathbf{v}\left(t+1\right)'\right]$$
$$+E\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\mathbf{v}\left(t+1\right)'\right)\right]$$

Recall that $\mathbf{v}\left(t+1\right) = \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$ so we get

$$= \mathbf{h_x} E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] (\mathbf{h_x} \otimes \mathbf{h_x})' + \mathbf{0} + E\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right]$$

$$= \mathbf{h_x} E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] (\mathbf{h_x} \otimes \mathbf{h_x})' + \mathbf{0} + \sigma\boldsymbol{\eta} E\left[\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'\right] (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t+1}^f \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right)'\right]\right) = (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x})\, vec\left(E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)$$
$$+vec\left(\sigma\boldsymbol{\eta} E\left[\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'\right] (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)$$

because $vec(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A})\, vec(\mathbf{B})$

$\Updownarrow$

$$\left(E\left[\mathbf{x}_{t+1}^f \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right)'\right]\right) = \left(\mathbf{I}_{n_x^3} - (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} vec\left(\sigma\boldsymbol{\eta} E\left[\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'\right] (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'\right)$$

Finding $E\left[\mathbf{x}_t^f (\mathbf{x}_t^s)'\right]$

$$E\left[\mathbf{x}_{t+1}^f (\mathbf{x}_{t+1}^s)'\right] = E\left[\left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{h_x}\mathbf{x}_t^s + \tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right]$$

$$= E\left[\mathbf{h_x}\mathbf{x}_t^f\left((\mathbf{x}_t^s)'\mathbf{h}_\mathbf{x}' + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\tilde{\mathbf{H}}_{\mathbf{xx}}' + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}'\sigma^2\right)\right]$$

$$= \mathbf{h_x} E\left[\mathbf{x}_t^f (\mathbf{x}_t^s)'\right]\mathbf{h}_\mathbf{x}' + \mathbf{h_x} E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}'$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_{t+1}^f (\mathbf{x}_{t+1}^s)'\right]\right) = (\mathbf{h_x} \otimes \mathbf{h_x})\, vec\left(E\left[\mathbf{x}_t^f (\mathbf{x}_t^s)'\right]\right) + vec\left(\mathbf{h_x} E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}'\right)$$

$\Updownarrow$

$$vec\left(E\left[\mathbf{x}_t^f (\mathbf{x}_t^s)'\right]\right) = \left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} vec\left(\mathbf{h_x} E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\tilde{\mathbf{H}}_{\mathbf{xx}}'\right)$$

3.5 Decomposition: The total variance of the state variables

The previous subsection have computed $Var(\mathbf{z}_t)$ and $Var(\mathbf{y}_t^s)$. We next discuss how the total variance of the state variables should be computed. Starting from our decomposition, we have

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s$$

Hence, it is natural to compute moments of $\mathbf{x}_t$ based on this decomposition. Hence, for the variance we get

$$\begin{aligned} Var(\mathbf{x}_t) &= Var\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) \\ &= Var\left(\mathbf{x}_t^f\right) + Var(\mathbf{x}_t^s) + Cov\left(\mathbf{x}_t^f, \mathbf{x}_t^s\right) + Cov\left(\mathbf{x}_t^s, \mathbf{x}_t^f\right) \end{aligned}$$

Note that this procedure is fully consistent with the one adopted for the control variables. To realize that, let us consider the case where one element in $\mathbf{y}_t$ simply reproduces one state variable. Then $\mathbf{g_x} = \mathbf{0}$ except $\mathbf{g_x}(k, j) = 1$ if we want to have the k'th control variable to reproduce the j'th state variable, while all remaining derivatives of $\mathbf{g}$ are zero. Hence, we have

$$y(k)_t = \mathbf{x}_t^f + \mathbf{x}_t^s.$$

Hence, $Var\left(y\left(k\right)_t\right) = Var\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)$.

One potential short-coming of this procedure for computing the moments of $\mathbf{x}_t$ might be that we do not include all higher order terms. In the case of the variance, one could believe that we omit the fourth order term $Cov\left(\mathbf{x}_t^s, \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$ in our expression of $Var\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)$. However, this is not the case. To realize this, recall

$$\mathbf{x}_{t+1}^f = \mathbf{h}_\mathbf{x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$\mathbf{x}_{t+1}^s = \mathbf{h}_\mathbf{x}\mathbf{x}_t^s + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

The law of motion for the total state variable is

$$\begin{aligned} & \mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s \\ = \; & \mathbf{h}_\mathbf{x}\left(\mathbf{x}_t^s + \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \end{aligned}$$

We compute second moments of $\mathbf{x}_t^s + \mathbf{x}_t^f$ based on the first line in this expression. But, of course, we could equally well have computed the moments based on the second line in this expression. Adopting this alternative approach, we obtain

$$\begin{aligned} Var\left(\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s\right) = \; & \mathbf{h}_\mathbf{x} Var\left(\mathbf{x}_t^s + \mathbf{x}_t^f\right)\mathbf{h}_\mathbf{x} + \frac{1}{4}\mathbf{H}_{\mathbf{xx}} Var\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{H}_{\mathbf{xx}}' + \sigma^2\boldsymbol{\eta}\boldsymbol{\eta}' \\ & + \frac{1}{2}\mathbf{h}_\mathbf{x} Cov\left(\mathbf{x}_t^s + \mathbf{x}_t^f, \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\mathbf{H}_{\mathbf{xx}}' \\ & + \frac{1}{2}\mathbf{H}_{\mathbf{xx}} Cov\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f, \mathbf{x}_t^s + \mathbf{x}_t^f\right)\mathbf{h}_\mathbf{x}' \end{aligned}$$

It is evident that this expression for $Var\left(\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s\right)$ includes the term $Cov\left(\mathbf{x}_t^s, \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$. Using this expression to solve directly for $Var\left(\mathbf{x}_t^s + \mathbf{x}_t^f\right)$ gives exactly the same expression for $Var\left(\mathbf{x}_t^s + \mathbf{x}_t^f\right)$ as the one we first suggested (which we obtain without solving more equations!)

3.6 The auto-correlations

This section derives the auto-correlations for the states and the control variables.

3.6.1 The innovations

We start by showing that $\boldsymbol{\xi}_{t+1}$ and $\boldsymbol{\xi}_{t+1+s}$ are uncorrelated for $s = 1, 2, ...$ To see this note that
$E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1+s}'\right] =$

$$E\left[\begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \end{bmatrix} \begin{matrix} \boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)' & \left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)' \end{matrix}\right]$$

$$= E\left[\begin{matrix} \boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1+s}' & \boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' \\ \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' \\ \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' \\ \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1+s}' & \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' \end{matrix}\right.$$

$$
\left.\begin{array}{cc}
\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^{f}\right)' & \boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_{t+s}^{f}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^{f}\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{x}_{t+s}^{f}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^{f}\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\right)\left(\mathbf{x}_{t+s}^{f}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
\left(\mathbf{x}_{t}^{f}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^{f}\right)' & \left(\mathbf{x}_{t}^{f}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_{t+s}^{f}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'
\end{array}\right]
$$

$$
=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

3.6.2 The auto-covariances

Recall that we have

$$
\mathbf{z}_t=\left[\begin{array}{c}
\mathbf{x}_t^f \\
\mathbf{x}_t^s \\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f
\end{array}\right]
$$

$$
\mathbf{z}_{t+1}=\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1}
$$

$$
\mathbf{y}_t^s=\mathbf{D}\mathbf{z}_t+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2
$$

To find the one period auto-correlation, i.e. $Cov\left(\mathbf{z}_{t+1},\mathbf{z}_t\right)$, we have

$$
Cov\left(\mathbf{z}_{t+1},\mathbf{z}_t\right)=Cov\left(\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right)=\mathbf{A}Cov\left(\mathbf{z}_t,\mathbf{z}_t\right)=\mathbf{A}Var\left(\mathbf{z}_t\right)
$$

because $Cov\left(\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right)=0$ as shown above. And for two periods

$Cov\left(\mathbf{z}_{t+2},\mathbf{z}_t\right)=Cov\left(\mathbf{c}+\mathbf{A}\mathbf{z}_{t+1}+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)$

$=Cov\left(\mathbf{A}\left(\mathbf{c}+\mathbf{A}\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+1}\right)+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)$

$=Cov\left(\mathbf{A}^2\mathbf{z}_t+\mathbf{B}\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)$

$=Cov\left(\mathbf{A}^2\mathbf{z}_t,\mathbf{z}_t\right)$

$=\mathbf{A}^2Cov\left(\mathbf{z}_t,\mathbf{z}_t\right)$

$=\mathbf{A}^2Var\left(\mathbf{z}_t\right)$

Here, we use the fact that $Cov\left(\boldsymbol{\xi}_{t+2},\mathbf{z}_t\right)=0$. This follows from the same arguements as above, that is consider

$$
E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+2}'\right]=E\left[\left[\begin{array}{c}
\mathbf{x}_t^f \\
\mathbf{x}_t^s \\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f
\end{array}\right]\left[\begin{array}{cccc}
\boldsymbol{\epsilon}_{t+2}' & \left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)'
\end{array}\right]\right]
$$

$$
=E\left[\begin{array}{cccc}
\mathbf{x}_t^f\boldsymbol{\epsilon}_{t+2}' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \\
\mathbf{x}_t^s\boldsymbol{\epsilon}_{t+2}' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\boldsymbol{\epsilon}_{t+2}-vec\left(\mathbf{I}_{ne}\right)\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+1}^f\otimes\boldsymbol{\epsilon}_{t+2}\right)'
\end{array}\right]
$$

$$
=0
$$

Hence, in the general case

$Cov\left(\mathbf{z}_{t+l},\mathbf{z}_t\right)=\mathbf{A}^lVar\left(\mathbf{z}_t\right)$

For the control variables:

$$Cov\left(\mathbf{y}_{t+l}^{s},\mathbf{y}_{t}^{s}\right)=Cov\left(\mathbf{D}\mathbf{z}_{t+l}+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^{2},\mathbf{D}\mathbf{z}_{t}+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^{2}\right)$$

$$=Cov\left(\mathbf{D}\mathbf{z}_{t+l},\mathbf{D}\mathbf{z}_{t}\right)$$

$$=\mathbf{D}Cov\left(\mathbf{z}_{t+l},\mathbf{z}_{t}\right)\mathbf{D}'$$

$$=\mathbf{D}\mathbf{A}^{l}Var\left(\mathbf{z}_{t}\right)\mathbf{D}'$$

4 Stastical properties: Third order approximation

4.1 Covariance-stationary

Proposition 1:

The pruned third order approximation for $\mathbf{x}_t^f, \mathbf{x}_t^s, \mathbf{x}_t^{rd}$, and $\mathbf{y}_t^{rd}$ is covariance-stationary if

1. the DSGE model has a unique stable equilibrium, i.e. all eigenvalue of $\mathbf{h}_{\mathbf{x}}$ have modulus less than 1
2. $\boldsymbol{\epsilon}_{t+1}$ has finite sixth moment

Proof

Note first that

$$x_{t+1}^{rd}(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\,\mathbf{x}_t^{rd}+\tfrac{2}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xx}}(j,:,:)\left(\mathbf{x}_t^s\right)$$
$$+\tfrac{1}{6}\left(\mathbf{x}_t^f\right)'\begin{bmatrix}\left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,1,:,:)\left(\mathbf{x}_t^f\right)\\ \dots \\ \left(\mathbf{x}_t^f\right)'\mathbf{h}_{\mathbf{xxx}}(j,n_x,:,:)\left(\mathbf{x}_t^f\right)\end{bmatrix}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}(j,:)\,\sigma^2\mathbf{x}_t^f+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

$\Updownarrow$

$$\mathbf{x}_{t+1}^{rd}=\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^{rd}+2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

where $\tilde{\mathbf{H}}_{\mathbf{xx}}\equiv\frac{1}{2}reshape\left(\mathbf{h}_{\mathbf{xx}},n_x,n_x^2\right)$ and $\tilde{\mathbf{H}}_{\mathbf{xxx}}\equiv\frac{1}{6}reshape\left(\mathbf{h}_{\mathbf{xxx}},n_x,n_x^3\right)$. So we need to find the law of motion of $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)$ and $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$.

Hence,

$$\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right)=\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$=\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s+\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

using $(\mathbf{A}+\mathbf{B})\otimes(\mathbf{C}+\mathbf{D})=\mathbf{A}\otimes\mathbf{C}+\mathbf{A}\otimes\mathbf{D}+\mathbf{B}\otimes\mathbf{C}+\mathbf{B}\otimes\mathbf{D}$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\left(\mathbf{x}_t^f\otimes\sigma^2\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\sigma^2\right)$$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\mathbf{AC}\otimes\mathbf{BD}$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \mathbf{x}_t^f$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+1}$$

Recall from above

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

So

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left((\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\right.$$
$$\left. + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right)$$

$$= \mathbf{h_x}\mathbf{x}_t^f \otimes (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\mathbf{x}_t^f \otimes (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ \mathbf{h_x}\mathbf{x}_t^f \otimes (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\mathbf{x}_t^f \otimes (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$
$$+ \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) - E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right]\right)$$
$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right]$$

Thus we can construct the following extended system

$$\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^{rd} \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_x \times 1} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta})\, vec\left(\mathbf{I}_{n_e}\right) \\ \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 \\ \mathbf{0}_{n_x^2 \times 1} \\ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})\, E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right] \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{h_x} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} & \mathbf{0}_{n_x \times n_x^3} \\ \mathbf{0}_{n_x \times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} & \mathbf{0}_{n_x \times n_x^3} \\ \mathbf{0}_{n_x^2 \times n_x} & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{h_x} \otimes \mathbf{h_x} & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{0}_{n_x^2 \times n_x^2} & \mathbf{0}_{n_x^2 \times n_x^3} \\ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x \times n_x} & \mathbf{0}_{n_x \times n_x^2} & \mathbf{h_x} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}} \\ \left(\mathbf{h_x} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) & \mathbf{0}_{n_x^2 \times n_x} & \mathbf{0}_{n_x^2 \times n_x^2} & \mathbf{0}_{n_x^2 \times n_x} & (\mathbf{h_x} \otimes \mathbf{h_x}) & \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) \\ \mathbf{0}_{n_x^3 \times n_x} & \mathbf{0}_{n_x^3 \times n_x} & \mathbf{0}_{n_x^3 \times n_x^2} & \mathbf{0}_{n_x^3 \times n_x} & \mathbf{0}_{n_x^3 \times n_x^2} & (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$

$$
+\begin{bmatrix}
\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}) & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}
\end{bmatrix.
$$

$$
\begin{matrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}
\end{matrix}\right]
$$

$$
\times\begin{bmatrix}
\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}) - E\left[(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\right]
\end{bmatrix}
$$

$\Updownarrow$

$$
\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1} \tag{38}
$$

The absolute value of the eigenvalues in $\mathbf{h_x}$ are all strictly less then one by assumption. Accordingly, all eigenvalues of $\mathbf{A}$ are also strictly less than one. To see this note first that

$p(\lambda) = |\mathbf{A}-\lambda\mathbf{I}|$

$$
= \left|\begin{bmatrix}
\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I} & \tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\
\mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3} \\
\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{h_x}-\lambda\mathbf{I} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}} \\
\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}} \\
\mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}
\end{bmatrix}\right|
$$

$$
= \left|\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}\right|
$$

where we let

$$
\mathbf{B}_{11} \equiv \begin{bmatrix}
\mathbf{h_x}-\lambda\mathbf{I} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\
\mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}
\end{bmatrix}
$$

$$
\mathbf{B}_{12} \equiv \begin{bmatrix}
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\
\mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3}
\end{bmatrix}
$$

$$
\mathbf{B}_{21} \equiv \begin{bmatrix}
\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\
\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} \\
\mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2}
\end{bmatrix}
$$

$$\mathbf{B}_{22} \equiv \begin{bmatrix} \mathbf{h_x}-\lambda\mathbf{I} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}} \\ \mathbf{0}_{n_x^2 \times n_x} & (\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I} & \left(\mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right) \\ \mathbf{0}_{n_x^3 \times n_x} & \mathbf{0}_{n_x^3 \times n_x^2} & (\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I} \end{bmatrix}$$

$= |\mathbf{B}_{11}|\,|\mathbf{B}_{22}|$

using $\begin{vmatrix} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{vmatrix} = |\mathbf{U}|\,|\mathbf{Y}|$ where $\mathbf{U}$ is $m \times m$ and $\mathbf{Y}$ is $n \times n$

$= |\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{B}_{22}|$

using the results from the second order approximation

$= |\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x} - \lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|$

using the rule on block determinants repeatly on $\mathbf{B}_{22}$

Hence, the eigenvalue λ solves the problem

$p(\lambda) = 0$

$\Updownarrow$

$|\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}|\,|\mathbf{h_x} - \lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}|\,|(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}| = 0$

$\Updownarrow$

$|\mathbf{h_x}-\lambda\mathbf{I}| = 0$ or $|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}| = 0$ or $|(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})-\lambda\mathbf{I}| = 0$

The absolute value of all eigenvalues to the first problem are strictly less than one. That is $|\lambda_i| < 1$ $i = 1, 2, ..., n_x$. This is also the case for the second problem because the eigenvalues to $\mathbf{h_x}\otimes\mathbf{h_x}$ are $\lambda_i\lambda_j$ for $i = 1, 2, ..., n_x$ and $j = 1, 2, ..., n_x$. The same arguement ensures that this is also the case for the third problem.

Thus, the system in (38) is covariance stationary if $\boldsymbol{\xi}_{t+1}$ has finite first and second moment. It follows directly that $E\left[\boldsymbol{\xi}_{t+1}\right] = \mathbf{0}$ and $\boldsymbol{\xi}_{t+1}$ has finite second moments if $\boldsymbol{\epsilon}_{t+1}$ has a sixth moment. The latter holds by assumption.

For the control variables we have

$$y_t^{rd}(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_t^f + 2\mathbf{x}_t^s\right)$$
$$+\frac{1}{6}\left(\mathbf{x}_t^f\right)'\begin{bmatrix} \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,1,:,:)\left(\mathbf{x}_t^f\right) \\ ... \\ \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,n_x,:,:)\left(\mathbf{x}_t^f\right) \end{bmatrix} + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}(i,:)\sigma^2\mathbf{x}_t^f + \frac{1}{6}g_{\sigma\sigma\sigma}(i,1)\sigma^3$$

$\Updownarrow$

$$\mathbf{y}_t^{rd} = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right) + 2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$$
$$+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

where $\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2}reshape\left(\mathbf{g_{xx}}, n_y, n_x^2\right)$ and $\tilde{\mathbf{G}}_{\mathbf{xxx}} \equiv \frac{1}{6}reshape\left(\mathbf{g_{xxx}}, n_y, n_x^3\right)$

$\Updownarrow$

$$\mathbf{y}_t^{rd} = \begin{bmatrix} \mathbf{g_x} + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{g_x} & \tilde{\mathbf{G}}_{\mathbf{xx}} & \mathbf{g_x} & 2\tilde{\mathbf{G}}_{\mathbf{xx}} & \tilde{\mathbf{G}}_{\mathbf{xxx}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^s \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f \end{bmatrix} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

$= \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$
That is $\mathbf{y}_t^{rd}$ is linear function of $\mathbf{z}_t$ and $\mathbf{y}_t^{rd}$ is therefore also covariance-stationary.

Q.E.D.

4.2 Method 1: Formulas for the first and second moments

This section computes first and second moments using the representation of the second-order system stated above. This method is fairly direct but has the computational disadvantage of requiring a lot of memory because we work directly with the big $\mathbf{B}$ matrix.

The system

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t^{rd} = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

The mean values are

$$E\left[\mathbf{z}_t\right] = \left(\mathbf{I}_{3n_x+2n_x^2+n_x^3} - \mathbf{A}\right)^{-1}\mathbf{c}.$$

$$E\left[\mathbf{y}_t^{rd}\right] = \mathbf{D}E\left[\mathbf{z}_t\right] + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

For the variances we first have as above that
$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$

$= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$
$+E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$
$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$

$= \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}'$
$+\mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$
$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$

and
$E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)'$

$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$

Hence,
$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}'$
$+\mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$
$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$
$-\left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' - \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}$

$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}'$
$+\mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}'$
$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$
$\Updownarrow$

$Var\left[\mathbf{z}_{t+1}\right] = \mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}'$
$+\mathbf{A}\left(E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\boldsymbol{\xi}_{t+1}'\right]\right)\mathbf{B}' + \mathbf{B}\left(E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right] - E\left[\boldsymbol{\xi}_{t+1}\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}'$
$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$

Notice that $E\left[\mathbf{z}_t\right]E\left[\boldsymbol{\xi}'_{t+1}\right]=0$ because $E\left[\boldsymbol{\xi}'_{t+1}\right]=0$
$\Updownarrow$

$$Var\left[\mathbf{z}_{t+1}\right]=\mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}'+\mathbf{B}Var\left[\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'+\mathbf{A}Cov\left[\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'+\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]\mathbf{A}'$$

Moreover,

$$Var\left[\mathbf{y}_t^{rd}\right]=\mathbf{D}Var\left[\mathbf{z}_t\right]\mathbf{D}'$$

Contrary to a second-order approximation, we have that $Cov\left[\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right]\neq 0$. This is seen as follows

$$E\left[\mathbf{z}_t\boldsymbol{\xi}'_{t+1}\right]=E\left[\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^{rd}\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{array}\right]\right.$$

$$\times\left[\begin{array}{l}\boldsymbol{\epsilon}'_{t+1}\quad\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\quad\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\quad\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\quad\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\quad\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\quad\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\quad\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\quad\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\quad\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)'\end{array}\right]$$

$$=\left[\begin{array}{cccccccccccc}
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{1,9} & r_{1,10} & r_{1,11} & 0_{n_x\times n_e^3}\\
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{2,9} & r_{2,10} & r_{2,11} & 0_{n_x\times n_e^3}\\
0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & r_{3,9} & r_{3,10} & r_{3,11} & 0_{n_x^2\times n_e^3}\\
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & r_{4,9} & r_{4,10} & r_{4,11} & 0_{n_x\times n_e^3}\\
0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & r_{5,9} & r_{5,10} & r_{5,1} & 0_{n_x^2\times n_e^3}\\
0_{n_x^3\times n_e} & 0_{n_x^3\times n_e^2} & 0_{n_x^3\times n_e n_x} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_e n_x^2} & 0_{n_x^3\times n_x^2 n_e} & 0_{n_x^3\times n_x^2 n_e} & r_{6,9} & r_{6,10} & r_{6,11} & 0_{n_x^3\times n_e^3}
\end{array}\right]$$

$$=\left[\begin{array}{ccc}0 & \mathbf{R} & 0\end{array}\right]$$

We now compute the non-zero elements in this matrix

1) The value of $r_{1,9}$

$$r_{1,9}=E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

```
E_xf_xfeps2 = zeros(nx, nx × ne × ne)
for gama1 = 1 : nx
    index2 = 0
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            for phi2 = 1 : ne
                index2 = index2 + 1
                if phi1 == phi2
                    E_xf_xfeps2(phi1, index2) = E_xf_xf(gama1, gama2)
                end
```

end

end

end

end

2) The value of $r_{1,10}$

$$r_{1,10} = E\left[\mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

3) The value of $r_{1,11}$

$$r_{1,11} = E\left[\mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

4) The value of $r_{2,9}$

$$r_{2,9} = E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

5) The value of $r_{1,10}$

$$r_{2,10} = E\left[\mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

6) The value of $r_{2,11}$

$$r_{2,11} = E\left[\mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

7) The value of $r_{3,9}$

$$r_{3,9} = E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

8) The value of $r_{3,10}$

$$r_{3,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

9) The value of $r_{3,11}$

$$r_{3,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

10) The value of $r_{4,9}$

$$r_{4,9} = E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

11) The value of $r_{4,10}$

$$r_{4,10} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

12) The value of $r_{4,11}$

$$r_{4,11} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

13) The value of $r_{5,9}$

$$r_{5,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

14) The value of $r_{5,10}$

$$r_{5,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

15) The value of $r_{5,11}$

$$r_{5,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

16) The value of $r_{6,9}$

$$r_{6,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_t^f(\gamma_4,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right]$$

17) The value of $r_{6,10}$

$$r_{6,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_4,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

18) The value of $r_{6,11}$

$$r_{6,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Notice that all the required moments needed to compute these 18 terms are available from the covariance matrix at second order. Hence we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$. This is done below.

4.2.1 Efficient computing of $\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]$

The matrix $\mathbf{B}$ is very big and we therefore by hand try to simplify the summations $\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]$. Note that such a simplified expression is also useful when computing auto-correlations. We first note that

$$\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right] = \mathbf{B}E\left[\boldsymbol{\xi}_{\mathbf{t+1}}\mathbf{z}'_{\mathbf{t}}\right]$$

$$= \mathbf{B}\begin{bmatrix} 0 \\ \mathbf{R}' \\ 0 \end{bmatrix}$$

because $E\left[\mathbf{z}_t\boldsymbol{\xi}'_{t+1}\right] = \begin{bmatrix} 0 & \mathbf{R} & 0 \end{bmatrix}$

$$= \begin{bmatrix} \sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}) & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x} \end{bmatrix.$$

$$\begin{matrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta} \end{matrix} \Bigg]$$

$$\times \begin{bmatrix} 0 \\ \mathbf{R}' \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} \end{bmatrix}\mathbf{R}' \end{bmatrix}$$

$$= \begin{bmatrix} 0_{n_x\times(3n_x+2n_x^2+n_x^3)} \\ 0_{n_x\times(3n_x+2n_x^2+n_x^3)} \\ 0_{n_x^2\times(3n_x+2n_x^2+n_x^3)} \\ 0_{n_x\times(3n_x+2n_x^2+n_x^3)} \\ 0_{n_x^2\times(3n_x+2n_x^2+n_x^3)} \\ \begin{bmatrix} \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} \end{bmatrix}\mathbf{R}' \end{bmatrix}$$

We see that $\begin{bmatrix} \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} \end{bmatrix}$ has dimensions $n_x^3 \times 3\left(n_x n_e^2\right)$ and $\mathbf{R}$ has dimensions $\left(3n_x + 2n_x^2 + n_x^3\right) \times \left(3n_x^2 n_e\right)$. Thus
$\begin{bmatrix} \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} \end{bmatrix}\mathbf{R}'$ has dimensions $n_x^3 \times \left(3n_x + 2n_x^2 + n_x^3\right)$

Hence,
$Cov\left[\mathbf{z}_t, \boldsymbol{\xi}_{t+1}\right]\mathbf{B}'$

$$= \left(\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right]\right)'$$

$$= \begin{bmatrix} 0_{nn\times n_x} & 0_{nn\times n_x} & 0_{nn\times n_x^2} & 0_{nn\times n_x} & 0_{nn\times n_x^2} & \mathbf{R}\begin{bmatrix} \mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta} & \sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x} \end{bmatrix}' \end{bmatrix}$$

where $nn = \left(3n_x + 2n_x^2 + n_x^3\right)$

4.2.2 Computing $Var\left[\boldsymbol{\xi}_{t+1}\right]$

We start by noticing that

$$E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right] = E\left[\begin{bmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right] \end{bmatrix}\right.$$

$$\times\left[\begin{matrix} \boldsymbol{\epsilon}_{t+1}' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)' & \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)' \\ \left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)' \end{matrix}\right]$$

$$= \begin{bmatrix} p_{1,1} & p_{1,2} & 0 & 0 & p_{1,5} & p_{1,6} & p_{1,7} & p_{1,8} & 0 & 0 & 0 & p_{1,12} \\ & p_{2,2} & 0 & 0 & p_{2,5} & p_{2,6} & p_{2,7} & p_{2,8} & 0 & 0 & 0 & p_{2,12} \\ & & p_{3,3} & p_{3,4} & p_{3,5} & p_{3,6} & p_{3,7} & p_{3,8} & p_{3,9} & p_{3,10} & p_{3,11} & p_{3,12} \\ & & & p_{4,4} & p_{4,5} & p_{4,6} & p_{4,7} & p_{4,8} & p_{4,9} & p_{4,10} & p_{4,11} & p_{4,12} \\ & & & & p_{5,5} & p_{5,6} & p_{5,7} & p_{5,8} & p_{5,9} & p_{5,10} & p_{5,11} & p_{5,12} \\ & & & & & p_{6,6} & p_{6,7} & p_{6,8} & p_{6,9} & p_{6,10} & p_{6,11} & p_{6,12} \\ & & & & & & p_{7,7} & p_{7,8} & p_{7,9} & p_{7,10} & p_{7,11} & p_{7,12} \\ & & & & & & & p_{8,8} & p_{8,9} & p_{8,10} & p_{8,11} & p_{8,12} \\ & & & & & & & & p_{9,9} & p_{9,10} & p_{9,11} & p_{9,12} \\ & & & & & & & & & p_{10,10} & p_{10,11} & p_{10,12} \\ & & & & & & & & & & p_{11,11} & 0 \\ & & & & & & & & & & & p_{12,12} \end{bmatrix}$$

Only stating the elements on and above the diagonal. We first notice that $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$ can be computed as:

```
E_eps3 = zeros(ne × ne × ne, 1)
index = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        for phi3 = 1 : ne
            index = index + 1
            if phi1 == phi2  &&  phi1 == phi3
                E_eps3(index, 1) = m^3 (ε_{t+1} (phi1))
            end
        end
    end
end
```

We next compute all the elements in this matrix. The method is illustrated below

1) for $p_{1,1}$
$E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right] = \mathbf{I}$

2) for $p_{1,2}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\right]=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Hence the quasi MATLAB codes are :

$E\_eps\_eps2 = zeros(ne,(ne)^2)$
$for\ phi1 = 1 : ne$
  $index2 = 0$
  $for\ phi2 = 1 : ne$
    $for\ phi3 = 1 : ne$
      $index2 = index2 + 1$
      $if\ (phi1 = phi2 = phi3)$
        $E\_eps\_eps2(phi1, index2) = m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
      end
    end
  end
end

3) for $p_{1,5}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\right]=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\right]$$

$$=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^s\right)'\right)\right]$$

$$=E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^s\right)'\right]$$

$$=\mathbf{I}\otimes E\left[\left(\mathbf{x}_t^s\right)'\right]$$

4) for $p_{1,6}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\right]$$

$$=E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

$$=\mathbf{I}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

5) for $p_{1,7}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[1\otimes\boldsymbol{\epsilon}_{t+1}\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right]$$

$$=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right]$$

$$=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}$$

6) for $p_{1,8}$

$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=$$

$$E\left[\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)'\right]$$

Thus, the quasi Matlab codes are

```
E_eps_xfepsxf = zeros(ne, nx × ne × nx)
for phi1 = 1 : ne
    index2 = 0
    for phi2 = 1 : ne
        for gama1 = 1 : nx
            for phi2 = 1 : ne
                for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 == phi2
                    E_eps_xfepsxf(phi1, index2) = E_xf_xf(gama1, gama2)
                end
            end
        end
    end
end
```

....

4.3 Method 2: Formulas for the first and second moments

This section computes first and second moments using a slightly different representation of the third-order system than stated above. (Basically, this was the first representation we considered for computing these moments). The advantage of this method is that it compared to Method 1 is less memory intensive because some of the matrix multiplications are done by hand.

We first recall that

$$\begin{aligned}\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right)&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\\&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

We also know that

$$\begin{aligned}\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

so

$$\begin{aligned}\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f&=\left(\mathbf{h_x}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\right.\\&\left.+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right)\end{aligned}$$

$$\begin{aligned}&=\mathbf{h_x}\mathbf{x}_t^f\otimes\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\mathbf{h_x}\mathbf{x}_t^f\otimes\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\mathbf{h_x}\mathbf{x}_t^f\otimes\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\mathbf{h_x}\mathbf{x}_t^f\otimes\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

$$
\begin{aligned}
&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\
&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\
&\\
&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\mathbf{u}_{t+1}
\end{aligned}
$$

where $\mathbf{u}_{t+1}\equiv\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$

$$
\begin{aligned}
&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\
&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\
&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)
\end{aligned}
$$

Thus we can construct the following extended system

$$
\begin{bmatrix}
\mathbf{x}_{t+1}^f\\
\mathbf{x}_{t+1}^s\\
\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\\
\mathbf{x}_{t+1}^{rd}\\
\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\\
\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{h_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\
\mathbf{0}_{n_x\times n_x} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3}\\
\mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x^3}\\
\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{h_x} & 2\tilde{\mathbf{H}}_{\mathbf{xx}} & \tilde{\mathbf{H}}_{\mathbf{xxx}}\\
\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) & \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x^2} & \mathbf{0}_{n_x^2\times n_x} & \left(\mathbf{h_x}\otimes\mathbf{h_x}\right) & \left(\mathbf{h_x}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\\
\mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \mathbf{0}_{n_x^3\times n_x} & \mathbf{0}_{n_x^3\times n_x^2} & \left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)
\end{bmatrix}
$$

$$
\times
\begin{bmatrix}
\mathbf{x}_t^f\\
\mathbf{x}_t^s\\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^{rd}\\
\mathbf{x}_t^f\otimes\mathbf{x}_t^s\\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
\mathbf{0}_{n_x\times 1}\\
\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\\
\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\\
+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\\
\mathbf{0}_{n_x^2\times 1}\\
\mathbf{0}_{n_x^3\times 1}+E\left[\mathbf{u}_{t+1}\right]
\end{bmatrix}
+
\begin{bmatrix}
\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\
\mathbf{0}_{n_x\times 1}\\
\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\\
\mathbf{0}_{n_x\times 1}\\
\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\\
\mathbf{u}_{t+1}-E\left[\mathbf{u}_{t+1}\right]
\end{bmatrix}
$$

$\Updownarrow$

$$
\mathbf{z}_{t+1}=\mathbf{c}+\mathbf{A}\mathbf{z}_t+\tilde{\boldsymbol{\xi}}_{t+1}
$$

Hence, $\tilde{\boldsymbol{\xi}}_{t+1}\equiv\mathbf{B}\boldsymbol{\xi}_{t+1}$. The expression for the controls are as before, i.e.

$$
\mathbf{y}_t=\mathbf{D}\mathbf{z}_t+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3
$$

The mean values are

$$
E\left[\mathbf{z}_t\right]=\left(\mathbf{I}_{3n_x+2n_x^2+n_x^3}-\mathbf{A}\right)^{-1}\mathbf{c}.
$$

$$
E\left[\mathbf{y}_t\right]=\mathbf{D}E\left[\mathbf{z}_t\right]+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3
$$

We showed above that

$$Var\left[\mathbf{z}_{t+1}\right]=\mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}'+\mathbf{B}Var\left[\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'+\mathbf{A}Cov\left[\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'+\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]\mathbf{A}'$$

which is equivalent to

$$Var\left[\mathbf{z}_{t+1}\right]=\mathbf{A}Var[\mathbf{z_t}]\,\mathbf{A}'+Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]+\mathbf{A}Cov\left[\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'+\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]\mathbf{A}'$$

We have already known how to compute the $\mathbf{A}Cov\left[\mathbf{z}_t,\boldsymbol{\xi}_{t+1}\right]\mathbf{B}'$ and $\mathbf{B}Cov\left[\boldsymbol{\xi}_{t+1},\mathbf{z}_t\right]\mathbf{A}'$. Hence we only need to compute $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]$. Recall from above that

$$\tilde{\boldsymbol{\xi}}_{t+1}\equiv\left[\begin{array}{c}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\ \mathbf{0}_{n_x\times 1}\\ \mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\\ \mathbf{0}_{n_x\times 1}\\ \left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\\ \mathbf{u}_{t+1}-E\left[\mathbf{u}_{t+1}\right]\end{array}\right]$$

where

$$\mathbf{v}\left(t+1\right)\equiv\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

and

$$\begin{aligned}\mathbf{u}_{t+1}\quad\equiv\quad&\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

Note that

$$E\left[\mathbf{u}_{t+1}\right]=\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$$

because $\boldsymbol{\epsilon}_{t+1}$ is iid, $E\left[\boldsymbol{\epsilon}_{t+1}\right]=\mathbf{0}$, and $E\left[\mathbf{x}_t^f\right]=\mathbf{0}$. Note also that $E\left[\mathbf{u}_{t+1}\right]$ can be coded directly as:

```
E_eps3 = zeros(ne × ne × ne, 1)
index = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        for phi3 = 1 : ne
            index = index + 1
            if phi1 == phi2  &&  phi1 == phi3
                E_eps3(index, 1) = m^3 (ϵ_{t+1} (phi1))
            end
        end
    end
end
```

Hence,

$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]=E[\tilde{\boldsymbol{\xi}}_{t+1}\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)']$

$$
= E\left[\begin{array}{c}
\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\
\mathbf{0}_{n_x\times 1} \\
\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right) \\
\mathbf{0}_{n_x\times 1} \\
\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1} \\
\mathbf{u}_{t+1}-E\left[\mathbf{u}_{t+1}\right]
\end{array}\right]
$$

$$
\times\left[\begin{array}{ccccc}
\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{1\times n_x} & \mathbf{v}\left(t+1\right)'-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)' & \mathbf{0}_{1\times n_x} &
\end{array}\right.
$$

$$
\left.\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)' \quad \mathbf{u}_{t+1}'-E\left[\mathbf{u}_{t+1}'\right]\right]
$$

$$
Var\left(\tilde{\boldsymbol{\xi}}_{t+1}\right)=E\left[\begin{array}{cccccc}
\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}' & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\
Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{31} & \mathbf{0}_{n_x^2\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36} \\
\mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} & \mathbf{0}_{n_x\times n_x^3} \\
Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{51} & \mathbf{0}_{n_x^2\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{53} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56} \\
Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{61} & \mathbf{0}_{n_x^3\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{63} & \mathbf{0}_{n_x\times n_x} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{65} & Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66}
\end{array}\right]
$$

where we have defined:

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13}\equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{v}\left(t+1\right)'-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}\equiv E[\left(\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{v}\left(t+1\right)'-vec\left(\mathbf{I}_{n_e}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15}\equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16}\equiv E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{u}_{t+1}'-E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$
\begin{aligned}
Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}\equiv {} & E[\left(\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right) \\
& \times\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]
\end{aligned}
$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}\equiv E[\left(\mathbf{v}\left(t+1\right)-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)\right)\left(\mathbf{u}_{t+1}'-E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$
\begin{aligned}
Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55}\equiv {} & E[\left(\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right) \\
& \times\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]
\end{aligned}
$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}\equiv E[\left(\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{u}_{t+1}'-E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66}\equiv E[\left(\mathbf{u}_{t+1}-E\left[\mathbf{u}_{t+1}\right]\right)\left(\mathbf{u}_{t+1}'-E\left[\mathbf{u}_{t+1}'\right]\right)]$$

We have already derived the expressions for $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{13}$ and $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{33}$, and we will now compute the remaining terms.

4.3.1 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15}$ has dimensions $n_x \times n_x^2$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{15} = E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$$= E[\sigma\boldsymbol{\eta}(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}_{t+1}'\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \sigma\boldsymbol{\eta}(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$$= E[\sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \sigma\boldsymbol{\eta}\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)']$$

$=$

1) $\sigma\boldsymbol{\eta}\left(\mathbf{I}_{n_e}\otimes E[\mathbf{x}_t^s]'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$

2) $+\sigma\boldsymbol{\eta}\left(\mathbf{I}_{n_e}\otimes\left(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

3) $+\sigma\boldsymbol{\eta}\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

Checking the dimensions:

Term 1: $(n_x\times n_e)(n_e\times n_e n_x)(n_x n_x\times n_e n_x)'$ ok

Term 2: $(n_x\times n_e)(n_e\times n_e n_x n_x)(n_x n_x\times n_e n_x n_x)'$ ok

Term 3: $(n_x\times n_e)(n_x n_x\times n_e)'$ ok

4.3.2 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16}$ has dimensions $n_x \times n_x^3$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{16} = E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\mathbf{u}_{t+1}']$$

$$\begin{aligned}
= E[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}(&\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})' \\
&+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})' + (\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')]
\end{aligned}$$

$$= \sigma\boldsymbol{\eta}E\left(\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$

$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

$$=\sigma\boldsymbol{\eta}E\left((1\otimes\boldsymbol{\epsilon}_{t+1})\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}'_{t+1}\right)\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\mathbf{0}$$
$$+\sigma\boldsymbol{\eta}E\left[(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\mathbf{0}$$
$$+\mathbf{0}+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

$$=\sigma\boldsymbol{\eta}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\right](\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right](\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

$$=\sigma\boldsymbol{\eta}\left(E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right](\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\sigma\boldsymbol{\eta}E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right](\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})']$$

Hence we only need to compute directly the terms $E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ and $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$.
We first note that
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

And
$$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}'_{t+1}\right)\right]$$
$$=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left\{\{\epsilon_{t+1}(1,\phi_3)\}_{\phi_3=1}^{n_e}\epsilon_{t+1}(1,\phi_4)\right\}_{\phi_4=1}^{n_e}\right)\right]$$
$$=E\left[\begin{array}{c}\epsilon_{t+1}(\phi_1,1)\\ \epsilon_{t+1}(\phi_2,1)\\ \dots\\ \epsilon_{t+1}(\phi_4,1)\end{array}\left(\left\{\epsilon_{t+1}(1,\phi_2)\left\{\{\epsilon_{t+1}(1,\phi_3)\}_{\phi_3=1}^{n_e}\epsilon_{t+1}(1,\phi_4)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)\right]$$

Thus, the quasi Matlab codes are
$E\_eps\_eps3 = zeros(ne, (ne)^3)$
$for\ phi1 = 1 : ne$
  $index2 = 0$
  $for\ phi2 = 1 : ne$
    $for\ phi3 = 1 : ne$
      $for\ phi4 = 1 : ne$

```
% second moments
if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
      E_eps_eps3(phi1, index2) = 1
elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
      E_eps_eps3(phi1, index2) = 1
elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
      E_eps_eps3(phi1, index2) = 1
% fourth moments
elseif(phi1 == phi2 && phi1 == phi3 && phi1 == phi4)
      E_eps_eps3(phi1, index2) = m^4 (ε_{t+1} (phi1))
end
    end
  end
 end
end
```

4.3.3 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}$ has dimensions $n_x^2 \times n_x^2$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{35}$$

$$\equiv E[(\mathbf{v}(t+1) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e}))\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[\left((\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right) + (\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right) + (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}) - (\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})\right)$$
$$\times\left((\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})' + \left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)' + \boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[$$
$$(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e})\,\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$]$$

$$= E[$$

$$(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$-0$$
$$-0$$
$$-0$$
$$]$$

$$=E[$$
$$(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+0$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+0$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$]$$

$$=E[$$
$$(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes\mathbf{x}_t^f(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$
$$]$$

Note that

$$(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$=(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\otimes 1)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \boldsymbol{\epsilon}'_{t+1} \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\boldsymbol{\epsilon}'_{t+1} \otimes 1) \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$= (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left((\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \right) \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

So

$Var\left[\boldsymbol{\xi}_{t+1}\right]_{35} =$

1) $(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)'\right](\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

2) $+(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

3) $+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\mathbf{I}_{n_e} \otimes E\left[\mathbf{x}_t^f (\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

4) $+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})\left(\mathbf{I}_{n_e} \otimes E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

5) $+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)'\right](\sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$

6) $+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(E\left[(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right] \otimes E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

7) $+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})\left(E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right]\right)\left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

Checking the dimensions:

Term 1: $(n_x n_x \times n_x n_e)(n_x n_e \times n_e n_x)(n_e n_x \times n_x n_x)$ *ok*

Term 2: $(n_x n_x \times n_x n_e)\left(n_x n_e \times n_e n_x^2\right)\left(n_e n_x^2 \times n_x^2\right)$ *ok*

Term 3: $(n_x n_x \times n_e n_x)(n_e n_x \times n_x n_e)(n_e n_x \times n_x n_x)$ *ok*

Term 4: $\left(n_x^2 \times n_e n_x\right)(n_e n_x \times n_e n_x n_x)(n_e n_x n_x \times n_x n_x)$ *ok*

Term 5: $\left(n_x^2 \times n_e^2\right)\left(n_e^2 \times n_e n_x\right)(n_e n_x \times n_x n_x)$ *ok*

Term 6: $\left(n_x^2 \times n_e^2\right)\left(n_e^2 \times n_e n_x^2\right)(n_e n_x n_x \times n_x n_x)$ *ok*

Term 7: $\left(n_x^2 \times n_e^2\right)\left(n_e^2 \times n_e\right)(n_e \times n_x n_x)$ *ok*

We then need to show how to compute the following matrices

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)'\right] = E\left[\left\{x_t^f(\gamma_1, 1)\,\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2, 1)\,\mathbf{x}_t^s\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2, 1)\left\{x_t^s(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are

```
E_xfeps_epsxs = zeros(nx ne, ne nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2
                    E_xfeps_epsxs (index1, index2) = E_xf_xs(gama1, gama2)
                end
            end
```

end
end
end
where $E\_xf\_xs = E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^s\right)'\right] = reshape(\left(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right]\right)', nx, nx)$. This is so because

$$E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right] = E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right]$$

$$= E\begin{bmatrix} x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x} \\ x_t^f(\gamma_2,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x} \\ \dots \\ x_t^f(n_e,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x} \end{bmatrix}$$

So simply doing (for a 2 by 2 matrix)
$reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right], nx, nx) = E\begin{bmatrix} x^f(1,1)\,x^s(1,1) & x^f(2,1)\,x^s(1,1) \\ x^f(1,1)\,x^s(2,1) & x^f(2,1)\,x^s(2,1) \end{bmatrix}$
and we therefore need to transpose $E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right]$ in the expression above.

And
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x^2}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfeps\_epsxfxf = zeros(nxne, ne\,(nx)^2)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfeps\_epsxfxf(index1, index2) = E\_xf\_xf\_xf(gama1, gama2, gama3)$
end
end
end
end
end
end
where $E\_xf\_xf\_xf = reshape(\left(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right]\right), nx, nx, nx)$

And
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\right]$

$$= E\left[\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_eps2_epsxs = zeros(n_e n_e, n_e n_x)
index1 = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for phi3 = 1 : ne
            for gama1 = 1 : nx
                index2 = index2 + 1
                if phi1 = phi2 && phi1 == phi3
                    E_eps2_epsxs (index1, index2) = E_xs(gama1, 1) × m^3 (ϵ_{t+1} (phi1))
                end
            end
        end
    end
end
```

Finally:
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right] = E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon'_{t+1}(1,\phi_2)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$

$$= E\left[\left\{\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon'_{t+1}(1,\phi_2)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right]$$

Thus the quasi Matlab codes are:

```
E_eps2_eps = zeros((ne)^2 , ne)
for phi2 = 1 : ne
    index1 = 0
    for phi1 = 1 : ne
        for phi3 = 1 : ne
            index1 = index1 + 1
            if phi1 == phi2 && phi1 == phi3
                E_eps2_eps(index1, phi2) = m^3 (ϵ_{t+1} (phi1))
            end
        end
    end
end
```

4.3.4 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}$ has dimensions $n_x^2\times n_x^3$.

$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}\equiv E[(\mathbf{v}(t+1)-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec(\mathbf{I}_{n_e}))\left(\mathbf{u}'_{t+1}-E\left[\mathbf{u}'_{t+1}\right]\right)]$

$$
\begin{aligned}
&= E[\Big(\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) - \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\Big) \\
&\quad \times \left(\mathbf{u}'_{t+1} - E\left[\mathbf{u}'_{t+1}\right]\right)]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\Big(\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) - \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\Big)\mathbf{u}'_{t+1}] \\
{}' \quad & -E[\Big(\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) - \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\Big) E\left[\mathbf{u}'_{t+1}\right]]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\Big(\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) - \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\Big)\mathbf{u}'_{t+1}] \\
{}' \quad & -E[\left(0+0+0\right) E\left[\mathbf{u}'_{t+1}\right]]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)\mathbf{u}'_{t+1} \\
&\quad + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)\mathbf{u}'_{t+1} \\
&\quad + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\mathbf{u}'_{t+1} \\
&\quad - \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right)\mathbf{u}'_{t+1}]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)\mathbf{u}'_{t+1} \\
&\quad + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)\mathbf{u}'_{t+1} \\
&\quad + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\mathbf{u}'_{t+1}] \\
&\quad - \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right) E\left[\mathbf{u}'_{t+1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\\
&\quad \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)(\Big(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)' \\
&\qquad + \Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)'\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)' + \Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)' \\
&\qquad + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\Big)'\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)' + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}\right)' \\
&\qquad + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)'\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}\right)' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)')
\end{aligned}
$$

$$
\begin{aligned}
&\quad \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)(\Big(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)' \\
&\qquad + \Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)'\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)' + \Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)' \\
&\qquad + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\Big)'\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)' + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}\right)' \\
&\qquad + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)'\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}\right)' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)')
\end{aligned}
$$

$$
\begin{aligned}
&\quad \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)(\Big(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)' \\
&\qquad + \Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)'\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)' + \Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)' \\
&\qquad + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\Big)'\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)' + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}\right)' \\
&\qquad + \Big(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\Big)'\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}\right)' + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)')]
\end{aligned}
$$

$$
- \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right) vec\left(\mathbf{I}_{n_e}\right) E\left[\mathbf{u}'_{t+1}\right]
$$

$$
\begin{aligned}
&= E[\\
&\quad \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\Big(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)(\Big(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\Big)'\left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)'
\end{aligned}
$$

$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+0)$$

$$(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+0)$$

$$(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+0$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+0$$
$$+0+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')]$$

$$-(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\,vec\,(\mathbf{I}_{n_e})\,E\left[\mathbf{u}'_{t+1}\right]$$

$$=E[$$
$$(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}'_{t+1}\right)(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f\right)'\otimes(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\right)(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})']$$

$$- (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \, vec\left(\mathbf{I}_{n_e}\right) E\left[\mathbf{u}_{t+1}'\right]$$

$$= E[$$

$$(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\boldsymbol{\epsilon}_{t+1}\right) (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \left(\mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'\right) (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

$$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' \otimes \left(\mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}' \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' \otimes \mathbf{x}_t^f \left(\mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (1 \otimes (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})) \left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \otimes \boldsymbol{\epsilon}_{t+1}'\right) (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) ((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \otimes 1) \left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right) (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\boldsymbol{\epsilon}_{t+1}' \otimes \boldsymbol{\epsilon}_{t+1}'\right)\right) (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})']$$

$$- (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \, vec\left(\mathbf{I}_{n_e}\right) E\left[\mathbf{u}_{t+1}'\right]$$

$$=$$

1) $$(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] \otimes \mathbf{I}_{n_e}\right) (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})'$$

2) $$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})'$$

3) $$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) \left(E\left[\mathbf{x}_t^f \left(\mathbf{x}_t^f\right)'\right] \otimes E\left[\boldsymbol{\epsilon}_{t+1} (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'\right]\right) (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'$$

4) $$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1}' \otimes \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right)\right] (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})'$$

5) $$+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}) E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})'$$

6) $$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\right]\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$$

7) $$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$

8) $$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$

9) $$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

10) $$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$

11) $$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$$

12) $$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$$

13) $$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right]\right)\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$$

14) $$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$$

15) $$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right]\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$$

16 $$+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}'\otimes\left(\boldsymbol{\epsilon}_{t+1}'\otimes\boldsymbol{\epsilon}_{t+1}'\right)\right)\right]\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)'$$

17) $$-\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)vec\left(\mathbf{I}_{n_e}\right)E\left[\mathbf{u}_{t+1}'\right]$$

Hence, we need to compute the remaining matrices directly. This is done below where the number relates to the row in the expression for $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{36}$

1) None

2)

$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$=E\left[\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfeps_xfepsxf = zeros(nx × ne, nx × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfeps_xfepsxf(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)
```

end
end
end
end
end
end
where $E\_xf\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx, nx)$

3)
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}'_{t+1}\right] = \left(E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)'$
but $E\left[\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]$ is already computed

4)
$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfeps\_epsxfxf = zeros(nx\times ne, ne\times nx\times nx)$
$index1 = 0$
$for\ gama1 = 1:nx$
$for\ phi1 = 1:ne$
$index1 = index1+1$
$index2 = 0$
$for\ phi2 = 1:ne$
$for\ gama2 = 1:nx$
$for\ gama3 = 1:nx$
$index2 = index2+1$
$if\ phi1 == phi2$
$E\_xfeps\_epsxfxf(index1,index2) = E\_xf\_xf\_xf(gama1,gama2,gama3)$
end
end
end
end
end
end
where $E\_xf\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx, nx)$

5)
$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfeps\_epsxfeps = zeros(nx \times ne, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
  $for\ phi1 = 1 : ne$
    $index1 = index1 + 1$
    $index2 = 0$
    $for\ phi2 = 1 : ne$
      $for\ gama2 = 1 : nx$
        $for\ phi3 = 1 : ne$
          $index2 = index2 + 1$
          $if\ phi1 == phi2\ \&\&\ phi1 == phi3$
            $E\_xfeps\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^3(\epsilon_{t+1}(phi1))$
          end
        end
      end
    end
  end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

6)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left((\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' \otimes \left(\mathbf{x}_t^f\right)'\right)\right]$

$$= E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$
$$= E\left[\left(\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2, 1)\left\{\epsilon_{t+1}(\phi_3, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfeps\_eps2xf = zeros(nx \times ne, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
  $for\ phi1 = 1 : ne$
    $index1 = index1 + 1$
    $index2 = 0$
    $for\ phi2 = 1 : ne$
      $for\ phi3 = 1 : ne$
        $for\ gama2 = 1 : nx$
          $index2 = index2 + 1$
          $if\ phi1 == phi2\ \&\&\ phi1 == phi3$
            $E\_xfeps\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^3(\epsilon_{t+1}(phi1))$
          end
        end
      end
    end
  end
end
where $E\_xf\_xf = reshape(E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right], nx, nx)$

7)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_xfxfeps = zeros(ne × nx, nx × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxf_xfxfeps(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

8)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_3,1\right)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_xfepsxf = zeros(ne × nx, nx × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxf_xfepsxf(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
```

end

9)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_xfeps2 = zeros(ne × nx, nx × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_epsxf_xfeps2(index1, index2) = E_xf_xf(gama1, gama2) × m^3 (ϵ_{t+1} (phi1))
                    end
                end
            end
        end
    end
end
```

10)

$$\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)$$

where $E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=reshape(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right],nx,(nx)^2)$

11)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxf_epsxfeps = zeros(ne × nx, ne × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                for phi3 = 1 : ne
```

$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_epsxf\_xfeps2(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end

12) none

13)
$\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] \otimes E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right]\right)$
Here we only need to compute
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}_{t+1}'\right] = E\left[\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\left\{\epsilon_{t+1}'\left(1, \phi_3\right)\right\}_{\phi_3=1}^{n_e}\right]$
Thus the quasi Matlab codes are:
$E\_eps2\_eps = zeros(ne \times ne, ne)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_eps2\_eps(index1, index2) = m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
But we already know E_eps2_eps from previous derivations.
14)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_eps2\_xfepsxf = zeros(ne \times ne, nx \times ne \times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama1 = 1 : nx$
$for\ phi3 = 1 : ne$

$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_eps2\_xfepsxf(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end

15)
$$\left(E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1}\right] \otimes E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]\right)$$
None since we know $E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\boldsymbol{\epsilon}'_{t+1}\right]$

16)
$$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}'_{t+1} \otimes \left(\boldsymbol{\epsilon}'_{t+1} \otimes \boldsymbol{\epsilon}'_{t+1}\right)\right)\right]$$

$$= E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{\epsilon_{t+1}\left(\phi_4, 1\right)\left\{\epsilon_{t+1}\left(\phi_5, 1\right)\right\}_{\phi_5=1}^{n_e}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_eps2_eps3 = zeros(ne*ne,ne*ne*ne);
index1 = 0;
    for phi1=1:ne
        for phi2=1:ne
        index1 = index1 + 1;
        index2 = 0;
        for phi3=1:ne
            for phi4=1:ne
                for phi5=1:ne
                    index2 = index2 + 1;
                    % Second order moments times third order moments
                    if phi1 == phi2 & & phi2 == phi3 & & phi4 == phi5 & & phi1 ~= phi4
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi3 & & phi3 == phi4 & & phi2 == phi5 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi4 & & phi4 == phi5 & & phi2 == phi3 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi3 == phi4 & & phi4 == phi5 & & phi1 == phi2 & & phi1 ~= phi3
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi3);
                    elseif phi2 == phi3 & & phi3 == phi4 & & phi1 == phi5 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi2);
                    elseif phi1 == phi3 & & phi3 == phi5 & & phi2 == phi4 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi2 & & phi1 == phi4 & & phi3 == phi5 & & phi1 ~= phi3
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
                    elseif phi1 == phi2 & & phi1 == phi5 & & phi3 == phi4 & & phi1 ~= phi3
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi1);
```

```
                    elseif phi2 == phi4 & & phi2 == phi5 & & phi1 == phi3 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi2);
                    elseif phi2 == phi3 & & phi2 == phi5 & & phi1 == phi4 & & phi1 ~= phi2
                    E_eps2_eps3(index1,index2) = vectorMom3(1,phi2);
                    % Fifth order moments
                    elseif phi1 == phi2 & & phi2 == phi3 & & phi3 == phi4 & & phi4 == phi5
                        E_eps2_eps3(index1,index2) = vectorMom5(1,phi1);
                    end
                end
            end
        end
    end
end
```

17) none

4.3.5 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55}$ has dimensions $n_x^2 \times n_x^2$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{55} \equiv E[\left(\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$\times\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)]$$

$$= E[$$
$$\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'+\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'\right)$$
$$]$$

$$= E[$$
$$\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

]

$$= E[$$

$$(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes 1\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes 1\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

]

$$= E[$$

$$(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\mathbf{x}_t^s(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\mathbf{x}_t^s\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes(\mathbf{x}_t^s)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

]

$$=(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$$

$$=$$

1) $(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$

2) $+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

3) $+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})\left(\mathbf{I}_{n_e}\otimes E\left[\mathbf{x}_t^s\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)')$

4) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$

5) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

6) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

7) $+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[(\mathbf{x}_t^s)'\right]\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$

8) $+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)'$

9) $+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)'$

Note here that we already know $E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$ and $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$ from the variance of the states using a second order approximation. This is because

$$Var\left[\mathbf{x}_t^s\right]=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)'\right]$$
$$=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left((\mathbf{x}_t^s)'-E\left[\mathbf{x}_t^s\right]'\right)\right]$$
$$=E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'-\mathbf{x}_t^sE\left[\mathbf{x}_t^s\right]'-E\left[\mathbf{x}_t^s\right](\mathbf{x}_t^s)'+E\left[\mathbf{x}_t^s\right]E\left[\mathbf{x}_t^s\right]'\right]$$
$$=E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]-E\left[\mathbf{x}_t^s\right]E\left[\mathbf{x}_t^s\right]'$$
$$\Updownarrow$$
$$Var\left[\mathbf{x}_t^s\right]+E\left[\mathbf{x}_t^s\right]E\left[\mathbf{x}_t^s\right]'=E\left[\mathbf{x}_t^s(\mathbf{x}_t^s)'\right]$$

and

$$Var\left(\mathbf{x}_t^s,\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)'\right]$$
$$=E\left[\left(\mathbf{x}_t^s-E\left[\mathbf{x}_t^s\right]\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\right]$$
$$=E[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'-\mathbf{x}_t^sE\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]-E\left[\mathbf{x}_t^s\right]\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'+E\left[\mathbf{x}_t^s\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]]$$
$$=E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]-E\left[\mathbf{x}_t^s\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$
$$\Updownarrow$$
$$Var\left(\mathbf{x}_t^s,\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)+E\left[\mathbf{x}_t^s\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

and

$$Var\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]=E\left[\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)'\right]$$

$$=E\left[\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\right)\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\right]$$

$$=E[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'-\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$
$$-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'+E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]]$$

$$=E[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)']-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$
$$-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]+E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$

$$=E[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)']-E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$$
$$\Updownarrow$$
$$Var\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]+E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right]E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)']$$

4.3.6 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}$ has dimensions $n_x^2\times n_x^3$.

$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{56}$

$$=E[\left((\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{u}'_{t+1}-E\left[\mathbf{u}'_{t+1}\right]\right)]$$

$$=E[\left((\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)\mathbf{u}'_{t+1}$$
$$-\left((\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right)E\left[\mathbf{u}'_{t+1}\right]]$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\mathbf{u}'_{t+1}+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\mathbf{u}'_{t+1}+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\mathbf{u}'_{t+1}$$
$$-(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)E\left[\mathbf{u}'_{t+1}\right]-\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)E\left[\mathbf{u}'_{t+1}\right]-\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}E\left[\mathbf{u}'_{t+1}\right]]$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\mathbf{u}'_{t+1}+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\mathbf{u}'_{t+1}+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\mathbf{u}'_{t+1}]$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\mathbf{u}'_{t+1}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\mathbf{u}'_{t+1}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\mathbf{u}'_{t+1}]$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

using the definition of $\mathbf{u}_{t+1}$ where

$$\mathbf{u}_{t+1}'\equiv\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

$$=E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'\otimes\left(\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})'$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(1\otimes\boldsymbol{\epsilon}_{t+1})\left(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\otimes\boldsymbol{\epsilon}'_{t+1}\right)(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}})'$$
$$+0$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(\boldsymbol{\epsilon}_{t+1}\otimes 1)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}})'$$

$$
\begin{aligned}
&+0 \\
&+0 \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)'\big) \\
&]
\end{aligned}
$$

$$
\begin{aligned}
= E[&(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)' \\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)' \\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)' \\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)' \\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)' \\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)' \\
&+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes 1\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)' \\
\\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}'_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes 1\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right) \\
\\
&+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)' \\
&+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)'\big) \\
]&
\end{aligned}
$$

$$
\begin{aligned}
1)\quad & = E[(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)' \\
2)\quad & \qquad +(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)' \\
3)\quad & \qquad +(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)' \\
4)\quad & \qquad +(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{I}_{n_e}\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)' \\
5)\quad & \qquad +(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)' \\
6)\quad & \qquad +(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'
\end{aligned}
$$

7) $+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\mathbf{x}_t^s\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)'$

8) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$

9) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$

10) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)'$

11) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{I}_{n_e}\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$

12) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)'$

13) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x}\right)'$

14) $+\left(\sigma\boldsymbol{\eta}\otimes\tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)$

15) $+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\otimes\mathbf{I}_{n_e}\right)\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)'$

16) $+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)'$

17) $+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\mathbf{I}_{n_e}\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)'$

18) $+\left(\sigma\boldsymbol{\eta}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta}\right)')$

]

We thus need to explain how to compute each of these terms

1)

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_xfxfeps = zeros(ne × nx, nx × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxs_xfxfeps(index1, index2) = E_xs_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
```

end
end
where $E\_xs\_xf\_xf = reshape(E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right], nx, nx, nx)$

2)
$E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_xfepsxf = zeros(ne × nx, nx × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_epsxs_xfepsxf(index1, index2) = E_xs_xf_xf(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

3)
$E\left[(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^s(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxs_xfeps2 = zeros(ne × nx, nx × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2 && phi1 == phi3
                        E_epsxs_xfeps2(index1, index2) = E_xs_xf(gama1, gama2) × m^3 (ε_{t+1} (phi1))
```

end
end
end
end
end
end

4)
None

5)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^s\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_epsxs\_epsxfeps = zeros(ne\times nx, ne\times nx\times ne)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$\quad for\ gama1 = 1 : nx$
$\quad\quad index1 = index1 + 1$
$\quad\quad index2 = 0$
$\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$\quad\quad\quad\quad\quad\quad E\_epsxs\_epsxfeps(index1, index2) = E\_xs\_xf(gama1, gama2)\times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$
$\quad\quad end$
$\quad end$
end

6)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right)\right]=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

7)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\mathbf{x}_t^s\right)\right]=E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\mathbf{x}_t^s\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

8)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$=E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_3,1\right)\left\{x_t^f\left(\gamma_4,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_xfxfeps = zeros(ne × nx × nx, nx × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for gama4 = 1 : nx
                    for phi2 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2
                        E_epsxfxf_xfxfeps(index1, index2) = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

where $E\_xf\_xf\_xf\_xf = reshape(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right], nx, nx, nx, nx)$

9)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_3, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\left\{x_t^f(\gamma_4, 1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_xfepsxf = zeros(ne × nx × nx, nx × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if phi1 == phi2
                        E_epsxfxf_xfepsxf(index1, index2) = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

10)

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_xfeps2 = zeros(ne × nx × nx, nx × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    for phi3 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2 && phi1 == phi3
                        E_epsxfxf_xfeps2(index1, index2) = E_xf_xf_xf(gama1, gama2, gama3)×m^3 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

11)

None as $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$ is known

12)

$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_epsxfeps = zeros(ne × nx × nx, ne × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for phi3 = 1 : ne
                        index2 = index2 + 1
```

$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_epsxfxf\_epsxfeps(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

13)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right)\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ and $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right] = reshape(E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right], (nx)^2, nx)$

14)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\otimes\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\right] = E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]\otimes E\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]$
Note that we already know $E\left[\boldsymbol{\epsilon}_{t+1}\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

15) None

16)
$E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right)\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_eps\_xfepsxf = zeros(ne, nx \times ne \times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama1 = 1 : nx$
$for\ phi2 = 1 : ne$
$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_eps\_xfepsxf(index1, index2) = E\_xf\_xf(gama1, gama2)$
end
end
end
end
end

17)
None

18)
None

4.3.7 For $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66}$

Note first that $Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66}$ has dimensions $n_x^3 \times n_x^3$.

$$Var\left[\tilde{\boldsymbol{\xi}}_{t+1}\right]_{66} = E[(\mathbf{u}_{t+1} - E\left[\mathbf{u}_{t+1}\right])\left(\mathbf{u}_{t+1}' - E\left[\mathbf{u}_{t+1}'\right]\right)]$$

$$= E[\mathbf{u}_{t+1}\mathbf{u}_{t+1}' - \mathbf{u}_{t+1}E\left[\mathbf{u}_{t+1}'\right] - E\left[\mathbf{u}_{t+1}\right]\mathbf{u}_{t+1}' + E\left[\mathbf{u}_{t+1}\right]E\left[\mathbf{u}_{t+1}'\right]]$$

$$= E\left[\mathbf{u}_{t+1}\mathbf{u}_{t+1}'\right] - E\left[\mathbf{u}_{t+1}\right]E\left[\mathbf{u}_{t+1}'\right]$$

We already know $E\left[\mathbf{u}_{t+1}\right]$ so we only need to compute the first term. Hence

$$E\left[\mathbf{u}_{t+1}\mathbf{u}_{t+1}'\right] = E[$$
$$((\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)$$
$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)$$
$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}))$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

$$= E[$$
$$(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$

$$
\begin{aligned}
&\quad + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) \\
&\quad (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&\quad + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&+ (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&\quad (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&\quad + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\
&\quad (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&\quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&\quad + (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})' (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&\quad (\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&\quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)' (\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'
\end{aligned}
$$

$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
$$]$$

$$=E[$$
$$(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})$$
$$(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'$$
$$+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$

$$+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})$$

$$
\begin{aligned}
&\Big(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})'\Big)
\end{aligned}
$$

$$
\begin{aligned}
+&(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \\
&\Big(\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&+0\Big)
\end{aligned}
$$

$$
\begin{aligned}
+&(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \\
&\Big(\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})'\Big)
\end{aligned}
$$

$$
\begin{aligned}
+&(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \\
&\Big(\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&+\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})'
\end{aligned}
$$

$+0)$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})\\
&\quad(\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&\quad+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&\quad+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\\
&\quad+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'\\
&\quad+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'\\
&\quad+\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'\\
&\quad+0)
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})\\
&\quad((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
&\quad+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
&\quad+0\\
&\quad+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'\\
&\quad+0\\
&\quad+0\\
&\quad+(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')\\
&]
\end{aligned}
$$

$$
\begin{aligned}
&=E[\\
&\quad(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})\\
1)&\qquad(\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
2)&\qquad+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
3)&\qquad+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\\
4)&\qquad+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'\\
5)&\qquad+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'\\
6)&\qquad+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})'\\
7)&\qquad+\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&\quad+(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\\
8)&\qquad(\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'\\
9)&\qquad+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'\\
10)&\qquad+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta})'\\
11)&\qquad+\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'
\end{aligned}
$$

$$
\begin{aligned}
&12) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&13) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&14) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&+(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}) \\
&15) \quad (\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&16) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&17) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&18) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&19) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&20) \quad + \left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})')
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}) \\
&21) \quad (\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&22) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&23) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&24) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&25) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&26) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})' \\
&27) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta})')
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}) \\
&28) \quad (\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&29) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&30) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})' \\
&31) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x})' \\
&32) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta})' \\
&33) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x})')
\end{aligned}
$$

$$
\begin{aligned}
&+(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}) \\
&34) \quad (\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta})' \\
&35) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x})' \\
&36) \quad + \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta})'
\end{aligned}
$$

$$37) \quad +\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$38) \quad +\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta})'$$
$$39) \quad +\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\mathbf{h_x})')$$

$$+(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})$$
$$40) \quad ((\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta})'$$
$$41) \quad +(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})'$$
$$42) \quad +(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x})'$$
$$43) \quad +(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1})'(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\otimes\boldsymbol{\sigma\eta})')$$
]

We next derive how to compute the moments in these terms.

1)
$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_xfxfeps = zeros(nx × nx × ne, nx × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for gama3 = 1 : nx
                for gama4 = 1 : nx
                    for phi2 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2
                        E_xfxfeps_xfxfeps(index1, index2) = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

2)
$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

$E\_xfxfeps\_xfepsxf = zeros(nx \times nx \times ne, nx \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$\quad for\ gama2 = 1 : nx$
$\quad\quad for\ phi1 = 1 : ne$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad\quad for\ gama4 = 1 : nx$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ phi1 == phi2$
$\quad\quad\quad\quad\quad\quad\quad E\_xfxfeps\_xfepsxf(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf\_xf(gama1, gama2, gama3, gama4)$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$
$\quad\quad end$
$\quad end$
end

3)

$$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)'\right]$$

Thus the quasi Matlab codes are:

$E\_xfxfeps\_xfeps2 = zeros(nx \times nx \times ne, nx \times ne \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$\quad for\ gama2 = 1 : nx$
$\quad\quad for\ phi1 = 1 : ne$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$\quad\quad\quad\quad\quad\quad\quad E\_xfxfeps\_xfeps2(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}(phi1)\right)$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$

end
end
end

4)
$$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_epsxfxf = zeros(nx × nx × ne, ne × nx × nx)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_xfxfeps_epsxfxf(index1, index2)
                            = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

5)
$$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E\left[\left(\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_1,1)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_epsxfeps = zeros(nx × nx × ne, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for phi3 = 1 : ne
```

$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfxfeps\_epsxfeps(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

6)
$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E\left[\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{x_t^f\left(\gamma_3,1\right)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfxfeps\_eps2xf = zeros(nx \times nx \times ne, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ phi3 = 1 : ne$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfxfeps\_eps2xf(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

7)
$E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1}\left(\phi_1,1\right)\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{\epsilon_{t+1}\left(\phi_4,1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfxfeps_eps3 = zeros(nx × nx × ne, ne × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_xfxfeps_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_xfxfeps_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_xfxfeps_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif(phi1 == phi2 && phi1 == phi3 && phi1 == phi4)
                            E_xfxfeps_eps3(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

8)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]'$$

where we aldready know $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$ from 2).

9)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$$

$$= E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right) \times \left(\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_xfepsxf = zeros(nx × ne × nx, nx × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
```

$index1 = index1 + 1$
$index2 = 0$
$for\ gama3 = 1 : nx$
$for\ phi2 = 1 : ne$
$for\ gama4 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xfepsxf\_xfepsxf(index1, index2)$
$= E\_xf\_xf\_xf\_xf(gama1, gama2, gama3, gama4)$
end
end
end
end
end
end
end

10)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{x_t^f(\gamma_3, 1)\left\{\epsilon_{t+1}(\phi_2, 1)\left\{\epsilon_{t+1}(\phi_3, 1)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right)']$$

Thus the quasi Matlab codes are:
$E\_xfepsxf\_xfeps2 = zeros(nx \times ne \times nx, nx \times ne \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ gama3 = 1 : nx$
$for\ phi2 = 1 : ne$
$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfepsxf\_xfeps2(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3(\epsilon_{t+1}(phi1))$
end
end
end
end
end
end
end

11)

$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]$

$$= E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{x_t^f(\gamma_4,1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_epsxfxf = zeros(nx × ne × nx, ne × nx × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_xfepsxf_epsxfxf(index1, index2)
                            = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

12)

$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{x_t^f(\gamma_3,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_epsxfeps = zeros(nx × ne × nx, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
```

$for\ phi3 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfepsxf\_epsxfeps(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
end
end
end
end
end
end
end

13)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E[\left(\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times \left(\left(\left\{\epsilon_{t+1}(\phi_2, 1)\left\{\epsilon_{t+1}(\phi_3, 1)\left\{x_t^f(\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)'\right]$$

Thus the quasi Matlab codes are:
$E\_xfepsxf\_eps2xf = zeros(nx \times ne \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi2 = 1 : ne$
$for\ phi3 = 1 : ne$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfepsxf\_eps2xf(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3 (\epsilon_{t+1} (phi1))$
end
end
end
end
end
end
end

14)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{x_t^f(\gamma_1, 1)\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$

$$\times\left(\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfepsxf_eps3 = zeros(nx × ne × nx, ne × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1˜ = phi4)
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1˜ = phi2)
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1˜ = phi2)
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        if phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_xfepsxf_eps3(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

15)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 3).

16)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 10)

17)

$$E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$

$$\times \left(\left\{ x_t^f (\gamma_2, 1) \left\{ \epsilon_{t+1} (\phi_3, 1) \left\{ \epsilon_{t+1} (\phi_4, 1) \right\}_{\phi_4=1}^{n_e} \right\}_{\phi_3=1}^{n_e} \right\}_{\gamma_2=1}^{n_x} \right)']$$

Thus the quasi Matlab codes are:

```
E_xfeps2_xfeps2 = zeros(nx × ne × ne, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for gama2 = 1 : nx
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1˜ = phi4)
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1˜ = phi2)
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1˜ = phi2)
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_xfeps2_xfeps2(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

18)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right)' \right]$$

$$= E[\left(\left\{ x_t^f (\gamma_1, 1) \left\{ \epsilon_{t+1} (\phi_1, 1) \left\{ \epsilon_{t+1} (\phi_2, 1) \right\}_{\phi_2=1}^{n_e} \right\}_{\phi_1=1}^{n_e} \right\}_{\gamma_1=1}^{n_x} \right)$$

$$\times \left(\left\{ \epsilon_{t+1} (\phi_3, 1) \left\{ x_t^f (\gamma_2, 1) \left\{ x_t^f (\gamma_3, 1) \right\}_{\gamma_3=1}^{n_x} \right\}_{\gamma_2=1}^{n_x} \right\}_{\phi_3=1}^{n_e} \right)']$$

Thus the quasi Matlab codes are:

```
E_xfeps2_epsxfxf = zeros(nx × ne × ne, ne × nx × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
```

$for\ gama2 = 1 : nx$
$for\ gama3 = 1 : nx$
$index2 = index2 + 1$
$if\ (phi1 == phi2\ \&\&\ phi1 == phi3$
$E\_xfeps2\_epsxfxf(index1, index2)$
$= E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end
end
end

19)

$$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$= E[\left(\left\{x_t^f\left(\gamma_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_1, 1\right)\left\{\epsilon_{t+1}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right)$$
$$\times\left(\left\{\epsilon_{t+1}\left(\phi_3, 1\right)\left\{x_t^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1}\left(\phi_4, 1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:
$E\_xfeps2\_epsxfeps = zeros(nx \times ne \times ne, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index1 = index1 + 1$
$index2 = 0$
$for\ phi3 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ phi4 = 1 : ne$
$index2 = index2 + 1$
% second moments of innovations
$if\ (phi1 == phi2\ \&\&\ phi3 == phi4\ \&\&\ phi1\tilde{}= phi4)$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2)$
$elseif\ (phi1 == phi3\ \&\&\ phi2 == phi4\ \&\&\ phi1\tilde{}= phi2)$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2)$
$elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{}= phi2)$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2)$
% fourth moments of innovations
$elseif\ phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4$
$E\_xfeps2\_epsxfeps(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^4\left(\epsilon_{t+1}\left(phi1\right)\right)$
end
end
end
end
end

end
end

20)
$E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right]$

$$= E[\left(\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right) \times \left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_xfeps2_eps2xf = zeros(nx × ne × ne, ne × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for phi4 = 1 : ne
                    for gama2 = 1 : nx
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_xfeps2_eps2xf(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

21)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ from 4).

22)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]'$

where we already know $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ from 11)

23)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right]$ from 18)

24)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\right] =$

$$= E[\left(\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_2, 1)\left\{x_t^f(\gamma_3, 1)\left\{x_t^f(\gamma_4, 1)\right\}_{\gamma_4=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_epsxfxf = zeros(ne × nx × nx, ne × nx × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    for gama4 = 1 : nx
                        index2 = index2 + 1
                        if (phi1 == phi2
                            E_epsxfxf_epsxfxf(index1, index2)
                            = E_xf_xf_xf_xf(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

25)
$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)'\right] =$

$$= E[\left(\left\{\epsilon_{t+1}(\phi_1, 1)\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$

$$\times \left(\left\{ \epsilon_{t+1}(\phi_2, 1) \left\{ x_t^f(\gamma_3, 1) \left\{ \epsilon_{t+1}(\phi_3, 1) \right\}_{\phi_3=1}^{n_e} \right\}_{\gamma_3=1}^{n_x} \right\}_{\phi_2=1}^{n_e} \right)']$$

Thus the quasi Matlab codes are:

$E\_epsxfxf\_epsxfeps = zeros(ne \times nx \times nx, ne \times nx \times ne)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$\quad for\ gama1 = 1 : nx$
$\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ (phi1 == phi2\ \&\&\ phi1 == phi3$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfxf\_epsxfeps(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3(\epsilon_{t+1}(phi1))$
$\quad\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad\quad end$
$\quad\quad\quad\quad end$
$\quad\quad\quad end$
$\quad\quad end$
$\quad end$
end

26)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \right)' \right]$$

$$= E[\left(\left\{ \epsilon_{t+1}(\phi_1, 1) \left\{ x_t^f(\gamma_1, 1) \left\{ x_t^f(\gamma_2, 1) \right\}_{\gamma_2=1}^{n_x} \right\}_{\gamma_1=1}^{n_x} \right\}_{\phi_1=1}^{n_e} \right)$$
$$\times \left(\left\{ \epsilon_{t+1}(\phi_2, 1) \left\{ \epsilon_{t+1}(\phi_3, 1) \left\{ x_t^f(\gamma_3, 1) \right\}_{\gamma_3=1}^{n_x} \right\}_{\phi_3=1}^{n_e} \right\}_{\phi_2=1}^{n_e} \right)']$$

Thus the quasi Matlab codes are:

$E\_epsxfxf\_eps2xf = zeros(ne \times nx \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ phi1 = 1 : ne$
$\quad for\ gama1 = 1 : nx$
$\quad\quad for\ gama2 = 1 : nx$
$\quad\quad\quad index1 = index1 + 1$
$\quad\quad\quad index2 = 0$
$\quad\quad\quad for\ phi2 = 1 : ne$
$\quad\quad\quad\quad for\ phi3 = 1 : ne$
$\quad\quad\quad\quad\quad for\ gama3 = 1 : nx$
$\quad\quad\quad\quad\quad\quad index2 = index2 + 1$
$\quad\quad\quad\quad\quad\quad if\ (phi1 == phi2\ \&\&\ phi1 == phi3$
$\quad\quad\quad\quad\quad\quad\quad E\_epsxfxf\_eps2xf(index1, index2)$
$\quad\quad\quad\quad\quad\quad\quad = E\_xf\_xf\_xf(gama1, gama2, gama3) \times m^3(\epsilon_{t+1}(phi1))$
$\quad\quad\quad\quad\quad\quad end$

end
end
end
end
end
end

27)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{\epsilon_{t+1}\left(\phi_4,1\right)\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfxf_eps3 = zeros(ne × nx × nx, ne × ne × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for gama2 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi2 = 1 : ne
                for phi3 = 1 : ne
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1˜ = phi4)
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1˜ = phi2)
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1˜ = phi2)
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_epsxfxf_eps3(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

28)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 5).

29)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 2)

30)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 19).

31)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$$

where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 25).

32)

$$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$$

$$=E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{x_t^f(\gamma_2,1)\left\{\epsilon_{t+1}(\phi_4,1)\right\}_{\phi_4=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfeps_epsxfeps = zeros(ne × nx × ne, ne × nx × ne)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for gama2 = 1 : nx
                    for phi4 = 1 : ne
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_epsxfeps_epsxfeps(index1, index2) = E_xf_xf(gama1, gama2)×m^4(ε_{t+1}(phi1))
                        end
```

end
end
end
end
end
end

33)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$=E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{x_t^f(\gamma_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_3,1)\left\{\epsilon_{t+1}(\phi_4,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_epsxfeps_eps2xf = zeros(ne × nx × ne, ne × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for gama1 = 1 : nx
        for phi2 = 1 : ne
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for phi4 = 1 : ne
                    for gama2 = 1 : nx
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_epsxfeps_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_epsxfeps_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi4 && phi2 == phi3 && phi1~ = phi2)
                            E_epsxfeps_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        % fourth moments of innovations
                        elseif phi1 == phi2 && phi1 == phi3 && phi1 == phi4
                            E_epsxfeps_eps2xf(index1, index2) = E_xf_xf(gama1, gama2) × m^4 (ε_{t+1} (phi1))
                        end
                    end
                end
            end
        end
    end
end
```

34)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 6).

35)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 13).

36)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 20).

37)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 26).

38)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]=E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]'$
where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$ from 33).

39)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right]$

$$=E[\left(\left\{\epsilon_{t+1}\left(\phi_1,1\right)\left\{\epsilon_{t+1}\left(\phi_2,1\right)\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}\left(\phi_3,1\right)\left\{\epsilon_{t+1}\left(\phi_4,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_4=1}^{n_e}\right\}_{\phi_3=1}^{n_e}\right)']$$

Thus the quasi Matlab codes are:

```
E_eps2xf_eps2xf = zeros(ne × ne × nx, ne × ne × nx)
index1 = 0
for phi1 = 1 : ne
    for phi2 = 1 : ne
        for gama1 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for phi3 = 1 : ne
                for phi4 = 1 : ne
                    for gama2 = 1 : nx
                        index2 = index2 + 1
                        % second moments of innovations
                        if (phi1 == phi2 && phi3 == phi4 && phi1~ = phi4)
                            E_eps2xf_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
                        elseif (phi1 == phi3 && phi2 == phi4 && phi1~ = phi2)
                            E_eps2xf_eps2xf(index1, index2) = E_xf_xf(gama1, gama2)
```

$elseif\ (phi1 == phi4\ \&\&\ phi2 == phi3\ \&\&\ phi1\tilde{}= phi2)$
$E\_eps2xf\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2)$
% fourth moments of innovations
$elseif\ phi1 == phi2\ \&\&\ phi1 == phi3\ \&\&\ phi1 == phi4$
$E\_eps2xf\_eps2xf(index1, index2) = E\_xf\_xf(gama1, gama2) \times m^4\left(\epsilon_{t+1}(phi1)\right)$
end
end
end
end
end
end
end

40)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right] = E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 7).

41)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$
where we already know $E\left[\left(\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 14).

42)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right] = E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]'$
where we already know $E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$ from 27).

43)
$E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)'\right]$

$$= E[\left(\left\{\epsilon_{t+1}(\phi_1,1)\left\{\epsilon_{t+1}(\phi_2,1)\left\{\epsilon_{t+1}(\phi_3,1)\right\}_{\phi_3=1}^{n_e}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right)$$
$$\times\left(\left\{\epsilon_{t+1}(\phi_4,1)\left\{\epsilon_{t+1}(\phi_5,1)\left\{\epsilon_{t+1}(\phi_6,1)\right\}_{\phi_6=1}^{n_e}\right\}_{\phi_5=1}^{n_e}\right\}_{\phi_4=1}^{n_e}\right)']$$

The codes are given in the matlab file. (too big for displaying)

4.4 Method 3: Simple formulas for first and second moments

This section shows how to compute mean values up to third order in a very direct manner. As in the case of the second-order approximation, the advantage of Method 3 is that we do not recompute terms which are already known at a lower approximation order. As a result, the matrices which must be inverted are here smaller than in Method 1 and 2.

4.4.1 First moments

This section derives the unconditional mean value of $\mathbf{y}_t$ and $\mathbf{x}_t$. Recall

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}$$

Thus, we only need to find $E\left[\mathbf{x}_t^{rd}\right]$. Here

$$E\left[\mathbf{x}_{t+1}^{rd}\right] = \mathbf{h_x}E\left[\mathbf{x}_t^{rd}\right] + 2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 E\left[\mathbf{x}_t^f\right] + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$\Updownarrow$$
$$\left(\mathbf{I}_{n_x} - \mathbf{h_x}\right)E\left[\mathbf{x}_t^{rd}\right] = 2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

because $\mathbf{x}_{t+1}^{rd}$ is stationary and $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$

$$\Updownarrow$$
$$E\left[\mathbf{x}_t^{rd}\right] = \left(\mathbf{I}_{n_x} - \mathbf{h_x}\right)^{-1}\left(2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$

To compute $\tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$ recall that

$$\begin{aligned}\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f &= \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\\
&+ \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\\
&+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right)\\
&+ \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

Hence

$$E\left[\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right] = \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$$
$$\Updownarrow$$
$$\left(\mathbf{I}_{n_x^3} - \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\right)E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right] = \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$$

because $\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f$ is stationary

$$\Updownarrow$$
$$E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right] = \left(\mathbf{I}_{n_x^3} - \left(\mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}\right)\right)^{-1}\left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}\right)E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$$

Note that this term is zero if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero.

To compute $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right]$ recall that

$$\begin{aligned}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) &= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\\
&+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

So

$$E\left[\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right)\right] = \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] + \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$$
$$\Updownarrow$$
$$\left(\mathbf{I}_{n_x^2} - \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\right)E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right] = \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right)E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$$

because $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)$ is stationary
$\Updownarrow$

$$E\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right] = \left(\mathbf{I}_{n_x^2} - (\mathbf{h_x} \otimes \mathbf{h_x})\right)^{-1} \left(\mathbf{h_x} \otimes \tilde{\mathbf{H}}_{\mathbf{xx}}\right) E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$$

Note that this term is zero if $E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] = \mathbf{0}$, which is the case if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero.

For the control variables, we have

$$E\left[\mathbf{y}_t^{rd}\right] = \mathbf{g_x}\left(E\left[\mathbf{x}_t^f\right] + E\left[\mathbf{x}_t^s\right] + E\left[\mathbf{x}_t^{rd}\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + 2E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right]$$
$$+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 E\left[\mathbf{x}_t^f\right] + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{g_x}\left(E\left[\mathbf{x}_t^s\right] + E\left[\mathbf{x}_t^{rd}\right]\right) + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + 2E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right]\right)$$
$$+\tilde{\mathbf{G}}_{\mathbf{xxx}} E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right] + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

Now recall, if all third moments of $\boldsymbol{\epsilon}_{t+1}$ are zero, then $\mathbf{g}_{\sigma\sigma\sigma} = \mathbf{0}$ and $\mathbf{h}_{\sigma\sigma\sigma} = \mathbf{0}$. Hence, we have the following

Corollary 1 *The mean value in a third order approximation is identical to the mean value in a second order approximation if all third moments of* $\boldsymbol{\epsilon}_{t+1}$ *are zero.*

4.4.2 Second moments

We start by noticing that

$$Var\left(\mathbf{z}_t\right) = E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)'\right]$$

$$= E\left[\left(\mathbf{z}_t - E\left[\mathbf{z}_t\right]\right)\left(\mathbf{z}_t' - E\left[\mathbf{z}_t'\right]\right)\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t' - \mathbf{z}_t E\left[\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]\mathbf{z}_t' + E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right]$$

$$= E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]$$

and

$$E\left[\mathbf{z}_t\mathbf{z}_t'\right] = E\left[\begin{bmatrix}\mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\end{bmatrix}\begin{bmatrix}\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^{rd}\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)'\end{bmatrix}\right]$$

$$= E\begin{bmatrix}
\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^{rd}\right)' \\
\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^{rd}\right)' \\
\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)' \\
\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^s\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^{rd}\right)' \\
\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_t^{rd}\right)' \\
\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)'
\end{bmatrix}$$

$$
\begin{array}{cc}
\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'
\end{array}
\Bigg]
$$

Hence, we need to find the following terms:

- $\mathbf{x}_t^f\left(\mathbf{x}_t^{rd}\right)'$, $\mathbf{x}_t^s\left(\mathbf{x}_t^{rd}\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)'$, $\mathbf{x}_t^{rd}\left(\mathbf{x}_t^{rd}\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^{rd}\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^{rd}\right)'$
- $\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'$
- $\mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$, $\mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$, $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$

All these terms are easy to compute using the procedure outlined above and previous results.

4.5 The auto-correlations

This section derives the auto-correlations for the states and the control variables.

4.5.1 The innovations

We first show that $Cov\left(\boldsymbol{\xi}_{t+1},\boldsymbol{\xi}_{t+1+s}\right)\neq\mathbf{0}$ for $s=1,2,3,...$ To see this recall that

$$
E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1+s}'\right]=E\left[\begin{bmatrix}
\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}-vec\left(\mathbf{I}_{n_e}\right) \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\boldsymbol{\epsilon}_{t+1} \\
\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f \\
\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)-E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]
\end{bmatrix}\right.
$$

$$
\times\left[\begin{array}{l}
\boldsymbol{\epsilon}_{t+1+s}' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}-vec\left(\mathbf{I}_{n_e}\right)\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \\
\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^s\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\mathbf{x}_{t+s}^f\right)' \\
\left(\mathbf{x}_{t+s}^f\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \quad \left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)-E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)'
\end{array}\right]\Bigg]
$$

We now inspect each of the rows in turn. Here, we need the following result that
$\mathbf{x}_{t+1}^f=\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$

$$
\begin{aligned}
\mathbf{x}^f_{t+2} &= \mathbf{h_x}\mathbf{x}^f_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\
&= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\
&= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}
\end{aligned}
$$

...

$$
\mathbf{x}^f_{t+s} = \mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t + \textstyle\sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}
$$

1) Row with $\boldsymbol{\epsilon}_{t+1}$

Consider the sub-matrix

$$
\begin{aligned}
E\{\boldsymbol{\epsilon}_{t+1} \Big[\; & \boldsymbol{\epsilon}'_{t+1+s} \quad (\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right))' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \\
& \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^s_{t+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\otimes\mathbf{x}^f_{t+s}\right)' \\
& \left(\mathbf{x}^f_{t+s}\otimes\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \quad \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
& \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]\right)' \; \Big]\}
\end{aligned}
$$

$$
\begin{aligned}
= E\{\boldsymbol{\epsilon}_{t+1} \big[\; & \mathbf{0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0' \\
& 0 \quad 0 \quad \left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)' \\
& \left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}^f_{t+s}\right)' \quad 0 \; \Big]\}
\end{aligned}
$$

Hence, we only need to study the term of the form

$$
E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{x}^f_{t+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]
$$

$$
\begin{aligned}
&= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t + \textstyle\sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right] \\
&= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s} + \textstyle\sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right] \\
&= E\left[\boldsymbol{\epsilon}_{t+1}\left(\mathbf{h}^s_{\mathbf{x}}\mathbf{x}^f_t\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right] \\
&\quad + E\left[\boldsymbol{\epsilon}_{t+1}\left(\textstyle\sum_{i=1}^s \mathbf{h}^{s-i}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right] \\
&= 0 + E\left[\boldsymbol{\epsilon}_{t+1}\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right] \\
&= E\left[\left(\boldsymbol{\epsilon}_{t+1}\otimes 1\right)\left(\boldsymbol{\epsilon}'_{t+1}\boldsymbol{\eta}'\sigma\otimes\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right)\right] \\
&= E\left[\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}'_{t+1}\boldsymbol{\eta}'\sigma\otimes\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right] \\
&= \mathbf{I}_{n_e}\boldsymbol{\eta}'\sigma\otimes vec\left(\mathbf{I}_{n_e}\right)'
\end{aligned}
$$

2)

To be completed

4.5.2 The covariances

Recall that we have

$$\mathbf{z}_t = \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}$$

$$\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}$$

$$\mathbf{y}_t^{rd} = \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

To find the one period auto-covariances, i.e. $Cov\left(\mathbf{z}_{t+1}, \mathbf{z}_t\right)$, we have

$Cov\left(\mathbf{z}_{t+1}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right)$

$= \mathbf{A}Cov\left(\mathbf{z}_t, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right)$

And for two periods

$Cov\left(\mathbf{z}_{t+2}, \mathbf{z}_t\right) = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{z}_{t+1} + \mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{c} + \mathbf{A}\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right) + \mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{c} + \mathbf{A}\mathbf{c} + \mathbf{A}^2\mathbf{z}_t + \mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+1} + \mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{A}^2\mathbf{z}_t, \mathbf{z}_t\right) + Cov\left(\mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + Cov\left(\mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$

$= \mathbf{A}^2Cov\left(\mathbf{z}_t, \mathbf{z}_t\right) + \mathbf{A}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$

or

$Cov\left(\mathbf{z}_{t+2}, \mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_{t+1}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right)$

And for three periods

$Cov\left(\mathbf{z}_{t+3}, \mathbf{z}_t\right) = Cov\left(\mathbf{c} + \mathbf{A}\mathbf{z}_{t+2} + \mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{c} + \mathbf{A}\left(\mathbf{c} + \mathbf{A}\mathbf{c} + \mathbf{A}^2\mathbf{z}_t + \mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+1} + \mathbf{B}\boldsymbol{\xi}_{t+2}\right) + \mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{c} + \mathbf{A}\mathbf{c} + \mathbf{A}^2\mathbf{c} + \mathbf{A}^3\mathbf{z}_t + \mathbf{A}^2\mathbf{B}\boldsymbol{\xi}_{t+1} + \mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+2} + \mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$

$= Cov\left(\mathbf{A}^3\mathbf{z}_t, \mathbf{z}_t\right) + Cov\left(\mathbf{A}^2\mathbf{B}\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + Cov\left(\mathbf{A}\mathbf{B}\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) + Cov\left(\mathbf{B}\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$

$= \mathbf{A}^3Var\left(\mathbf{z}_t\right) + \mathbf{A}^2\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+1}, \mathbf{z}_t\right) + \mathbf{A}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+2}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$

$= \mathbf{A}^3Var\left(\mathbf{z}_t\right) + \sum_{i=1}^{3}\mathbf{A}^{3-i}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+i}, \mathbf{z}_t\right)$

or

$Cov\left(\mathbf{z}_{t+3}, \mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_{t+2}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+3}, \mathbf{z}_t\right)$

Hence in general

$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}^sVar\left(\mathbf{z}_t\right) + \sum_{i=1}^{s}\mathbf{A}^{s-i}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+i}, \mathbf{z}_t\right)$

$\Updownarrow$

$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}^sVar\left(\mathbf{z}_t\right) + \sum_{j=0}^{s-1}\mathbf{A}^{s-(j+1)}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+j+1}, \mathbf{z}_t\right)$

$i = j + 1$ so $j = i - 1$

$\Updownarrow$

$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}^sVar\left(\mathbf{z}_t\right) + \sum_{j=0}^{s-1}\mathbf{A}^{s-1-j}\mathbf{B}Cov\left(\boldsymbol{\xi}_{t+j+1}, \mathbf{z}_t\right)$

or

$Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right) = \mathbf{A}Cov\left(\mathbf{z}_{t+s-1}, \mathbf{z}_t\right) + \mathbf{B}Cov\left(\boldsymbol{\xi}_{t+s}, \mathbf{z}_t\right)$

For the control variables:
$Cov\left(\mathbf{y}_{t+s}^{rd}, \mathbf{y}_{t}^{rd}\right) = Cov\left(\mathbf{D}\mathbf{z}_{t+s} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2, \mathbf{D}\mathbf{z}_t + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2\right)$

$$= Cov\left(\mathbf{D}\mathbf{z}_{t+s}, \mathbf{D}\mathbf{z}_t\right)$$

$$= \mathbf{D}Cov\left(\mathbf{z}_{t+s}, \mathbf{z}_t\right)\mathbf{D}'$$

Thus we only need to compute $Cov\left(\boldsymbol{\xi}_{t+s}, \mathbf{z}_t\right) = E\left[\boldsymbol{\xi}_{t+s}\mathbf{z}_t'\right]$

4.5.3 Computing $Cov\left(\boldsymbol{\xi}_{t+s}, \mathbf{z}_t\right)$

We consider $E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1+s}'\right]$ and note that $E\left[\boldsymbol{\xi}_{t+1+s}\mathbf{z}_t'\right] = \left(E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1+s}'\right]\right)'$.

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1+s}'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^{rd} \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\right.$$

$$\times \Big[\ \boldsymbol{\epsilon}_{t+1+s}' \quad \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} - vec\left(\mathbf{I}_{n_e}\right)\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'$$
$$\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^s\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \mathbf{x}_{t+s}^f\right)'$$
$$\left(\mathbf{x}_{t+s}^f \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)' \quad \left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)' \quad \left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'$$
$$\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)' \quad \left(\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right) - E\left[\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]\right)'\ \Big]$$

$$= \begin{bmatrix}
0_{n_x \times n_e} & 0_{n_x \times n_e^2} & 0_{n_x \times n_e n_x} & 0_{n_x \times n_x n_e} & 0_{n_x \times n_x n_e} & 0_{n_x \times n_e n_x^2} & 0_{n_x \times n_x^2 n_e} & 0_{n_x \times n_x^2 n_e} & r_{1,9} & r_{1,10} & r_{1,11} & 0_{n_x \times n_e^3} \\
0_{n_x \times n_e} & 0_{n_x \times n_e^2} & 0_{n_x \times n_e n_x} & 0_{n_x \times n_x n_e} & 0_{n_x \times n_x n_e} & 0_{n_x \times n_e n_x^2} & 0_{n_x \times n_x^2 n_e} & 0_{n_x \times n_x^2 n_e} & r_{2,9} & r_{2,10} & r_{2,11} & 0_{n_x \times n_e^3} \\
0_{n_x^2 \times n_e} & 0_{n_x^2 \times n_e^2} & 0_{n_x^2 \times n_e n_x} & 0_{n_x^2 \times n_x n_e} & 0_{n_x^2 \times n_x n_e} & 0_{n_x^2 \times n_e n_x^2} & 0_{n_x^2 \times n_x^2 n_e} & 0_{n_x^2 \times n_x^2 n_e} & r_{3,9} & r_{3,10} & r_{3,11} & 0_{n_x^2 \times n_e^3} \\
0_{n_x \times n_e} & 0_{n_x \times n_e^2} & 0_{n_x \times n_e n_x} & 0_{n_x \times n_x n_e} & 0_{n_x \times n_x n_e} & 0_{n_x \times n_e n_x^2} & 0_{n_x \times n_x^2 n_e} & 0_{n_x \times n_x^2 n_e} & r_{4,9} & r_{4,10} & r_{4,11} & 0_{n_x \times n_e^3} \\
0_{n_x^2 \times n_e} & 0_{n_x^2 \times n_e^2} & 0_{n_x^2 \times n_e n_x} & 0_{n_x^2 \times n_x n_e} & 0_{n_x^2 \times n_x n_e} & 0_{n_x^2 \times n_e n_x^2} & 0_{n_x^2 \times n_x^2 n_e} & 0_{n_x^2 \times n_x^2 n_e} & r_{5,9} & r_{5,10} & r_{5,1} & 0_{n_x^2 \times n_e^3} \\
0_{n_x^3 \times n_e} & 0_{n_x^3 \times n_e^2} & 0_{n_x^3 \times n_e n_x} & 0_{n_x^3 \times n_x n_e} & 0_{n_x^3 \times n_x n_e} & 0_{n_x^3 \times n_e n_x^2} & 0_{n_x^3 \times n_x^2 n_e} & 0_{n_x^3 \times n_x^2 n_e} & r_{6,9} & r_{6,10} & r_{6,11} & 0_{n_x^3 \times n_e^3}
\end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathbf{R} & 0 \end{bmatrix}$$

We now compute the non-zero elements in this matrix

1) The value of $r_{1,9}$
$r_{1,9} = E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\left\{x_t^f\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_1, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are
$E\_xf\_xfeps2 = zeros(nx, nx \times ne \times ne)$

$for\ gama1 = 1 : nx$
    $index2 = 0$
    $for\ gama2 = 1 : nx$
        $for\ phi1 = 1 : ne$
            $for\ phi2 = 1 : ne$
                $index2 = index2 + 1$
                $if\ phi1 == phi2$
                    $E\_xf\_xfeps2(gama1, index2) = E\_xf\_xfS(gama1, gama2)$
                end
            end
        end
    end
end

where $E\_xf\_xfS = E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]$

2) The value of $r_{1,10}$

$$r_{1,10} = E\left[\mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{x_{t+s}^f\left(\gamma_2,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

$E\_xf\_epsxfeps = zeros(nx, ne\times nx\times ne)$
$for\ gama1 = 1 : nx$
    $index2 = 0$
    $for\ phi1 = 1 : ne$
        $for\ gama2 = 1 : nx$
            $for\ phi2 = 1 : ne$
                $index2 = index2 + 1$
                $if\ phi1 == phi2$
                    $E\_xf\_epsxfeps(gama1, index2) = E\_xf\_xfS(gama1, gama2)$
                end
            end
        end
    end
end

3) The value of $r_{1,11}$

$$r_{1,11} = E\left[\mathbf{x}_t^f\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\left\{x_{t+s}^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

$E\_xf\_eps2xf = zeros(nx, ne\times ne\times nx)$
$for\ gama1 = 1 : nx$
    $index2 = 0$
    $for\ phi1 = 1 : ne$
        $for\ phi2 = 1 : ne$

$for\ gama2 = 1 : nx$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xf\_eps2xf(gama1, index2) = E\_xf\_xfS(gama1, gama2)$
end
end
end
end
end

4) The value of $r_{2,9}$

$$r_{2,9} = E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_1, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are
$E\_xs\_xfeps2 = zeros(nx, nx \times ne \times ne)$
$for\ gama1 = 1 : nx$
$index2 = 0$
$for\ gama2 = 1 : nx$
$for\ phi1 = 1 : ne$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xs\_xfeps2(gama1, index2) = E\_xs\_xfS(gama1, gama2)$
end
end
end
end
end
where $E\_xs\_xfS = E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]$

5) The value of $r_{2,10}$

$$r_{2,10} = E\left[\mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1, 1\right)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}\left(\phi_1, 1\right)\left\{x_{t+s}^f\left(\gamma_2, 1\right)\left\{\epsilon_{t+1+s}\left(\phi_2, 1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are
$E\_xs\_epsxfeps = zeros(nx, ne \times nx \times ne)$
$for\ gama1 = 1 : nx$
$index2 = 0$
$for\ phi1 = 1 : ne$
$for\ gama2 = 1 : nx$
$for\ phi2 = 1 : ne$
$index2 = index2 + 1$
$if\ phi1 == phi2$
$E\_xs\_epsxfeps(gama1, index2) = E\_xs\_xfS(gama1, gama2)$

end
end
end
end
end

6) The value of $r_{2,11}$

$$r_{2,11} = E\left[\mathbf{x}_t^s\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left\{x_t^s\left(\gamma_1,1\right)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\left\{x_{t+s}^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xs_eps2xf = zeros(nx, ne × ne × nx)
for gama1 = 1 : nx
    index2 = 0
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 == phi2
                    E_xs_eps2xf(gama1, index2) = E_xs_xfS(gama1, gama2)
                end
            end
        end
    end
end
```

7) The value of $r_{3,9}$

$$r_{3,9} = E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^f\left(\gamma_1,1\right)\left\{x_t^f\left(\gamma_2,1\right)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f\left(\gamma_3,1\right)\left\{\epsilon_{t+1+s}\left(\phi_1,1\right)\left\{\epsilon_{t+1+s}\left(\phi_2,1\right)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxf_xfeps2 = zeros(nx × nx, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama3 = 1 : nx
            for phi1 = 1 : ne
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfxf_xfeps2(index1, index2) = E_xf_xf_xfS(gama1, gama2, gama3)
                    end
                end
```

end
end
end
end
where $E\_xf\_xf\_xfS = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

8) The value of $r_{3,10}$

$$r_{3,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxf_epsxfeps = zeros(nx × nx, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    index1 = index1 + 1
    for gama2 = 1 : nx
        index2 = 0
        for phi1 = 1 : ne
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfxf_epsxfeps(index1, index2) = E_xf_xf_xfS(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

9) The value of $r_{3,11}$

$$r_{3,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'\right]$$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^f(\gamma_2,1)_{\gamma_2=1}^{n_x}\right\}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^f(\gamma_3,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxf_eps2xf = zeros(nx × nx, ne × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi1 = 1 : ne
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
```

```
                if phi1 == phi2
                    E_xfxf_eps2xf(index1, index2) = E_xf_xf_xfS(gama1, gama2, gama3)
                end
            end
        end
    end
end
end
```

10) The value of $r_{4,9}$

$$r_{4,9} = E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^f(\gamma_2,1)\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

```
E_xrd_xfeps2 = zeros(nx, nx × ne × ne)
for gama1 = 1 : nx
    index2 = 0
    for gama2 = 1 : nx
        for phi1 = 1 : ne
            for phi2 = 1 : ne
                index2 = index2 + 1
                if phi1 == phi2
                    E_xrd_xfeps2(gama1, index2) = E_xrd_xfS(gama1, gama2)
                end
            end
        end
    end
end
```

where $E\_xrd\_xfS = E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]$

11) The value of $r_{4,10}$

$$r_{4,10} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_2,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xrd_epsxfeps = zeros(nx, ne × nx × ne)
for gama1 = 1 : nx
    index2 = 0
    for phi1 = 1 : ne
        for gama2 = 1 : nx
            for phi2 = 1 : ne
                index2 = index2 + 1
                if phi1 == phi2
                    E_xrd_epsxfeps(gama1, index2) = E_xrd_xfS(gama1, gama2)
                end
```

end
end
end
end

12) The value of $r_{4,11}$

$$r_{4,11} = E\left[\mathbf{x}_t^{rd}\left(\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\mathbf{x}_{t+s}^{f}\right)'\right]$$

$$= E\left[\left\{x_t^{rd}(\gamma_1,1)\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^{f}(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xrd_eps2xf = zeros(nx, ne × ne × nx)
for gama1 = 1 : nx
    index2 = 0
    for phi1 = 1 : ne
        for phi2 = 1 : ne
            for gama2 = 1 : nx
                index2 = index2 + 1
                if phi1 == phi2
                    E_xrd_eps2xf(gama1, index2) = E_xrd_xfS(gama1, gama2)
                end
            end
        end
    end
end
```

13) The value of $r_{5,9}$

$$r_{5,9} = E\left[\left(\mathbf{x}_t^{f}\otimes\mathbf{x}_t^{s}\right)\left(\mathbf{x}_{t+s}^{f}\otimes\boldsymbol{\epsilon}_{t+1+s}\otimes\boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E\left[\left\{x_t^{f}(\gamma_1,1)\left\{x_t^{s}(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{x_{t+s}^{f}(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxs_xfeps2 = zeros(nx × nx, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for gama3 = 1 : nx
            for phi1 = 1 : ne
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfxs_xfeps2(index1, index2) = E_xf_xs_xfS(gama1, gama2, gama3)
                    end
                end
            end
```

end
end
end
where $E\_xf\_xs\_xfS = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

14) The value of $r_{5,10}$
$r_{5,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{x_{t+s}^f(\gamma_3,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxs_epsxfeps = zeros(nx × nx, ne × nx × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi1 = 1 : ne
            for gama3 = 1 : nx
                for phi2 = 1 : ne
                    index2 = index2 + 1
                    if phi1 == phi2
                        E_xfxs_epsxfeps(index1, index2) = E_xf_xs_xfS(gama1, gama2, gama3)
                    end
                end
            end
        end
    end
end
```

15) The value of $r_{5,11}$
$r_{5,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'\right]$

$$= E\left[\left\{x_t^f(\gamma_1,1)\left\{x_t^s(\gamma_2,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}\left\{\epsilon_{t+1+s}(\phi_1,1)\left\{\epsilon_{t+1+s}(\phi_2,1)\left\{x_{t+s}^f(\gamma_3,1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right]$$

Thus, the quasi Matlab codes are

```
E_xfxs_eps2xf = zeros(nx × nx, ne × ne × nx)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        index1 = index1 + 1
        index2 = 0
        for phi1 = 1 : ne
            for phi2 = 1 : ne
                for gama3 = 1 : nx
                    index2 = index2 + 1
                    if phi1 == phi2
```

$E\_xfxs\_eps2xf(index1, index2) = E\_xf\_xs\_xfS(gama1, gama2, gama3)$
end
end
end
end
end
end

16) The value of $r_{6,9}$

$$r_{6,9} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E[\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\left\{x_t^f(\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}$$
$$\times\left\{x_{t+s}^f(\gamma_3, 1)\left\{\epsilon_{t+1+s}(\phi_1, 1)\left\{\epsilon_{t+1+s}(\phi_2, 1)\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}]$$

Thus, the quasi Matlab codes are

```
E_xfxfxf_xfeps2 = zeros(nx × nx × nx, nx × ne × ne)
index1 = 0
for gama1 = 1 : nx
    for gama2 = 1 : nx
        for gama3 = 1 : nx
            index1 = index1 + 1
            index2 = 0
            for gama4 = 1 : nx
                for phi1 = 1 : ne
                    for phi2 = 1 : ne
                        index2 = index2 + 1
                        if phi1 == phi2
                            E_xfxfxf_xfeps2(index1, index2)
                            = E_xf_xf_xf_xfS(gama1, gama2, gama3, gama4)
                        end
                    end
                end
            end
        end
    end
end
```

where $E\_xf\_xf\_xf\_xfS = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

17) The value of $r_{6,10}$

$$r_{6,10} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f \otimes \boldsymbol{\epsilon}_{t+1+s}\right)'\right]$$

$$= E[\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\left\{x_t^f(\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}$$
$$\times\left\{\epsilon_{t+1+s}(\phi_1, 1)\left\{x_{t+s}^f(\gamma_3, 1)\left\{\epsilon_{t+1+s}(\phi_2, 1)\right\}_{\phi_2=1}^{n_e}\right\}_{\gamma_3=1}^{n_x}\right\}_{\phi_1=1}^{n_e}]$$

Thus, the quasi Matlab codes are

$E\_xfxfxf\_epsxfeps = zeros(nx \times nx \times nx, ne \times nx \times ne)$
$index1 = 0$
$for\ gama1 = 1 : nx$
    $for\ gama2 = 1 : nx$
        $for\ gama3 = 1 : nx$
            $index1 = index1 + 1$
            $index2 = 0$
            $for\ phi1 = 1 : ne$
                $for\ gama4 = 1 : nx$
                    $for\ phi2 = 1 : ne$
                        $index2 = index2 + 1$
                        $if\ phi1 == phi2$
                            $E\_xfxfxf\_epsxfeps(index1, index2)$
                            $= E\_xf\_xf\_xf\_xfS(gama1, gama2, gama3, gama4)$
                        end
                    end
                end
            end
        end
    end
end

18) The value of $r_{6,11}$

$$r_{6,11} = E\left[\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1+s} \otimes \boldsymbol{\epsilon}_{t+1+s} \otimes \mathbf{x}_{t+s}^f\right)'\right]$$

$$= E[\left\{x_t^f(\gamma_1, 1)\left\{x_t^f(\gamma_2, 1)\left\{x_t^f(\gamma_3, 1)\right\}_{\gamma_3=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_1=1}^{n_x}$$
$$\times\left\{\epsilon_{t+1+s}(\phi_1, 1)\left\{\epsilon_{t+1+s}(\phi_2, 1)\left\{x_{t+s}^f(\gamma_3, 1)\right\}_{\gamma_2=1}^{n_x}\right\}_{\phi_2=1}^{n_e}\right\}_{\phi_1=1}^{n_e}]$$

Thus, the quasi Matlab codes are

$E\_xfxfxf\_eps2xf = zeros(nx \times nx \times nx, ne \times ne \times nx)$
$index1 = 0$
$for\ gama1 = 1 : nx$
    $for\ gama2 = 1 : nx$
        $for\ gama3 = 1 : nx$
            $index1 = index1 + 1$
            $index2 = 0$
            $for\ phi1 = 1 : ne$
                $for\ phi2 = 1 : ne$
                    $for\ gama4 = 1 : nx$
                        $index2 = index2 + 1$
                        $if\ phi1 == phi2$
                            $E\_xfxfxf\_eps2xf(index1, index2)$
                            $= E\_xf\_xf\_xf\_xfS(gama1, gama2, gama3, gama4)$
                        end
                    end
                end
            end

end

end

end

We know all the required moments, except $E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]$, $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$, and $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

a) For $E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]$

Recall that $\mathbf{x}_{t+s}^f=\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}$.

So

$$E\left[\mathbf{x}_t^f\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\mathbf{x}_t^f\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]=E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$$

b) For $E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$E\left[\mathbf{x}_t^s\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\mathbf{x}_t^s\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]=E\left[\mathbf{x}_t^s\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$$

c) For $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]=E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$$

d) For $E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_{t+s}^f\right)'\right]=E\left[\mathbf{x}_t^{rd}\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]$$

$$=E\left[\mathbf{x}_t^{rd}\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f\right)'\right]$$

$$=E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$$

So we only need to find $E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]$. Recall that

$$\mathbf{x}_{t+1}^{rd}=\mathbf{h}_\mathbf{x}\mathbf{x}_t^{rd}+2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

So

$$E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]$$

$$=E\left[\left(\mathbf{h}_\mathbf{x}\mathbf{x}_t^{rd}+2\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\tilde{\mathbf{H}}_{\mathbf{xxx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\left(\mathbf{x}_t^f\right)'\right]$$

$$=\mathbf{h}_\mathbf{x}E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right]+2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)'\right]$$
$$+\tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]$$

$\Updownarrow$

$$E\left[\mathbf{x}_t^{rd}\left(\mathbf{x}_t^f\right)'\right] = \left(\mathbf{I}-\mathbf{h_x}\right)^{-1}\left[2\tilde{\mathbf{H}}_{\mathbf{xx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)'\right]+\tilde{\mathbf{H}}_{\mathbf{xxx}}E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E\left[\mathbf{x}_t^f\left(\mathbf{x}_t^f\right)'\right]\right]$$

e) For $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\textstyle\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$$

f) For $E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_{t+s}^f\right)'\right]$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{h}_\mathbf{x}^s\mathbf{x}_t^f+\textstyle\sum_{i=1}^s\mathbf{h}_\mathbf{x}^{s-i}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+i}\right)'\right]$$

$$= E\left[\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\right)'\right]\left(\mathbf{h}_\mathbf{x}^s\right)'$$

5 The Dynare++ notation

This section presents the pruning method up to third order using the notation in Dynare and Dyanare++. The solution to DSGE models are in Dynare and Dynare++ given by

$$\mathbf{z}_t = \mathbf{f}\left(\mathbf{z}_{t-1},\mathbf{u}_t,\sigma\right) \tag{39}$$

where $\mathbf{z}_t$ contains all the endogenous variables (i.e. control variables and all state variables), and $\mathbf{u}_t$ with size $n_u\times 1$ is the vector of disturbances with the property $\mathbf{u}_t\sim\mathcal{NID}\left(\mathbf{0},\boldsymbol{\Sigma}\right)$. It is convenient to express this more general solution in a notation that is similar to the one used above. We therefore write (39) as

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_{t-1},\mathbf{u}_t,\sigma\right) \tag{40}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t,\mathbf{u}_{t+1},\sigma\right) \tag{41}$$

where $\mathbf{y}_t$ and $\mathbf{x}_t$ are as defined above. The key difference compared to the notation in Schmitt-Grohé & Uribe (2004) is that the function $\mathbf{g}$ depends on the innovations $\mathbf{u}_t$. Note also that the innovations may enter in a non-linear fashion in the $\mathbf{h}$ function. Below, it is useful to define

$$\mathbf{v}_{t,t+1}\equiv\begin{bmatrix}\mathbf{x}_t\\ \mathbf{u}_{t+1}\end{bmatrix} \tag{42}$$

where $\mathbf{v}_{t,t+1}$ has dimensions $n_v\times 1$. The first subscribt of $\mathbf{v}_{t,t+1}$ refers to the time index of $\mathbf{x}_t$ and the second to the time index of $\mathbf{u}_{t+1}$.

A first-order approximation (40) and (41) around the deterministic steady state is

$$\mathbf{y}_t = \mathbf{g}_\mathbf{v}\mathbf{v}_{t-1,t} \tag{43}$$

$$\mathbf{x}_{t+1} = \mathbf{h}_{\mathbf{v}} \mathbf{v}_{t,t+1} \tag{44}$$

A second-order approximation is

$$\mathbf{y}_t = \mathbf{g}_{\mathbf{v}} \mathbf{v}_{t-1,t} + \frac{1}{2} \mathbf{G}_{\mathbf{vv}} \left(\mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t} \right) + \frac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2 \tag{45}$$

and

$$\mathbf{x}_{t+1} = \mathbf{h}_{\mathbf{v}} \mathbf{v}_{t,t+1} + \frac{1}{2} \mathbf{H}_{\mathbf{vv}} \left(\mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1} \right) + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \tag{46}$$

A third-order approximation is

$$\begin{aligned} \mathbf{y}_t &= \mathbf{g}_{\mathbf{v}} \mathbf{v}_{t-1,t} + \frac{1}{2} \mathbf{G}_{\mathbf{vv}} \left(\mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t} \right) + \frac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2 \\ &+ \frac{1}{6} \mathbf{G}_{\mathbf{vvv}} \left(\mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t} \otimes \mathbf{v}_{t-1,t} \right) + \frac{3}{6} \mathbf{g}_{\sigma\sigma\mathbf{v}} \sigma^2 \mathbf{v}_{t-1,t} + \frac{1}{6} \mathbf{g}_{\sigma\sigma\sigma} \sigma^3 \end{aligned} \tag{47}$$

and

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{h}_{\mathbf{v}} \mathbf{v}_{t,t+1} + \frac{1}{2} \mathbf{H}_{\mathbf{vv}} \left(\mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1} \right) + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \\ &+ \frac{1}{2} \mathbf{H}_{\mathbf{vvv}} \left(\mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1} \otimes \mathbf{v}_{t,t+1} \right) + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{v}} \sigma^2 \mathbf{v}_{t,t+1} + \frac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3 \end{aligned} \tag{48}$$

6 Pruning scheme in Dynare++:

6.1 Second order approximation:

We start considering (46) which we write as

$$x_{t+1}(j,1) = \mathbf{h}_{\mathbf{v}}(j,:) \mathbf{v}_{t,t+1} + \left(\mathbf{v}_{t,t+1} \right)' \mathbf{h}_{\mathbf{vv}}(j,:,:) \mathbf{v}_{t,t+1} + \tfrac{1}{2} h_{\sigma\sigma}(j,1) \sigma^2$$

$$\Updownarrow$$

$$x_{t+1}(j,1) = \mathbf{h}_{\mathbf{v}}(j,:) \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_{t+1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_{t+1} \end{bmatrix}' \mathbf{h}_{\mathbf{vv}}(j,:,:) \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_{t+1} \end{bmatrix} + \tfrac{1}{2} h_{\sigma\sigma}(j,1) \sigma^2$$

for $j = 1, 2, ..., n_x$.

Let us now decompose the state vector $\mathbf{x}_t$ as

$$\mathbf{x}_t = \mathbf{x}_t^f + \mathbf{x}_t^s \tag{49}$$

Notice, that we do not need to compose the innovations as they are a first order effect in the system. For the subsequent decomposition let

$$\mathbf{h}_{\mathbf{v}} \equiv \begin{bmatrix} \mathbf{h}_{\mathbf{x}}(j,:) & \mathbf{h}_{\mathbf{u}}(j,:) \end{bmatrix} \tag{50}$$

$$\mathbf{h}_{\mathbf{vv}}(j,:,:) = \begin{bmatrix} \mathbf{h}_{\mathbf{xx}}(j,:,:) & \mathbf{h}_{\mathbf{xu}}(j,:,:) \\ \mathbf{h}_{\mathbf{ux}}(j,:,:) & \mathbf{h}_{\mathbf{uu}}(j,:,:) \end{bmatrix} \tag{51}$$

for $j = 1, 2, ..., n_x$.

Hence,

$$x_{t+1}^f(j,1) + x_{t+1}^s(j,1) = \begin{bmatrix} \mathbf{h}_{\mathbf{x}}(j,:) & \mathbf{h}_{\mathbf{u}}(j,:) \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f + \mathbf{x}_t^s \\ \mathbf{u}_{t+1} \end{bmatrix}$$

$$+ \tfrac{1}{2} \begin{bmatrix} \mathbf{x}_t^f + \mathbf{x}_t^s \\ \mathbf{u}_{t+1} \end{bmatrix}' \begin{bmatrix} \mathbf{h}_{\mathbf{xx}}(j,:,:) & \mathbf{h}_{\mathbf{xu}}(j,:,:) \\ \mathbf{h}_{\mathbf{ux}}(j,:,:) & \mathbf{h}_{\mathbf{uu}}(j,:,:) \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f + \mathbf{x}_t^s \\ \mathbf{u}_{t+1} \end{bmatrix} + \tfrac{1}{2} h_{\sigma\sigma}(j,1) \sigma^2$$

$$\Updownarrow$$

$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \mathbf{h_u}(j,:)\mathbf{u}_{t+1}$
$+\frac{1}{2}\begin{bmatrix} \left(\mathbf{x}^f_t\right)' + (\mathbf{x}^s_t)' & \mathbf{u}'_{t+1} \end{bmatrix}\begin{bmatrix} \mathbf{h_{xx}}(j,:,:) & \mathbf{h_{xu}}(j,:,:) \\ \mathbf{h_{ux}}(j,:,:) & \mathbf{h_{uu}}(j,:,:) \end{bmatrix}\begin{bmatrix} \mathbf{x}^f_t + \mathbf{x}^s_t \\ \mathbf{u}_{t+1} \end{bmatrix}$
$+\frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$
$\Updownarrow$

$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \mathbf{h_u}(j,:)\mathbf{u}_{t+1}$
$+\frac{1}{2}\begin{bmatrix} \left(\left(\mathbf{x}^f_t\right)' + (\mathbf{x}^s_t)'\right)\mathbf{h_{xx}}(j,:,:) + \mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:) & \left(\left(\mathbf{x}^f_t\right)' + (\mathbf{x}^s_t)'\right)\mathbf{h_{xu}}(j,:,:) + \mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:) \end{bmatrix}\begin{bmatrix} \mathbf{x}^f_t + \mathbf{x}^s_t \\ \mathbf{u}_{t+1} \end{bmatrix}$
$+\frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$
$\Updownarrow$

$x^f_{t+1}(j,1) + x^s_{t+1}(j,1) = \mathbf{h_x}(j,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \mathbf{h_u}(j,:)\mathbf{u}_{t+1}$
$+\frac{1}{2}\left(\left(\mathbf{x}^f_t\right)' + (\mathbf{x}^s_t)'\right)\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:)\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right)$
$+\frac{1}{2}\left(\left(\mathbf{x}^f_t\right)' + (\mathbf{x}^s_t)'\right)\mathbf{h_{xu}}(j,:,:)\mathbf{u}_{t+1} + \frac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:)\mathbf{u}_{t+1}$
$+\frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$

A law of motion for the first-order terms is thus

$$x^f_{t+1}(j,1) = \mathbf{h_x}(j,:)\mathbf{x}^f_t + \mathbf{h_u}(j,:)\mathbf{u}_{t+1} \tag{52}$$

for $j = 1, 2, ..., n_x$.

A law of motion for the second-order terms is thus

$$\begin{aligned} x^s_{t+1}(j,1) &= \mathbf{h_x}(j,:)\mathbf{x}^s_t + \frac{1}{2}\left(\mathbf{x}^f_t\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}^f_t\right) + \frac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{ux}}(j,:,:)\mathbf{x}^f_t \\ &\quad + \frac{1}{2}\left(\mathbf{x}^f_t\right)'\mathbf{h_{xu}}(j,:,:)\mathbf{u}_{t+1} + \frac{1}{2}\mathbf{u}'_{t+1}\mathbf{h_{uu}}(j,:,:)\mathbf{u}_{t+1} + \frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2 \end{aligned} \tag{53}$$

for $j = 1, 2, ..., n_x$. Note here, that non-linear shocks will imply that we have innovations to $x^s_{t+1}(j,1)$, i.e. if $\mathbf{h_{ux}}(j,:,:) \neq \mathbf{0}$ and $\mathbf{h_{uu}}(j,:,:) \neq \mathbf{0}$.

For the control variables, we introduce the following notation

$$\mathbf{g_v} \equiv \begin{bmatrix} \mathbf{g_x}(i,:) & \mathbf{g_u}(i,:) \end{bmatrix} \tag{54}$$

$$\mathbf{g_{vv}}(i,:,:) = \begin{bmatrix} \mathbf{g_{xx}}(i,:,:) & \mathbf{g_{xu}}(i,:,:) \\ \mathbf{g_{ux}}(i,:,:) & \mathbf{g_{uu}}(i,:,:) \end{bmatrix} \tag{55}$$

for $i = 1, 2, ..., n_y$. Thus
$y_t(i,1) = \mathbf{g_v}(i,:)\mathbf{v}_{t-1,t} + \frac{1}{2}(\mathbf{v}_{t-1,t})'\mathbf{g_{vv}}(i,:,:)\mathbf{v}_{t-1,t} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}(i,1)\sigma^2$
$\Updownarrow$

$y_t(i,1) = \begin{bmatrix} \mathbf{g_x}(i,:) & \mathbf{g_u}(i,:) \end{bmatrix}\begin{bmatrix} \mathbf{x}^f_{t-1} + \mathbf{x}^s_{t-1} \\ \mathbf{u}_t \end{bmatrix}$
$+\frac{1}{2}\begin{bmatrix} \mathbf{x}^f_{t-1} + \mathbf{x}^s_{t-1} \\ \mathbf{u}_t \end{bmatrix}'\begin{bmatrix} \mathbf{g_{xx}}(i,:,:) & \mathbf{g_{xu}}(i,:,:) \\ \mathbf{g_{ux}}(i,:,:) & \mathbf{g_{uu}}(i,:,:) \end{bmatrix}\begin{bmatrix} \mathbf{x}^f_{t-1} + \mathbf{x}^s_{t-1} \\ \mathbf{u}_t \end{bmatrix} + \frac{1}{2}g_{\sigma\sigma}(i,1)\sigma^2$
$\Updownarrow$

$$
\begin{aligned}
&y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\,\mathbf{u}_t \\
&\quad +\tfrac{1}{2}\left[\begin{array}{cc} \left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)' & \mathbf{u}_t' \end{array}\right]\left[\begin{array}{cc} \mathbf{g_{xx}}(i,:,:) & \mathbf{g_{xu}}(i,:,:) \\ \mathbf{g_{ux}}(i,:,:) & \mathbf{g_{uu}}(i,:,:) \end{array}\right]\left[\begin{array}{c} \mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s \\ \mathbf{u}_t \end{array}\right] + \tfrac{1}{2} g_{\sigma\sigma}(i,1)\,\sigma^2 \\
&\Updownarrow \\
&y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\,\mathbf{u}_t \\
&\quad +\tfrac{1}{2}\left[\begin{array}{cc} \left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xx}}(i,:,:) + \mathbf{u}_t'\mathbf{g_{ux}}(i,:,:) & \left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xu}}(i,:,:) + \mathbf{u}_t'\mathbf{g_{uu}}(i,:,:) \end{array}\right] \\
&\quad \times \left[\begin{array}{c} \mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s \\ \mathbf{u}_t \end{array}\right] \\
&\quad +\tfrac{1}{2} g_{\sigma\sigma}(i,1)\,\sigma^2 \\
&\Updownarrow \\
&y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\,\mathbf{u}_t \\
&\quad +\tfrac{1}{2}\left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xx}}(i,:,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{ux}}(i,:,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) \\
&\quad +\tfrac{1}{2}\left(\left(\mathbf{x}_{t-1}^f\right)' + \left(\mathbf{x}_{t-1}^s\right)'\right)\mathbf{g_{xu}}(i,:,:)\,\mathbf{u}_t + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{uu}}(i,:,:)\,\mathbf{u}_t \\
&\quad +\tfrac{1}{2} g_{\sigma\sigma}(i,1)\,\sigma^2
\end{aligned}
$$

for $i = 1, 2, ..., n_y$. We want to preserve terms up to second order, hence the pruned approximation is

$$
\begin{aligned}
y_t(i,1) \;=\; & \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\,\mathbf{u}_t \\
& +\frac{1}{2}\left(\mathbf{x}_{t-1}^f\right)'\mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_{t-1}^f + \frac{1}{2}\mathbf{u}_t'\mathbf{g_{ux}}(i,:,:)\,\mathbf{x}_{t-1}^f \\
& +\frac{1}{2}\left(\mathbf{x}_{t-1}^f\right)'\mathbf{g_{xu}}(i,:,:)\,\mathbf{u}_t + \frac{1}{2}\mathbf{u}_t'\mathbf{g_{uu}}(i,:,:)\,\mathbf{u}_t \\
& +\frac{1}{2} g_{\sigma\sigma}(i,1)\,\sigma^2
\end{aligned} \tag{56}
$$

for $i = 1, 2, ..., n_y$.

6.2 Second order approximation: a convenient representation

When coding the derived formulas it is convenient to use $\mathbf{v}_{t,t+1}$ directly, and the corresponding derivatives of $\mathbf{g}$ and $\mathbf{h}$, because this is how the output from Dynare and Dynare++ is stored. Hence, we can write

$$
\mathbf{x}_{t+1}^f = \mathbf{h_v}\left[\begin{array}{c} \mathbf{x}_t \\ \mathbf{u}_{t+1} \end{array}\right] \tag{57}
$$

and

$$
x_{t+1}^s(j,:) = \mathbf{h_v}(j,:)\left[\begin{array}{c} \mathbf{x}_t^s \\ \mathbf{0} \end{array}\right] + \frac{1}{2}\left[\begin{array}{c} \mathbf{x}_t^f \\ \mathbf{u}_{t+1} \end{array}\right]'\mathbf{h_{vv}}(j,:,:)\left[\begin{array}{c} \mathbf{x}_t^f \\ \mathbf{u}_{t+1} \end{array}\right] + \frac{1}{2} h_{\sigma\sigma}(j,:)\,\sigma^2 \tag{58}
$$

for $j = 1, 2, ..., n_x$. For the control variables we have

$$
y_t(i,1) = \mathbf{g_v}(i,:)\left(\left[\begin{array}{c} \mathbf{x}_{t-1}^f \\ \mathbf{u}_t \end{array}\right] + \left[\begin{array}{c} \mathbf{x}_{t-1}^s \\ \mathbf{0} \end{array}\right]\right) + \frac{1}{2}\left[\begin{array}{c} \mathbf{x}_{t-1}^f \\ \mathbf{u}_t \end{array}\right]'\mathbf{g_{vv}}(i,:,:)\left[\begin{array}{c} \mathbf{x}_{t-1}^f \\ \mathbf{u}_t \end{array}\right] + \frac{1}{2} g_{\sigma\sigma}(i,1)\,\sigma^2 \tag{59}
$$

for $i = 1, 2, ..., n_y$.

Using the kronecker representation (fast for MATLAB) we have

$$\mathbf{x}_{t+1}^{s}=\mathbf{h}_{\mathbf{v}}\left[\begin{array}{c}\mathbf{x}_{t}^{s}\\ \mathbf{0}\end{array}\right]+\tilde{\mathbf{H}}_{\mathbf{vv}}\left(\left[\begin{array}{c}\mathbf{x}_{t}^{f}\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t}^{f}\\ \mathbf{u}_{t+1}\end{array}\right]\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2} \tag{60}$$

where

$$\tilde{\mathbf{H}}_{\mathbf{vv}}\equiv\frac{1}{2}reshape\left(\mathbf{h}_{\mathbf{vv}},n_{x},n_{v}^{2}\right) \tag{61}$$

And

$$\mathbf{y}_{t}=\mathbf{g}_{\mathbf{v}}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^{f}\\ \mathbf{u}_{t}\end{array}\right]+\left[\begin{array}{c}\mathbf{x}_{t-1}^{s}\\ \mathbf{0}\end{array}\right]\right)+\tilde{\mathbf{G}}_{\mathbf{vv}}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^{f}\\ \mathbf{u}_{t}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^{f}\\ \mathbf{u}_{t}\end{array}\right]\right)+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^{2} \tag{62}$$

where

$$\tilde{\mathbf{G}}_{\mathbf{vv}}\equiv\frac{1}{2}reshape\left(\mathbf{g}_{\mathbf{vv}},n_{y},n_{v}^{2}\right) \tag{63}$$

6.3 Third order approximation:

Let us now decompose the state vector $\mathbf{x}_t$ as

$$\mathbf{x}_{t}=\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd} \tag{64}$$

Thus

$$\begin{aligned}&x_{t+1}(j,1)=\mathbf{h}_{\mathbf{v}}(j,:)\mathbf{v}_{t,t+1}+\tfrac{1}{2}(\mathbf{v}_{t,t+1})'\mathbf{h}_{\mathbf{vv}}(j,:,:)\mathbf{v}_{t,t+1}+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\sigma^{2}\\ &\quad+\tfrac{1}{6}(\mathbf{v}_{t,t+1})'\left[\begin{array}{c}(\mathbf{v}_{t,t+1})'\mathbf{h}_{\mathbf{vvv}}(j,1,:,:)\mathbf{v}_{t,t+1}\\ \ldots\\ (\mathbf{v}_{t,t+1})'\mathbf{h}_{\mathbf{vvv}}(j,n_{v},:,:)\mathbf{v}_{t,t+1}\end{array}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}(j,:)\sigma^{2}\mathbf{v}_{t,t+1}+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\sigma^{3}\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}&x_{t+1}^{f}(j,1)+x_{t+1}^{s}(j,1)+x_{t+1}^{rd}(j,1)=\left[\begin{array}{cc}\mathbf{h}_{\mathbf{x}}(j,:) & \mathbf{h}_{\mathbf{u}}(j,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\\ &\quad+\tfrac{1}{2}\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]'\left[\begin{array}{cc}\mathbf{h}_{\mathbf{xx}}(j,:,:) & \mathbf{h}_{\mathbf{xu}}(j,:,:)\\ \mathbf{h}_{\mathbf{ux}}(j,:,:) & \mathbf{h}_{\mathbf{uu}}(j,:,:)\end{array}\right]\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\sigma^{2}\\ &\quad+\tfrac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_{t}^{f}\right)'+(\mathbf{x}_{t}^{s})'+\left(\mathbf{x}_{t}^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_{t}^{f}\right)'+(\mathbf{x}_{t}^{s})'+\left(\mathbf{x}_{t}^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h}_{\mathbf{vvv}}(j,1,:,:)\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\\ \ldots\\ \left[\begin{array}{cc}\left(\mathbf{x}_{t}^{f}\right)'+(\mathbf{x}_{t}^{s})'+\left(\mathbf{x}_{t}^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h}_{\mathbf{vvv}}(j,n_{v},:,:)\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\end{array}\right]\\ &\quad+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}(j,:)\sigma^{2}\left[\begin{array}{c}\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]+\tfrac{1}{6}h_{\sigma\sigma\sigma}(j,1)\sigma^{3}\end{aligned}$$

$\Updownarrow$

$$\begin{aligned}&x_{t+1}^{f}(j,1)+x_{t+1}^{s}(j,1)+x_{t+1}^{rd}(j,1)=\mathbf{h}_{\mathbf{x}}(j,:)\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\right)+\mathbf{h}_{\mathbf{u}}(j,:)\mathbf{u}_{t+1}\\ &\quad+\tfrac{1}{2}\left(\left(\mathbf{x}_{t}^{f}\right)'+(\mathbf{x}_{t}^{s})'+\left(\mathbf{x}_{t}^{rd}\right)'\right)\mathbf{h}_{\mathbf{xx}}(j,:,:)\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\right)+\tfrac{1}{2}\mathbf{u}_{t+1}'\mathbf{h}_{\mathbf{ux}}(j,:,:)\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}+\mathbf{x}_{t}^{rd}\right)\\ &\quad+\tfrac{1}{2}\left(\left(\mathbf{x}_{t}^{f}\right)'+(\mathbf{x}_{t}^{s})'+\left(\mathbf{x}_{t}^{rd}\right)'\right)\mathbf{h}_{\mathbf{xu}}(j,:,:)\mathbf{u}_{t+1}+\tfrac{1}{2}\mathbf{u}_{t+1}'\mathbf{h}_{\mathbf{uu}}(j,:,:)\mathbf{u}_{t+1}\\ &\quad+\tfrac{1}{2}h_{\sigma\sigma}(j,1)\sigma^{2}\end{aligned}$$

$$+\frac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)'+\left(\mathbf{x}_t^s\right)'+\left(\mathbf{x}_t^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)'+\left(\mathbf{x}_t^s\right)'+\left(\mathbf{x}_t^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,1,:,:)\left[\begin{array}{c}\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\\ \ldots\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)'+\left(\mathbf{x}_t^s\right)'+\left(\mathbf{x}_t^{rd}\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,n_v,:,:)\left[\begin{array}{c}\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]\end{array}\right]$$
$$+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}(j,:)\,\sigma^2\left[\begin{array}{c}\mathbf{x}_t^f+\mathbf{x}_t^s+\mathbf{x}_t^{rd}\\ \mathbf{u}_{t+1}\end{array}\right]+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

Without reducing the large term for with $\mathbf{h_{vvv}}(j,:,:,:)$, it is straightforward to see that the law of motion for $x_{t+1}^{rd}(j,1)$ that only preserves third order terms is

$$x_{t+1}^{rd}(j,1)=\mathbf{h_x}(j,:)\,\mathbf{x}_t^{rd}+$$
$$+\frac{1}{2}\left(\mathbf{x}_t^s\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^f+\frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\,\mathbf{x}_t^s+\frac{1}{2}\mathbf{u}_{t+1}'\mathbf{h_{ux}}(j,:,:)\,\mathbf{x}_t^s$$
$$+\frac{1}{2}\left(\mathbf{x}_t^s\right)'\mathbf{h_{xu}}(j,:,:)\,\mathbf{u}_{t+1}$$
$$+\frac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,1,:,:)\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\\ \ldots\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\end{array}\right]\mathbf{h_{vvv}}(j,n_v,:,:)\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\end{array}\right]$$
$$+\frac{3}{6}\mathbf{h}_{\sigma\sigma\tilde{\mathbf{x}}}(j,:)\,\sigma^2\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]+\frac{1}{6}h_{\sigma\sigma\sigma}(j,1)\,\sigma^3$$

Using the convenient representation we thus have

$$\begin{aligned}\mathbf{x}_{t+1}^{rd} &= \mathbf{h_v}\left[\begin{array}{c}\mathbf{x}_t^{rd}\\ \mathbf{0}\end{array}\right]\\ &+2\tilde{\mathbf{H}}_{\mathbf{vv}}\left(\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_t^s\\ \mathbf{0}\end{array}\right]\right)\\ &+\tilde{\mathbf{H}}_{\mathbf{vvv}}\left(\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]\right)\\ &+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{v}}\sigma^2\left[\begin{array}{c}\mathbf{x}_t^f\\ \mathbf{u}_{t+1}\end{array}\right]+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^2\end{aligned}\tag{65}$$

It is by now straigthforward to see that an expression for $\mathbf{y}_t$ which only preserves up to third order terms are:

$$\begin{aligned}\mathbf{y}_t &= \mathbf{g_v}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]+\left[\begin{array}{c}\mathbf{x}_{t-1}^s\\ \mathbf{0}\end{array}\right]+\left[\begin{array}{c}\mathbf{x}_{t-1}^{rd}\\ \mathbf{0}\end{array}\right]\right)\\ &+\tilde{\mathbf{G}}_{\mathbf{vv}}\left(\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\right)+2\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^s\\ \mathbf{0}\end{array}\right]\right)\right)\\ &+\tilde{\mathbf{G}}_{\mathbf{vvv}}\left(\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\otimes\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]\right)\\ &+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{v}}\sigma^2\left[\begin{array}{c}\mathbf{x}_{t-1}^f\\ \mathbf{u}_t\end{array}\right]+\frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^2\end{aligned}\tag{66}$$

7 Dynare++ notation and statistical properties: second order

7.1 Co-variance stationarity

We start with the state variables. From above we have for the first-order effects that

$x_{t+1}^{f}(j,1)=\mathbf{h_x}(j,:)\mathbf{x}_t^f+\mathbf{h_u}(j,:)\mathbf{u}_{t+1}$

$\Downarrow$

$\mathbf{x}_{t+1}^{f}=\mathbf{h_x}\mathbf{x}_t^f+\mathbf{h_u}\mathbf{u}_{t+1}$

For the second-order effects

$$x_{t+1}^{s}(j,1)=\mathbf{h_x}(j,:)\mathbf{x}_t^s+\frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xx}}(j,:,:)\left(\mathbf{x}_t^f\right)+\frac{1}{2}\mathbf{u}_{t+1}'\mathbf{h_{ux}}(j,:,:)\mathbf{x}_t^f$$
$$+\frac{1}{2}\left(\mathbf{x}_t^f\right)'\mathbf{h_{xu}}(j,:,:)\mathbf{u}_{t+1}+\mathbf{u}_{t+1}'\mathbf{h_{uu}}(j,:,:)\mathbf{u}_{t+1}+\frac{1}{2}h_{\sigma\sigma}(j,1)\sigma^2$$

$\Downarrow$

$$\mathbf{x}_{t+1}^{s}=\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\mathbf{u}_{t+1}'\mathbf{h_{ux}}(j,:,:)\left(\mathbf{x}_t^f\right)+\left(\mathbf{x}_t^f\right)'\mathbf{h_{xu}}(j,:,:)\mathbf{u}_{t+1}+\tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$\Updownarrow$

$$\mathbf{x}_{t+1}^{s}=\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tilde{\mathbf{H}}_{\mathbf{ux}}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)+\tilde{\mathbf{H}}_{\mathbf{xu}}\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)+\tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$\Updownarrow$

$$\mathbf{x}_{t+1}^{s}=\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tilde{\mathbf{H}}_{\mathbf{ux}}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)+\tilde{\mathbf{H}}_{\mathbf{xu}}\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)+\tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec(\mathbf{\Sigma})+vec(\mathbf{\Sigma})\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$\Updownarrow$

$$\mathbf{x}_{t+1}^{s}=\mathbf{h_x}\mathbf{x}_t^s+\tilde{\mathbf{H}}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tilde{\mathbf{H}}_{\mathbf{ux}}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)+\tilde{\mathbf{H}}_{\mathbf{xu}}\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)+\tilde{\mathbf{H}}_{\mathbf{uu}}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec(\mathbf{\Sigma})\right)$$
$$+\tilde{\mathbf{H}}_{\mathbf{uu}}vec(\mathbf{\Sigma})+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

where we have defined

$\tilde{\mathbf{H}}_{\mathbf{xx}}\equiv\frac{1}{2}reshape\left(\mathbf{h_{xx}},n_x,n_x^2\right)$

$\tilde{\mathbf{H}}_{\mathbf{uu}}\equiv\frac{1}{2}reshape\left(\mathbf{h_{uu}},n_x,n_u^2\right)$

$\tilde{\mathbf{H}}_{\mathbf{ux}}\equiv\frac{1}{2}reshape\left(\mathbf{h_{ux}},n_x,n_un_x\right)$

$\tilde{\mathbf{H}}_{\mathbf{xu}}\equiv\frac{1}{2}reshape\left(\mathbf{h_{xu}},n_x,n_xn_u\right)$

Hence, we need to find the law of motions for $\mathbf{x}_t^f\otimes\mathbf{x}_t^f$

$$\mathbf{x}_t^f\otimes\mathbf{x}_t^f=\left(\mathbf{h_x}\mathbf{x}_{t-1}^f+\mathbf{h_u}\mathbf{u}_t\right)\otimes\left(\mathbf{h_x}\mathbf{x}_{t-1}^f+\mathbf{h_u}\mathbf{u}_t\right)$$

$$=\mathbf{h_x}\mathbf{x}_{t-1}^f\otimes\left(\mathbf{h_x}\mathbf{x}_{t-1}^f+\mathbf{h_u}\mathbf{u}_t\right)+\mathbf{h_u}\mathbf{u}_t\otimes\left(\mathbf{h_x}\mathbf{x}_{t-1}^f+\mathbf{h_u}\mathbf{u}_t\right)$$

$$=\mathbf{h_x}\mathbf{x}_{t-1}^f\otimes\mathbf{h_x}\mathbf{x}_{t-1}^f+\mathbf{h_x}\mathbf{x}_{t-1}^f\otimes\mathbf{h_u}\mathbf{u}_t+\mathbf{h_u}\mathbf{u}_t\otimes\mathbf{h_x}\mathbf{x}_{t-1}^f+\mathbf{h_u}\mathbf{u}_t\otimes\mathbf{h_u}\mathbf{u}_t$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_{t-1}^f\otimes\mathbf{x}_{t-1}^f\right)+\left(\mathbf{h_x}\otimes\mathbf{h_u}\right)\left(\mathbf{x}_{t-1}^f\otimes\mathbf{u}_t\right)$$
$$+\left(\mathbf{h_u}\otimes\mathbf{h_x}\right)\left(\mathbf{u}_t\otimes\mathbf{x}_{t-1}^f\right)+\left(\mathbf{h_u}\otimes\mathbf{h_u}\right)\left(\mathbf{u}_t\otimes\mathbf{u}_t\right)$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_{t-1}^f\otimes\mathbf{x}_{t-1}^f\right)+\left(\mathbf{h_x}\otimes\mathbf{h_u}\right)\left(\mathbf{x}_{t-1}^f\otimes\mathbf{u}_t\right)$$
$$+\left(\mathbf{h_u}\otimes\mathbf{h_x}\right)\left(\mathbf{u}_t\otimes\mathbf{x}_{t-1}^f\right)+\left(\mathbf{h_u}\otimes\mathbf{h_u}\right)\left(\mathbf{u}_t\otimes\mathbf{u}_t\right)$$

Thus

$$\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_x}\otimes\mathbf{h_u}\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)$$
$$+\left(\mathbf{h_u}\otimes\mathbf{h_x}\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h_u}\otimes\mathbf{h_u}\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec(\mathbf{\Sigma})+vec(\mathbf{\Sigma})\right)$$

$$
\begin{aligned}
&= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + (\mathbf{h_x} \otimes \mathbf{h_u}) \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right) \\
&\quad + (\mathbf{h_u} \otimes \mathbf{h_x}) \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right) + (\mathbf{h_u} \otimes \mathbf{h_u}) \left((\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1}) - vec(\mathbf{\Sigma})\right) \\
&\quad + (\mathbf{h_u} \otimes \mathbf{h_u}) \, vec(\mathbf{\Sigma})
\end{aligned}
$$

Thus we can set up the following system

$$
\begin{bmatrix} \mathbf{x}_{t+1}^f \\ \mathbf{x}_{t+1}^s \\ \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{H}}_{\mathbf{uu}} vec(\mathbf{\Sigma}) \\ (\mathbf{h_u} \otimes \mathbf{h_u}) \, vec(\mathbf{\Sigma}) \end{bmatrix} + \begin{bmatrix} \mathbf{h_x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}
$$

$$
+ \begin{bmatrix} \mathbf{h_u} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_{\mathbf{uu}} & \tilde{\mathbf{H}}_{\mathbf{ux}} & \tilde{\mathbf{H}}_{\mathbf{xu}} \\ \mathbf{0} & \mathbf{h_u} \otimes \mathbf{h_u} & (\mathbf{h_u} \otimes \mathbf{h_x}) & (\mathbf{h_u} \otimes \mathbf{h_u}) \end{bmatrix} \begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec(\mathbf{\Sigma}) \\ \mathbf{u}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \mathbf{u}_{t+1} \end{bmatrix}
$$

$\Updownarrow$

$$
\mathbf{z}_{t+1} = \mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}
$$

where we have defined

$$
\mathbf{c} \equiv \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{H}}_{\mathbf{uu}} vec(\mathbf{\Sigma}) \\ (\mathbf{h_u} \otimes \mathbf{h_u}) \, vec(\mathbf{\Sigma}) \end{bmatrix}
$$

$$
\mathbf{A} \equiv \begin{bmatrix} \mathbf{h_x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} & \tilde{\mathbf{H}}_{\mathbf{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} \end{bmatrix}
$$

$$
\mathbf{B} \equiv \begin{bmatrix} \mathbf{h_u} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_{\mathbf{uu}} & \tilde{\mathbf{H}}_{\mathbf{ux}} & \tilde{\mathbf{H}}_{\mathbf{xu}} \\ \mathbf{0} & \mathbf{h_u} \otimes \mathbf{h_u} & (\mathbf{h_u} \otimes \mathbf{h_x}) & (\mathbf{h_u} \otimes \mathbf{h_u}) \end{bmatrix}
$$

$$
\boldsymbol{\xi}_{t+1} \equiv \begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec(\mathbf{\Sigma}) \\ \mathbf{u}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \mathbf{u}_{t+1} \end{bmatrix}
$$

Similar arguments as presented above ensure that all eigenvalues of $\mathbf{A}$ have modulus less than one provided the same holds for $\mathbf{h_x}$.

For the control variables we have

$$
\begin{aligned}
&y_t(i,1) = \mathbf{g_x}(i,:)\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}(i,:)\,\mathbf{u}_t \\
&\quad + \tfrac{1}{2}\left(\mathbf{x}_{t-1}^f\right)' \mathbf{g_{xx}}(i,:,:)\,\mathbf{x}_{t-1}^f + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{ux}}(i,:,:)\,\mathbf{x}_{t-1}^f \\
&\quad + \tfrac{1}{2}\left(\mathbf{x}_{t-1}^f\right)' \mathbf{g_{xu}}(i,:,:)\,\mathbf{u}_t + \tfrac{1}{2}\mathbf{u}_t'\mathbf{g_{uu}}(i,:,:)\,\mathbf{u}_t \\
&\quad + \tfrac{1}{2}g_{\sigma\sigma}(i,1)\,\sigma^2
\end{aligned}
$$

$\Downarrow$

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{g_x}\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}\mathbf{u}_t + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\mathbf{x}_{t-1}^f \otimes \mathbf{x}_{t-1}^f\right) \\
&\quad + \tilde{\mathbf{G}}_{\mathbf{ux}}\left(\mathbf{u}_t \otimes \mathbf{x}_{t-1}^f\right) + \tilde{\mathbf{G}}_{\mathbf{xu}}\left(\mathbf{x}_{t-1}^f \otimes \mathbf{u}_t\right) + \tilde{\mathbf{G}}_{\mathbf{uu}}(\mathbf{u}_t \otimes \mathbf{u}_t) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{g_x}\left(\mathbf{x}_{t-1}^f + \mathbf{x}_{t-1}^s\right) + \mathbf{g_u}\mathbf{u}_t + \tilde{\mathbf{G}}_{\mathbf{xx}}\left(\mathbf{x}_{t-1}^f \otimes \mathbf{x}_{t-1}^f\right)
\end{aligned}
$$

$$+\tilde{\mathbf{G}}_{\mathbf{ux}}\left(\mathbf{u}_t \otimes \mathbf{x}_{t-1}^f\right) + \tilde{\mathbf{G}}_{\mathbf{xu}}\left(\mathbf{x}_{t-1}^f \otimes \mathbf{u}_t\right) + \tilde{\mathbf{G}}_{\mathbf{uu}}\left((\mathbf{u}_t \otimes \mathbf{u}_t) - vec\left(\mathbf{\Sigma}\right)\right)$$
$$+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{G}}_{\mathbf{uu}} vec\left(\mathbf{\Sigma}\right)$$

where we have defined

$$\tilde{\mathbf{G}}_{\mathbf{xx}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{xx}}, n_y, n_x^2\right)$$
$$\tilde{\mathbf{G}}_{\mathbf{uu}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{uu}}, n_y, n_u^2\right)$$
$$\tilde{\mathbf{G}}_{\mathbf{ux}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{ux}}, n_y, n_u n_x\right)$$
$$\tilde{\mathbf{G}}_{\mathbf{xu}} \equiv \frac{1}{2} reshape\left(\mathbf{g}_{\mathbf{xu}}, n_y, n_x n_u\right)$$

Thus

$$\mathbf{y}_t = \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tilde{\mathbf{G}}_{\mathbf{uu}} vec\left(\mathbf{\Sigma}\right) + \begin{bmatrix} \mathbf{g}_{\mathbf{x}} & \mathbf{g}_{\mathbf{x}} & \tilde{\mathbf{G}}_{\mathbf{xx}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1}^f \\ \mathbf{x}_{t-1}^s \\ \mathbf{x}_{t-1}^f \otimes \mathbf{x}_{t-1}^f \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{g}_{\mathbf{u}} & \tilde{\mathbf{G}}_{\mathbf{uu}} & \tilde{\mathbf{G}}_{\mathbf{ux}} & \tilde{\mathbf{G}}_{\mathbf{xu}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_t \\ \mathbf{u}_t \otimes \mathbf{u}_t - vec\left(\mathbf{\Sigma}\right) \\ \mathbf{u}_t \otimes \mathbf{x}_{t-1}^f \\ \mathbf{x}_{t-1}^f \otimes \mathbf{u}_t \end{bmatrix}$$

$\Updownarrow$

$$\mathbf{y}_t = \mathbf{d} + \mathbf{E}\mathbf{z}_t + \mathbf{F}\boldsymbol{\xi}_t$$

7.2 First and second moments

We also see that $E_t\left[\boldsymbol{\xi}_{t+1}\right] = \mathbf{0}$. Hence, the first and second moments for $\mathbf{z}_t$ are:

$$E\left[\mathbf{z}_t\right] = \left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{c}$$

and for the variances we have that

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)'\right]$$

$$= E\left[\left(\mathbf{c} + \mathbf{A}\mathbf{z}_t + \mathbf{B}\boldsymbol{\xi}_{t+1}\right)\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$= E\left[\mathbf{c}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\left(\mathbf{c}' + \mathbf{z}_t'\mathbf{A}' + \boldsymbol{\xi}_{t+1}'\mathbf{B}'\right)\right]$$

$$= E\left[\mathbf{c}\mathbf{c}' + \mathbf{c}\mathbf{z}_t'\mathbf{A}' + \mathbf{c}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{A}\mathbf{z}_t\mathbf{c}' + \mathbf{A}\mathbf{z}_t\mathbf{z}_t'\mathbf{A}' + \mathbf{A}\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$
$$+E\left[\mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{c}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\mathbf{A}' + \mathbf{B}\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\mathbf{B}'\right]$$

$$= \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}'$$
$$+\mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$+\mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

We then note that

$$E\left[\mathbf{z}_t\boldsymbol{\xi}_{t+1}'\right] = E\left[\begin{bmatrix} \mathbf{x}_t^f \\ \mathbf{x}_t^s \\ \mathbf{x}_t^f \otimes \mathbf{x}_t^f \end{bmatrix}\begin{bmatrix} \mathbf{u}_{t+1}' & \left(\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\left(\mathbf{\Sigma}\right)\right)' & \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right)' \end{bmatrix}\right]$$

$$= E \begin{bmatrix} \mathbf{x}_t^f \mathbf{u}_{t+1}' & \mathbf{x}_t^f \left(\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\left(\boldsymbol{\Sigma}\right)\right)' & \mathbf{x}_t^f \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^f \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right)' \\ \mathbf{x}_t^s \mathbf{u}_{t+1}' & \mathbf{x}_t^s \left(\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\left(\boldsymbol{\Sigma}\right)\right)' & \mathbf{x}_t^s \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right)' & \mathbf{x}_t^s \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right)' \\ \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \mathbf{u}_{t+1}' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\left(\boldsymbol{\Sigma}\right)\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right)' \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Thus

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] = \mathbf{c}\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$$= \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

Note also that

$$E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)'$$

$$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

So

$$E\left[\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t\right]' = \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' + \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' + \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$
$$- \left(\mathbf{c} + \mathbf{A}E\left[\mathbf{z}_t\right]\right)\mathbf{c}' - \mathbf{c}E\left[\mathbf{z}_t'\right]\mathbf{A}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}E\left[\mathbf{z}_t\mathbf{z}_t'\right]\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}' - \mathbf{A}E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\mathbf{A}'$$

$$= \mathbf{A}\left(E\left[\mathbf{z}_t\mathbf{z}_t'\right] - E\left[\mathbf{z}_t\right]E\left[\mathbf{z}_t'\right]\right)\mathbf{A}' + \mathbf{B}E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]\mathbf{B}'$$

$\Updownarrow$

$$Var\left(\mathbf{z}_t\right) = \mathbf{A}Var\left(\mathbf{z}_t\right)\mathbf{A}' + \mathbf{B}Var\left(\boldsymbol{\xi}_{t+1}\right)\mathbf{B}'$$

For the control we have directly that

$$E\left[\mathbf{y}_t\right] = \mathbf{d} + \mathbf{E}E\left[\mathbf{z}_t\right]$$

$$Var\left[\mathbf{y}_t\right] = \mathbf{E}Var\left[\mathbf{z}_t\right]\mathbf{E}' + \mathbf{F}Var\left(\boldsymbol{\xi}_t\right)\mathbf{F}'$$

where we use that $Cov\left(\mathbf{z}_t, \boldsymbol{\xi}_t\right) = 0$. Note that we trivially have $E\left[\mathbf{x}_t^f\right] = \mathbf{0}$ which we will use below. Hence, we only need to compute $Var\left(\boldsymbol{\xi}_{t+1}\right)$.

7.2.1 Computing $Var\left(\boldsymbol{\xi}_{t+1}\right)$

We have

$$Var\left(\boldsymbol{\xi}_{t+1}\right) = E\left[\boldsymbol{\xi}_{t+1}\boldsymbol{\xi}_{t+1}'\right]$$

$$= E\left[\begin{bmatrix} \mathbf{u}_{t+1} \\ \mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1} - vec\left(\boldsymbol{\Sigma}\right) \\ \mathbf{u}_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \mathbf{u}_{t+1} \end{bmatrix} \begin{matrix} \mathbf{u}_{t+1}' & \left(\mathbf{u}_{t+1} \otimes \mathbf{u}_{t+1}\right)' - vec\left(\mathbf{I}\right)' & \left(\mathbf{u}_{t+1} \otimes \mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{u}_{t+1}\right)' \end{matrix}\right]$$

$$
= E\left[\begin{array}{cc}
\mathbf{u}_{t+1}\mathbf{u}_{t+1}' & \mathbf{u}_{t+1}\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\boldsymbol{\Sigma}\right)\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\boldsymbol{\Sigma}\right)\right)\mathbf{u}_{t+1}' & \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\mathbf{I}\right)\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\boldsymbol{\Sigma}\right)\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)\mathbf{u}_{t+1}' & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\boldsymbol{\Sigma}\right)\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)\mathbf{u}_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\boldsymbol{\Sigma}\right)\right)'
\end{array}\right.
$$

$$
\left.\begin{array}{cc}
\mathbf{u}_{t+1}\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{u}_{t+1}\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\boldsymbol{\Sigma}\right)\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\boldsymbol{\Sigma}\right)\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)'
\end{array}\right]
$$

$$
= \left[\begin{array}{cc}
\boldsymbol{\Sigma} & E\left[\mathbf{u}_{t+1}\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)'\right] \\
E\left[\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)\mathbf{u}_{t+1}'\right] & E\left[\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}-vec\left(\boldsymbol{\Sigma}\right)\right)\left(\left(\mathbf{u}_{t+1}\otimes\mathbf{u}_{t+1}\right)-vec\left(\boldsymbol{\Sigma}\right)\right)'\right] \\
\mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right.
$$

$$
\left.\begin{array}{cc}
\mathbf{0} & \mathbf{0}' \\
\mathbf{0} & \mathbf{0} \\
\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)' \\
\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{u}_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)\left(\mathbf{x}_t^f\otimes\mathbf{u}_{t+1}\right)'
\end{array}\right]
$$

These elements can be coded directly as shown above.

8 Equivalence between the SGU-notation and the Dynare notation

This section shows the equivalence between the notation by (Schmitt-Grohé & Uribe (2004)), i.e. the SGU-notation, where innovations only enter linearly and the Dynare and Dynare ++ notation where innovations may enter in a non-linear fashion. The key observation is that the SGU-notation actually also includes the Dynare and Dynare ++ notation when extending the state vector accordingly. Recall from above that the SGU-notation reads:

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t, \sigma\right) \tag{67}$$

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \tag{68}$$

By lagging (68) by one period we get

$$\mathbf{x}_t = \mathbf{h}\left(\mathbf{x}_{t-1}, \sigma\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_t \tag{69}$$

and $\mathbf{v}_t \equiv \left[\begin{array}{cc}\mathbf{x}_{t-1} & \boldsymbol{\epsilon}_t\end{array}\right]$ can then be considered as the extended state vector .Hence, for this extended system we thus have

$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_{t-1}, \boldsymbol{\epsilon}_t, \sigma\right) \tag{70}$$

$$\mathbf{x}_t = \mathbf{h}_1\left(\mathbf{x}_{t-1}, \boldsymbol{\epsilon}_t, \sigma\right) \tag{71}$$

$$\boldsymbol{\epsilon}_{t+1} = \mathbf{u}_{t+1} \tag{72}$$

which is the Dynare and Dynare ++ notation with innovations entering nonlinearly. Accordingly, we can without loss of generality consider the SGU-notation.

We next illustrate how the Dynare-notation can be implemented with SGU-codes for the simple neoclassical model.The standard implementation reads:

$$f \equiv \begin{bmatrix} c_t + k_{t+1} - (1-\delta) k_t - a_t k_t^{\alpha} \\ c_t^{-\gamma} - \beta c_{t+1}^{-\gamma} \left(a_{t+1} \alpha k_{t+1}^{\alpha-1} + 1 - \delta \right) \\ \log a_{t+1} - \rho \log a_t \end{bmatrix}$$

where $\mathbf{x}_t \equiv \begin{bmatrix} k_t & a_t \end{bmatrix}$ and $\mathbf{y}_t \equiv [c_t]$. The equivalent Dynare-notation implementation is given by

$$f \equiv \begin{bmatrix} c_t + k_{t+1} - (1-\delta) k_t - a_t k_t^{\alpha} \\ c_t^{-\gamma} - \beta c_{t+1}^{-\gamma} \left(\exp \{ \rho \log a_t + \sigma \boldsymbol{\epsilon}_{t+1} \} \alpha k_{t+1}^{\alpha-1} + 1 - \delta \right) \\ \log a_t - \rho \log a_{t-1} - \sigma \boldsymbol{\epsilon}_t \\ \boldsymbol{\epsilon}_{t+1} \end{bmatrix}$$

where $\mathbf{x}_t \equiv \begin{bmatrix} k_t & a_{t-1} & \boldsymbol{\epsilon}_t \end{bmatrix}$ and $\mathbf{y}_t \equiv [c_t]$.

9 Existence of Skewness and Kurtosis

This section derives conditions for the existence of skewness and kurtosis in a linear system. We consider the system $\mathbf{x}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}$ where $\mathbf{A}$ is stable and $\mathbf{v}_{t+1}$ are mean-zero innovations. Thus, the pruned state-space representation for DSGE models belong to this class. For notational convience, the system is re-express in deviation from its mean as $(\mathbf{I} - \mathbf{A}) E[\mathbf{x}] = \mathbf{a}$ and therefore

$$\mathbf{x}_{t+1} = (\mathbf{I} - \mathbf{A}) E[\mathbf{x}] + \mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}$$
$$\Updownarrow$$
$$\mathbf{x}_{t+1} - E[\mathbf{x}] = \mathbf{A}(\mathbf{x}_t - E[\mathbf{x}]) + \mathbf{v}_{t+1}$$
$$\Updownarrow$$
$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}$$

We then have

$$\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} = (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{A}\mathbf{z}_t \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}$$

$$\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} = (\mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}) \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$= \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})$$

$$\begin{aligned} &= \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \\ &+ \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \\ &+ \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t \otimes \mathbf{v}_{t+1} \\ &+ \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \end{aligned}$$

Thus, to solve for $E[\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}]$ the innovations need to have a finite third moment. At second order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite sixth moment. At third order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite ninth moment.

$$\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}$$

$$
\begin{aligned}
&= (\mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \\
&+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \\
&+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \\
&+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}) \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) \\
&+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) \\
&+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) \\
&+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1}) + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes (\mathbf{Az}_t + \mathbf{v}_{t+1})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \\
&+\mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \\
&+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \\
&+\mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \\
&+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \\
&+\mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \\
&+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t \otimes \mathbf{v}_{t+1} \\
&+\mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{Az}_t + \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1} \otimes \mathbf{v}_{t+1}
\end{aligned}
$$

Thus, to solve for $E\left[\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}\right]$ the innovations need to have a finite fourth moment. At second order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite eight moment. At third order, $\mathbf{v}_{t+1}$ is function of $\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}$, meaning that $\boldsymbol{\epsilon}_{t+1}$ must have finite twelve moment.

10 Impulse response functions - the definition by Andreasen

This section derives closed-form solutions for the impulse response function in non-linear DSGE models. Note that this section uses the definition of an impulse response function suggested by Andreasen (2012). This definition is

$$
\begin{aligned}
IRF_{\mathbf{var}}\left(l, \boldsymbol{\nu}, \mathbf{w}_t\right) &= E_t\left[\mathbf{var}_{t+l} | \boldsymbol{\epsilon}_{t+1} + \boldsymbol{\nu}\right] \\
&\quad - E_t\left[\mathbf{var}_{t+l}\right]
\end{aligned}
$$

where $\boldsymbol{\epsilon}_{t+1}$ is stochastic (appologies for the bad notation!). To reduce the notational burden in the derivations below, we adopt the parsimonious notation

$$
IRF_{\mathbf{var}}\left(l, \boldsymbol{\nu}, \mathbf{w}_t\right) = E_t\left[\widetilde{\mathbf{var}}_{t+l}\right] - E_t\left[\mathbf{var}_{t+l}\right]
$$

in relation to the conditional expectation operators.

10.1 At first order

Recall that we have:

$$
\mathbf{x}_{t+1}^f = \mathbf{h}_{\mathbf{x}} \mathbf{x}_t^f + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}
$$

and

$$
\begin{aligned}
\mathbf{x}_{t+2}^f &= \mathbf{h}_{\mathbf{x}} \mathbf{x}_{t+1}^f + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+2} \\
&= \mathbf{h}_{\mathbf{x}} \left(\mathbf{h}_{\mathbf{x}} \mathbf{x}_t^f + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+2} \\
&= \mathbf{h}_{\mathbf{x}}^2 \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{x}_{t+3}^f &= \mathbf{h}_{\mathbf{x}} \mathbf{x}_{t+2}^f + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+3} \\
&= \mathbf{h}_{\mathbf{x}} \left(\mathbf{h}_{\mathbf{x}}^2 \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+2}\right) + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+3}
\end{aligned}
$$

$$= \mathbf{h}_{\mathbf{x}}^3 \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^2 \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+2} + \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+3}$$
$$= \mathbf{h}_{\mathbf{x}}^3 \mathbf{x}_t^f + \sum_{j=1}^{3} \mathbf{h}_{\mathbf{x}}^{3-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

In general
$$\mathbf{x}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

With a shock of $\boldsymbol{\nu}$ in period $t+1$, we have
$$\tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j} \right)$$
where we define $\boldsymbol{\delta}_t$ such that:
$\boldsymbol{\delta}_{t+j} = \boldsymbol{\nu}$ for $j = 1$
$\boldsymbol{\delta}_{t+j} = \mathbf{0}$ for $j \neq 1$

So
$$E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \right] = E_t \left[\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j} \right) - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right]$$

$$= \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}$$
using the definition of $\boldsymbol{\delta}_{t+j}$.

$$= \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu}$$
because $\boldsymbol{\delta}_{t+1} = \boldsymbol{\nu}$

and
$$E_t \left[\tilde{\mathbf{y}}_{t+l}^f - \mathbf{y}_{t+l}^f \right] = \mathbf{g}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \right]$$

10.2 At second order

We need to consider:
$$\mathbf{x}_{t+1}^s = \mathbf{h}_{\mathbf{x}} \mathbf{x}_t^s + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$

$$\mathbf{x}_{t+2}^s = \mathbf{h}_{\mathbf{x}} \mathbf{x}_{t+1}^s + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$
$$= \mathbf{h}_{\mathbf{x}} \left(\mathbf{h}_{\mathbf{x}} \mathbf{x}_t^s + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right) + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$
$$= \mathbf{h}_{\mathbf{x}}^2 \mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$

$$\mathbf{x}_{t+3}^s = \mathbf{h}_{\mathbf{x}} \mathbf{x}_{t+2}^s + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$
$$= \mathbf{h}_{\mathbf{x}} \left(\mathbf{h}_{\mathbf{x}}^2 \mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right)$$
$$+ \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$
$$= \mathbf{h}_{\mathbf{x}}^3 \mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}^2 \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \mathbf{h}_{\mathbf{x}}^2 \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$
$$+ \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right) + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$

$$= \mathbf{h}_{\mathbf{x}}^3 \mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}^2 \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right)$$
$$+ \mathbf{h}_{\mathbf{x}}^2 \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$

$$= \mathbf{h}_{\mathbf{x}}^3 \mathbf{x}_t^s + \sum_{j=0}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right) + \left(\sum_{j=0}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \right) \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$

and in general

$$\mathbf{x}_{t+l}^s = \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^s + \sum_{j=0}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right) + \left(\sum_{j=0}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \right) \tfrac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2$$

for $l = 1, 2, 3, ...$

Thus, to compute $E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right]$, we need to find $E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right]$. Hence, consider:

$$\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f = \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right)$$
$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

and

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j} \right)$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j} \right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j} \right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j} \right)$$

$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right)$$
$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right)$$

This means that:

$$E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right]$$

$$= E_t [\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right)$$
$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \right)$$
$$- \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f - \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}]$$

$$=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$

$$=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
because $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}$

Thus $E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$
Or (using another index)
$E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]=\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$
for $j=1,2,3,...$

Thus, we have in general
$$E_t\left[\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^s\right]=E_t\left[\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\right)-\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)\right]$$

$$=\left[\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]\right]$$
the shock hits in period $t+1$, so $\left(\tilde{\mathbf{x}}_t^f\otimes\tilde{\mathbf{x}}_t^f\right)=\mathbf{x}_t^f\otimes\mathbf{x}_t^f$

$$=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)$$

$$= \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\mathbf{h}_{\mathbf{x}}^{j} \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{j-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{j-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{j} \mathbf{x}_t^f + \left(\mathbf{h}_{\mathbf{x}}^{j-1} \otimes \mathbf{h}_{\mathbf{x}}^{j-1} \right) (\sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\nu}) \right)$$

If we restrict the focus and do the IRF's at the unconditional mean of $\mathbf{x}_t^f = \mathbf{0}$, then we get

$$E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right] = \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \frac{1}{2} \mathbf{H}_{\mathbf{xx}} \left(\left(\mathbf{h}_{\mathbf{x}}^{j-1} \otimes \mathbf{h}_{\mathbf{x}}^{j-1} \right) (\sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\nu}) \right)$$

When implementing the IRF, it may be useful to have a recursive expression. Here, it is must convenient to use $E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right] = \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right]$

So

$E_t \left[\tilde{\mathbf{x}}_{t+1}^s - \mathbf{x}_{t+1}^s \right] = 0$

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+2}^s - \mathbf{x}_{t+2}^s \right] &= \sum_{j=1}^{1} \mathbf{h}_{\mathbf{x}}^{1-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right] \\ &= \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right] \end{aligned}$$

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+3}^s - \mathbf{x}_{t+3}^s \right] &= \sum_{j=1}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f - \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \right] \\ &= \mathbf{h}_{\mathbf{x}} \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right] \\ &+ \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right] \\ &= \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+2}^s - \mathbf{x}_{t+2}^s \right] + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \right] \end{aligned}$$

So in general

$E_t \left[\tilde{\mathbf{x}}_{t+k}^s - \mathbf{x}_{t+k}^s \right] = \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^s - \mathbf{x}_{t+k-1}^s \right] + \tfrac{1}{2} \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^f - \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^f \right]$

For the total state variable:

$E_t \left[\tilde{\mathbf{x}}_{t+l} - \mathbf{x}_{t+l} \right] = E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right]$

For the control variables:

$\mathbf{y}_{t+l}^s = \mathbf{g}_{\mathbf{x}} \left(\mathbf{x}_{t+l}^f + \mathbf{x}_{t+l}^s \right) + \tfrac{1}{2} \mathbf{G}_{\mathbf{xx}} \left(\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right) + \tfrac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2$

$\tilde{\mathbf{y}}_{t+l}^s = \mathbf{g}_{\mathbf{x}} \left(\tilde{\mathbf{x}}_{t+l}^f + \tilde{\mathbf{x}}_{t+l}^s \right) + \tfrac{1}{2} \mathbf{G}_{\mathbf{xx}} \left(\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \right) + \tfrac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2$

$E_t \left[\tilde{\mathbf{y}}_{t+l}^s - \mathbf{y}_{t+l}^s \right] = \mathbf{g}_{\mathbf{x}} \left(E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s \right] \right) + \tfrac{1}{2} \mathbf{G}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right]$

10.3 At third order

At third order, we additionally need to consider:

$\mathbf{x}_{t+1}^{rd} = \mathbf{h}_{\mathbf{x}} \mathbf{x}_t^{rd} + \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s \right) + \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 \mathbf{x}_t^f + \tfrac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3$

and

$\mathbf{x}_{t+2}^{rd} = \mathbf{h}_{\mathbf{x}} \mathbf{x}_{t+1}^{rd} + \mathbf{H}_{\mathbf{xx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s \right) + \tfrac{1}{6} \mathbf{H}_{\mathbf{xxx}} \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \right) + \tfrac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 \mathbf{x}_{t+1}^f + \tfrac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3$

$$= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}_t^{rd} + \mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}^2\mathbf{x}_t^{rd} + \mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

and

$$\mathbf{x}_{t+3}^{rd} = \mathbf{h_x}\mathbf{x}_{t+2}^{rd} + \mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+2}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}[\mathbf{h_x}^2\mathbf{x}_t^{rd} + \mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3]$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+2}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}^3\mathbf{x}_t^{rd} + \mathbf{h_x}^2\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \mathbf{h_x}^2\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h_x}^2\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \mathbf{h_x}^2\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+1}^f + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+2}^f + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}^3\mathbf{x}_t^{rd} + \sum_{j=0}^{2}\mathbf{h_x}^{2-j}\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right)$$
$$+ \sum_{j=0}^{2}\mathbf{h_x}^{2-j}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right)$$
$$+ \sum_{j=0}^{2}\mathbf{h_x}^{2-j}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f$$
$$+ \sum_{j=0}^{2}\mathbf{h_x}^{2-j}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}^3\mathbf{x}_t^{rd} + \sum_{j=0}^{2}\mathbf{h_x}^{2-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]$$
$$+ \left(\sum_{j=0}^{2}\mathbf{h_x}^{2-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

In general

$$\mathbf{x}_{t+l}^{rd} = \mathbf{h_x}^l\mathbf{x}_t^{rd} + \sum_{j=0}^{l-1}\mathbf{h_x}^{l-1-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]$$
$$+ \left(\sum_{j=0}^{l-1}\mathbf{h_x}^{l-1-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

Thus

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^{rd} - \mathbf{x}_{t+l}^{rd}\right] = E_t[\sum_{j=0}^{l-1}\mathbf{h_x}^{l-1-j}\left[\mathbf{H_{xx}}\left(\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f \otimes \tilde{\mathbf{x}}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}_{t+j}^f\right]$$
$$-\{\sum_{j=0}^{l-1}\mathbf{h_x}^{l-1-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]\}]$$

$$= \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{s} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \right] \right)$$
$$+ \sum_{j=1}^{l-1} \mathbf{h}_{\mathbf{x}}^{l-1-j} \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \right]$$

as the shock hits the economy in period $t+1$

A recursive version:
$E_t \left[\tilde{\mathbf{x}}_{t+1}^{rd} - \mathbf{x}_{t+1}^{rd} \right] = 0$

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+2}^{rd} - \mathbf{x}_{t+2}^{rd} \right] &= \sum_{j=1}^{1} \mathbf{h}_{\mathbf{x}}^{1-j} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{s} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \right] \right) \\ &\quad + \sum_{j=1}^{1} \mathbf{h}_{\mathbf{x}}^{1-j} \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \right] \\ &= \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^{f} \otimes \tilde{\mathbf{x}}_{t+1}^{s} - \mathbf{x}_{t+1}^{f} \otimes \mathbf{x}_{t+1}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+1}^{f} - \mathbf{x}_{t+1}^{f} \right] \\ &\quad + \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^{f} \otimes \tilde{\mathbf{x}}_{t+1}^{f} \otimes \tilde{\mathbf{x}}_{t+1}^{f} - \mathbf{x}_{t+1}^{f} \otimes \mathbf{x}_{t+1}^{f} \otimes \mathbf{x}_{t+1}^{f} \right] \end{aligned}$$

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+3}^{rd} - \mathbf{x}_{t+3}^{rd} \right] &= \sum_{j=1}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{s} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \right] \right) \\ &\quad + \sum_{j=1}^{2} \mathbf{h}_{\mathbf{x}}^{2-j} \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \right] \\ &= \mathbf{h}_{\mathbf{x}} \left(\mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^{f} \otimes \tilde{\mathbf{x}}_{t+1}^{s} - \mathbf{x}_{t+1}^{f} \otimes \mathbf{x}_{t+1}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+1}^{f} - \mathbf{x}_{t+1}^{f} \right] \right) \\ &\quad + \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^{f} \otimes \tilde{\mathbf{x}}_{t+2}^{s} - \mathbf{x}_{t+2}^{f} \otimes \mathbf{x}_{t+2}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+2}^{f} - \mathbf{x}_{t+2}^{f} \right] \\ &\quad + \mathbf{h}_{\mathbf{x}} \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+1}^{f} \otimes \tilde{\mathbf{x}}_{t+1}^{f} \otimes \tilde{\mathbf{x}}_{t+1}^{f} - \mathbf{x}_{t+1}^{f} \otimes \mathbf{x}_{t+1}^{f} \otimes \mathbf{x}_{t+1}^{f} \right] \\ &\quad + \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^{f} \otimes \tilde{\mathbf{x}}_{t+2}^{f} \otimes \tilde{\mathbf{x}}_{t+2}^{f} - \mathbf{x}_{t+2}^{f} \otimes \mathbf{x}_{t+2}^{f} \otimes \mathbf{x}_{t+2}^{f} \right] \\ &= \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+2}^{rd} - \mathbf{x}_{t+2}^{rd} \right] \\ &\quad + \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^{f} \otimes \tilde{\mathbf{x}}_{t+2}^{s} - \mathbf{x}_{t+2}^{f} \otimes \mathbf{x}_{t+2}^{s} \right] + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+2}^{f} - \mathbf{x}_{t+2}^{f} \right] \\ &\quad + \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+2}^{f} \otimes \tilde{\mathbf{x}}_{t+2}^{f} \otimes \tilde{\mathbf{x}}_{t+2}^{f} - \mathbf{x}_{t+2}^{f} \otimes \mathbf{x}_{t+2}^{f} \otimes \mathbf{x}_{t+2}^{f} \right] \end{aligned}$$

So in general

$$\begin{aligned} E_t \left[\tilde{\mathbf{x}}_{t+k}^{rd} - \mathbf{x}_{t+k}^{rd} \right] &= \mathbf{h}_{\mathbf{x}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^{rd} - \mathbf{x}_{t+k-1}^{rd} \right] + \mathbf{H}_{\mathbf{xx}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^{f} \otimes \tilde{\mathbf{x}}_{t+k-1}^{s} - \mathbf{x}_{t+k-1}^{f} \otimes \mathbf{x}_{t+k-1}^{s} \right] \\ &\quad + \frac{3}{6} \mathbf{h}_{\sigma\sigma\mathbf{x}} \sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+k-1}^{f} - \mathbf{x}_{t+k-1}^{f} \right] \\ &\quad + \frac{1}{6} \mathbf{H}_{\mathbf{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+k-1}^{f} \otimes \tilde{\mathbf{x}}_{t+k-1}^{f} \otimes \tilde{\mathbf{x}}_{t+k-1}^{f} - \mathbf{x}_{t+k-1}^{f} \otimes \mathbf{x}_{t+k-1}^{f} \otimes \mathbf{x}_{t+k-1}^{f} \right] \end{aligned}$$

Thus, we know $E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \right]$. So we only need to compute $E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{f} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{f} \right]$ and $E_t \left[\tilde{\mathbf{x}}_{t+j}^{f} \otimes \tilde{\mathbf{x}}_{t+j}^{s} - \mathbf{x}_{t+j}^{f} \otimes \mathbf{x}_{t+j}^{s} \right]$. This is done in the next two subsections.

10.3.1 For $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$

Consider:

$$\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f = \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$

$$= (\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j})$$
$$\otimes \left(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$

$$= (\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+(\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$

And we therefore have

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$
$$+ \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \left(\boldsymbol{\epsilon}_{t+j} + \boldsymbol{\delta}_{t+j}\right)$$

$$= \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+ \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

This means that:
$E_t\left[\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{f}-\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{f}\right]$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]$$

$$-E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]$$

$$-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]$$

$$-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)$$

$$
\begin{aligned}
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]
\end{aligned}
$$

$$
\begin{aligned}
&-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)]
\end{aligned}
$$

$$
\begin{aligned}
&-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}
$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$\begin{aligned}
&= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+0+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+0+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(0+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\left(\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}]
\end{aligned}$$

$-E_t[0$
$+\mathbf{0}$
$+0$
$+0]$
Terms cannel out

$$\begin{aligned}
&= E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+0\\
&+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{0}\\
&+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+0\\
&+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+0
\end{aligned}$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{0}$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}]$$
using the zero-mean property of $\boldsymbol{\epsilon}_{t+j}$

$$\begin{aligned}
&=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right]\\
&+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}
\end{aligned}$$

$$\begin{aligned}
&=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]\\
&+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}
\end{aligned}$$
using that $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}$

We then note that:

$$\begin{aligned}
&\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}
\end{aligned}$$

$$\begin{aligned}
&=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)
\end{aligned}$$

$$= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$$= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

Thus
$E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right]$

$$= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$
$$+E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]$$
$$+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

The final three terms can be computed as follows:
$E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+...$$
$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}$$
$$+...$$
$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}]$$
because the innovations are independent across time

$$= \sum_{j=1}^{l} E_t[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= \sum_{j=1}^{l} \boldsymbol{\Omega}_j$$

Let $\boldsymbol{\Omega}_j \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. So

$$\boldsymbol{\Omega}_j \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right]$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\begin{array}{c} \mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \\ \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x} \end{array}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\begin{array}{c} \mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_2=1}^{n_e}\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \\ \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x} \end{array}\right\}_{\gamma_3=1}^{n_x}$$

$$= \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

because the innovations are indepdendent. This expression is directly implementable.

And

$$E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]$$

$$= \sum_{j=1}^{l} E_t[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}]$$

$$= \sum_{j=1}^{l} E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}]$$

$$= \sum_{j=1}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\right) E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right] \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$$= \sum_{j=1}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j}\right)\boldsymbol{\Lambda} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

where $\boldsymbol{\Lambda} \equiv E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. To compute $\boldsymbol{\Lambda}$ we note that

$$\boldsymbol{\Lambda} = E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

$$= E_t\left\{\left\{\sigma\boldsymbol{\eta}(\gamma_2,:)\,\boldsymbol{\epsilon}_{t+1}\sigma\boldsymbol{\eta}(\gamma_1,:)\,\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= E_t\left\{\left\{\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(\gamma_2,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\sum_{\phi_2=1}^{n_e}\sigma\boldsymbol{\eta}(\gamma_1,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= E_t\left\{\left\{\sigma \sum_{\phi_1=1}^{n_e}\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}\left(\gamma_2,\phi_1\right)\boldsymbol{\epsilon}_{t+1}\left(\phi_1,1\right)\sigma\boldsymbol{\eta}\left(\gamma_1,\phi_2\right)\boldsymbol{\epsilon}_{t+1}\left(\phi_2,1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{\left\{\sigma \sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}\left(\gamma_2,\phi_1\right)\sigma\boldsymbol{\eta}\left(\gamma_1,\phi_1\right)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{\left\{\sigma^2\boldsymbol{\eta}\left(\gamma_2,:\right)\boldsymbol{\eta}\left(\gamma_1,:\right)'\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}$$

And finally

$$E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes E_t[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\boldsymbol{\Lambda}$$

10.3.2 For $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)$

Recall from above that

$$\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s = \left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$= \left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$

Therefore:

$$\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^s = \left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$= \left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)[\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}]$$

$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$= \left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+1}^{f}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+2}$$

and

$$\mathbf{x}_{t+3}^{f}\otimes\mathbf{x}_{t+3}^{s}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+2}^{f}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)[\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t}^{f}$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+1}^{f}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+2}]$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+2}^{f}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{3}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t}^{f}$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+1}^{f}$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+2}]$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+2}^{f}$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{3}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+i}^{f}$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{s}\right)$$
$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$

And in general

$$\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{s}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+i}^{f}$$

$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{s}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$

We therefore have

$$\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{s}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\tilde{\mathbf{x}}_{t+i}^{f}$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)\otimes\tilde{\mathbf{x}}_{t+i}^{s}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)\otimes\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\right)$$

Thus

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{s}-\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{s}\right]$$

$$=E_t[\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\tilde{\mathbf{x}}_{t+i}^{f}$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)\otimes\tilde{\mathbf{x}}_{t+i}^{s}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\left(\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i}\right)\otimes\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\right)$$

$$-\{\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+i}^{f}$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{s}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)\}$$
$$]$$

$$=E_t[\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}\otimes\tilde{\mathbf{x}}_{t+i}^{f}-\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$
$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+i}^{f}-\mathbf{x}_{t+i}^{f}\right)$$

$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\delta}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left((\boldsymbol{\epsilon}_{t+1+i}+\boldsymbol{\delta}_{t+1+i})\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$

$$-\{\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\}$$
$$]$$

$$=E_t[\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\delta}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$
$$]$$

because $\mathbf{x}^s_{t+i}$ is a function of $\mathbf{x}^f_{t+i}$ which is a function of $\boldsymbol{\epsilon}_{t+i}$. The zero-mean iid innovations therefore implies that, $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)\right]=\mathbf{0}$ and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}\right)\right]=\mathbf{0}$
The same argument implies that $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\right]=\mathbf{0}$
and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)\right]=\mathbf{0}$

$$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right]$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right]$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^s_t\right]\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right]\right)$$
using $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}$ for else $\boldsymbol{\delta}_{t+1+i}=0$
and therefore the index for l starts at 1

$$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right]$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right]$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^s_t\right]\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes E_t\left[\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right]\right)$$

$$= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right]$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\nu}$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\nu} \otimes \mathbf{x}^s_t)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\nu} \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)$$

because the shock hits in period $t+1$, so $E_t\left[\tilde{\mathbf{x}}^s_t\right] = \mathbf{x}^s_t$ and $E_t\left[\tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^f_t\right] = \mathbf{x}^f_t \otimes \mathbf{x}^f_t$

$$= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right]$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} (\sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}^s_t)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)\right)$$

$$= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right]$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}^s_t + \sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)\right)$$

$$= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right]$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

A recursive version for the sum reads:

$$X_l = \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right]$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

So

$$X_1 = \sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$X_2 = \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right]$$
$$+ \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$= \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1}\right]$$

$$+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma \boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) X_1 + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f\right]$$

$$X_3 = \sum_{i=1}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right]$$
$$+ \sum_{i=1}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^2 \left(\sigma \boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$
$$= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f\right]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f\right]$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^2 \left(\sigma \boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) X_2 +$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) E_t \left[\tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f \otimes \tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) E_t \left[\tilde{\mathbf{x}}_{t+2}^f - \mathbf{x}_{t+2}^f\right]$$

Hence, in general

$$X_k = (\mathbf{h_x} \otimes \mathbf{h_x}) X_{k-1} + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^f \otimes \tilde{\mathbf{x}}_{t+k-1}^f - \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^f \otimes \mathbf{x}_{t+k-1}^f\right)$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+k-1}^f - \mathbf{x}_{t+k-1}^f\right)$$

10.3.3 Summarizing

At third order, the total effect on the state variables is:

$$E_t \left[\tilde{\mathbf{x}}_{t+l} - \mathbf{x}_{t+l}\right] = E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f\right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s\right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^{rd} - \mathbf{x}_{t+l}^{rd}\right]$$

For the control variables:

$$\mathbf{y}_{t+l}^{rd} = \mathbf{g_x} \left(\mathbf{x}_{t+l}^f + \mathbf{x}_{t+l}^s + \mathbf{x}_{t+l}^{rd}\right) + \tfrac{1}{2}\mathbf{G_{xx}} \left(\left(\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right) + 2\left(\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^s\right)\right)$$
$$+ \tfrac{1}{6}\mathbf{G_{xxx}} \left(\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+l}^f + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$
$$\tilde{\mathbf{y}}_{t+l}^{rd} = \mathbf{g_x} \left(\tilde{\mathbf{x}}_{t+l}^f + \tilde{\mathbf{x}}_{t+l}^s + \tilde{\mathbf{x}}_{t+l}^{rd}\right) + \tfrac{1}{2}\mathbf{G_{xx}} \left(\left(\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f\right) + 2\left(\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^s\right)\right)$$
$$+ \tfrac{1}{6}\mathbf{G_{xxx}} \left(\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}_{t+l}^f + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

So:

$$E_t \left[\tilde{\mathbf{y}}_{t+l}^{rd} - \mathbf{y}_{t+l}^{rd}\right]$$
$$= \mathbf{g_x} \left(E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f\right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s\right] + E_t \left[\tilde{\mathbf{x}}_{t+l}^{rd} - \mathbf{x}_{t+l}^{rd}\right]\right)$$
$$+ \tfrac{1}{2}\mathbf{G_{xx}} \left(E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right] + 2E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^s\right]\right)$$
$$+ \tfrac{1}{6}\mathbf{G_{xxx}} E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right] + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 E_t \left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f\right]$$

11 Impulse response functions - GIRF

This section derives closed-form solutions for the generalized impulse response function in non-linear DSGE models when defined as the GIRF. That is

$$\begin{aligned} GIRF_{\mathbf{var}}\left(l,\boldsymbol{\nu},\mathbf{w}_t\right) &= E_t\left[\mathbf{var}_{t+l}|\boldsymbol{\epsilon}_{t+1}=\boldsymbol{\nu}\right] \\ &\quad -E_t\left[\mathbf{var}_{t+l}\right] \end{aligned}$$

To reduce the notational burden in the derivations below, we adopt the parsimonious notation

$$IRF_{\mathbf{var}}\left(l,\boldsymbol{\nu},\mathbf{w}_t\right) = E_t\left[\widetilde{\mathbf{var}}_{t+l}\right] - E_t\left[\mathbf{var}_{t+l}\right]$$

in relation to the conditional expectation operators.

11.1 At first order

Recall that we have:

$\mathbf{x}^f_{t+1} = \mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$

and

$$\begin{aligned} \mathbf{x}^f_{t+2} &= \mathbf{h_x}\mathbf{x}^f_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}^f_{t+3} &= \mathbf{h_x}\mathbf{x}^f_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h_x}\left(\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h}^2_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{3}\mathbf{h}^{3-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \end{aligned}$$

In general

$$\mathbf{x}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

With a shock of $\boldsymbol{\nu}$ in period $t+1$, we have

$$\tilde{\mathbf{x}}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

where we define $\boldsymbol{\delta}_t$ such that:

$$\begin{aligned} \boldsymbol{\delta}_{t+j} &= \boldsymbol{\nu} && \text{for } j=1 \\ \boldsymbol{\delta}_{t+j} &= \boldsymbol{\epsilon}_{t+j} && \text{for } j\neq 1 \end{aligned}$$

Hence, agents know the size of the shock $\boldsymbol{\nu}$ at time $t+1$, and it is therefore in agents' information set.

So

$$\begin{aligned} E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] &= E_t\left[\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right] \\ &= \mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \end{aligned}$$

$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$
because $\boldsymbol{\delta}_{t+1} = \boldsymbol{\nu}$

and
$E_t\left[\tilde{\mathbf{y}}_{t+l}^f - \mathbf{y}_{t+l}^f\right] = \mathbf{g}_{\mathbf{x}} E_t\left[\tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f\right]$

11.2 At second order

We need to consider:

$$\mathbf{x}_{t+1}^s = \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$$\begin{aligned}
\mathbf{x}_{t+2}^s &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_{t+1}^s + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}^2\mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_{t+3}^s &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_{t+2}^s + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}^2\mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \\
&\quad + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}^3\mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}^2\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_{\mathbf{x}}^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&\quad + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}^3\mathbf{x}_t^s + \mathbf{h}_{\mathbf{x}}^2\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) \\
&\quad + \mathbf{h}_{\mathbf{x}}^2\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
&= \mathbf{h}_{\mathbf{x}}^3\mathbf{x}_t^s + \sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \left(\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2
\end{aligned}$$

and in general

$$\mathbf{x}_{t+l}^s = \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^s + \sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right) + \left(\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

for $l = 1, 2, 3, \ldots$

Thus, to compute $E_t\left[\tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^s\right]$, we need to find $E_t\left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f\right]$. Hence, consider:

$$\begin{aligned}
\mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f &= \left(\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \\
&= \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \\
&\quad + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}$$

and

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f = \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

This means that:
$E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]$

$$\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}$$

$$\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-0\\
&-0-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]\\
&\text{using } E_t\left[\boldsymbol{\epsilon}_{t+j}\right]=\mathbf{0}
\end{aligned}$$

$$\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}$$

$$\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]
\end{aligned}$$

$$\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\
&-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\
&-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]
\end{aligned}$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} + 0$$
$$+0 + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$
$$-\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} + 0$$
$$-0 - \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}$$
$$+0$$
$$-\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}$$
$$-0]$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}$$
$$-\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}]$$

$$= \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu}$$
$$-E_t \left[\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right]$$

So we only need to compute $E_t \left[\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right]$. We then note that $E_t \left[\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right]$

$$= E_t \left[\left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right) \left(\sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right)\right]$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$= \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right) E_t \left[\sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right]$$

$$= \left(\mathbf{h}_\mathbf{x}^{l-1} \otimes \mathbf{h}_\mathbf{x}^{l-1}\right) \boldsymbol{\Lambda}$$

where $\boldsymbol{\Lambda} \equiv E_t \left[\sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right]$. We then have $\boldsymbol{\Lambda} = E_t \left[\sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right]$

$$= E_t \left\{ \left\{ \sigma \boldsymbol{\eta}(\gamma_2, :) \boldsymbol{\epsilon}_{t+1} \sigma \boldsymbol{\eta}(\gamma_1, :) \boldsymbol{\epsilon}_{t+1} \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

$$= E_t \left\{ \left\{ \sigma \textstyle\sum_{\phi_1=1}^{n_e} \boldsymbol{\eta}(\gamma_2, \phi_1) \boldsymbol{\epsilon}_{t+1}(\phi_1, 1) \sum_{\phi_2=1}^{n_e} \sigma \boldsymbol{\eta}(\gamma_1, \phi_2) \boldsymbol{\epsilon}_{t+1}(\phi_2, 1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

$$= E_t \left\{ \left\{ \sigma \textstyle\sum_{\phi_1=1}^{n_e} \sum_{\phi_2=1}^{n_e} \boldsymbol{\eta}(\gamma_2, \phi_1) \boldsymbol{\epsilon}_{t+1}(\phi_1, 1) \sigma \boldsymbol{\eta}(\gamma_1, \phi_2) \boldsymbol{\epsilon}_{t+1}(\phi_2, 1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{ \left\{ \sigma \textstyle\sum_{\phi_1=1}^{n_e} \boldsymbol{\eta}(\gamma_2, \phi_1) \sigma \boldsymbol{\eta}(\gamma_1, \phi_1) \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

$$= \left\{ \left\{ \sigma^2 \boldsymbol{\eta}(\gamma_2, :) \boldsymbol{\eta}(\gamma_1, :)' \right\}_{\gamma_1=1}^{n_x} \right\}_{\gamma_2=1}^{n_x}$$

Thus
$E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}$
$-\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)\boldsymbol{\Lambda}$

$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)(\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})$
$-\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)\boldsymbol{\Lambda}$

$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{l-1}_{\mathbf{x}}\otimes\mathbf{h}^{l-1}_{\mathbf{x}}\right)((\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})-\boldsymbol{\Lambda})$

Or (using another index)
$E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]=\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)((\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})-\boldsymbol{\Lambda})$
for $j=1,2,3,...$

Thus, we have in general
$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=E_t\left[\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}\right)-\sum_{j=0}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right)\right]$

$=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
the shock hits in period $t+1$, so $\left(\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t\right)=\mathbf{x}^f_t\otimes\mathbf{x}^f_t$

$=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}+\mathbf{h}^{j-1}_{\mathbf{x}}\sigma\boldsymbol{\eta\nu}\otimes\mathbf{h}^j_{\mathbf{x}}\mathbf{x}^f_t+\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)((\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu})-\boldsymbol{\Lambda})\right)$

If we restrict the focus and do the GIRF's at the unconditional mean of $\mathbf{x}^f_t=\mathbf{0}$, then we get

$$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\left(\mathbf{h}^{j-1}_{\mathbf{x}}\otimes\mathbf{h}^{j-1}_{\mathbf{x}}\right)(\sigma\boldsymbol{\eta\nu}\otimes\sigma\boldsymbol{\eta\nu}-\boldsymbol{\Lambda})\right)$$

When implementing the GIRF, it may be useful to have a recursive expression. Here, it is must convenient to use
$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=\sum_{j=1}^{l-1}\mathbf{h}^{l-1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
So
$E_t\left[\tilde{\mathbf{x}}^s_{t+1}-\mathbf{x}^s_{t+1}\right]=0$

$E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]=\sum_{j=1}^{1}\mathbf{h}^{1-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
$=\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+1}\otimes\tilde{\mathbf{x}}^f_{t+1}-\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right]$

$E_t\left[\tilde{\mathbf{x}}^s_{t+3}-\mathbf{x}^s_{t+3}\right]=\sum_{j=1}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$
$=\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+1}\otimes\tilde{\mathbf{x}}^f_{t+1}-\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right]$
$+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$
$=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$

So in general
$E_t\left[\tilde{\mathbf{x}}^s_{t+k}-\mathbf{x}^s_{t+k}\right]=\mathbf{h_x}E_t\left[\tilde{\mathbf{x}}^s_{t+k-1}-\mathbf{x}^s_{t+k-1}\right]+\frac{1}{2}\mathbf{H_{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\right]$

For the total state variable:
$E_t\left[\tilde{\mathbf{x}}_{t+l}-\mathbf{x}_{t+l}\right]=E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]$

For the control variables:
$\mathbf{y}^s_{t+l}=\mathbf{g_x}\left(\mathbf{x}^f_{t+l}+\mathbf{x}^s_{t+l}\right)+\frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right)+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$
$\tilde{\mathbf{y}}^s_{t+l}=\mathbf{g_x}\left(\tilde{\mathbf{x}}^f_{t+l}+\tilde{\mathbf{x}}^s_{t+l}\right)+\frac{1}{2}\mathbf{G_{xx}}\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\right)+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$

$E_t\left[\tilde{\mathbf{y}}^s_{t+l}-\mathbf{y}^s_{t+l}\right]=\mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]\right)+\frac{1}{2}\mathbf{G_{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]$

11.3 At third order

At third order, we additionally need to consider:
$\mathbf{x}^{rd}_{t+1}=\mathbf{h_x}\mathbf{x}^{rd}_t+\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

and
$\mathbf{x}^{rd}_{t+2}=\mathbf{h_x}\mathbf{x}^{rd}_{t+1}+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$$=\mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^{rd}_t+\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$=\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^{rd}_t+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\mathbf{h_x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\mathbf{h_x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\mathbf{h_x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

and
$\mathbf{x}^{rd}_{t+3}=\mathbf{h_x}\mathbf{x}^{rd}_{t+2}+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+2}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$

$$=\mathbf{h_x}[\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^{rd}_t+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\mathbf{h_x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\mathbf{h_x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\mathbf{h_x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3]$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+2}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$=\mathbf{h}^3_{\mathbf{x}}\mathbf{x}^{rd}_t+\mathbf{h}^2_{\mathbf{x}}\mathbf{H_{xx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\mathbf{h}^2_{\mathbf{x}}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)+\mathbf{h}^2_{\mathbf{x}}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t+\mathbf{h}^2_{\mathbf{x}}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}\right)+\mathbf{h_x}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\otimes\mathbf{x}^f_{t+1}\right)+\mathbf{h_x}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1}+\mathbf{h_x}\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right)+\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right)+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+2}+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$=\mathbf{h}^3_{\mathbf{x}}\mathbf{x}^{rd}_t+\sum_{j=0}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^s_{t+j}\right)$$
$$+\sum_{j=0}^{2}\mathbf{h}^{2-j}_{\mathbf{x}}\frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right)$$

$$+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f$$
$$+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$=\mathbf{h}_{\mathbf{x}}^3\mathbf{x}_t^{rd}+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^s\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]$$
$$+\left(\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

In general

$$\mathbf{x}_{t+l}^{rd}=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^{rd}+\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^s\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]$$
$$+\left(\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

Thus

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^{rd}-\mathbf{x}_{t+l}^{rd}\right]=E_t[\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^s\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}_{t+j}^f\right]$$
$$-\{\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^s\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f\right]\}]$$

$$=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^s-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^s\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\right]\right)$$
$$+\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$

as the shock hits the economy in period $t+1$

A recursive version:

$$E_t\left[\tilde{\mathbf{x}}_{t+1}^{rd}-\mathbf{x}_{t+1}^{rd}\right]=0$$

$$E_t\left[\tilde{\mathbf{x}}_{t+2}^{rd}-\mathbf{x}_{t+2}^{rd}\right]=\sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^s-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^s\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\right]\right)$$
$$+\sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$
$$=\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^s-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right]$$
$$+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right]$$

$$E_t\left[\tilde{\mathbf{x}}_{t+3}^{rd}-\mathbf{x}_{t+3}^{rd}\right]=\sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^s-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^s\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\right]\right)$$
$$+\sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$
$$=\mathbf{h}_{\mathbf{x}}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^s-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right]\right)$$
$$+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^s-\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^s\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\right]$$
$$+\mathbf{h}_{\mathbf{x}}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right]$$

$$+\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$

$$=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+2}-\mathbf{x}^{rd}_{t+2}\right]$$
$$+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}\right]+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\right]$$
$$+\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$

So in general

$$E_t\left[\tilde{\mathbf{x}}^{rd}_{t+k}-\mathbf{x}^{rd}_{t+k}\right]=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+k-1}-\mathbf{x}^{rd}_{t+k-1}\right]+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^s_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^s_{t+k-1}\right]$$
$$+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\right]$$
$$+\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\right]$$

Thus, we know $E_t\left[\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\right]$. So we only need to compute $E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^f_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right]$ and $E_t\left[\tilde{\mathbf{x}}^f_{t+j}\otimes\tilde{\mathbf{x}}^s_{t+j}-\mathbf{x}^f_{t+j}\otimes\mathbf{x}^s_{t+j}\right]$. This is done in the next two subsections. For these derivations recall that we define $\boldsymbol{\delta}_t$ such that:

$$\boldsymbol{\delta}_{t+j}=\boldsymbol{\nu}\quad\text{for } j=1$$
$$\boldsymbol{\delta}_{t+j}=\boldsymbol{\epsilon}_{t+j}\quad\text{for } j\neq 1$$

11.3.1 For $\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$

Consider:

$$\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}=\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$=(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})$$
$$\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$=(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

And we therefore have

$$\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

This means that:
$E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[0+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[+\mathbf{0}+\mathbf{0}$$
$$+\mathbf{0}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

using $E_t\left[\boldsymbol{\epsilon}_{t+i}\right]=0$ and cancelling the term $\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$

$$
\begin{aligned}
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&+\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}
\end{aligned}
$$

$$
\begin{aligned}
&-E_t[\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&+\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f+\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f+\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&+\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}
$$

We now evaluate the expressions on each of the four last lines:

$$
A_1\equiv E_t\left[\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\right]
$$

$$
\begin{aligned}
&=E_t[\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&\quad-\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&\quad+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&\quad-\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f\\
&\quad-\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h_x}^{l}\mathbf{x}_t^f]
\end{aligned}
$$

$$= E_t[\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$+ \left(\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$- \left(\sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f]$$

$$= E_t[\left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{0}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$+ \left(\mathbf{0} + \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{0}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{0} + \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f]$$

$$= E_t[\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$+$$
$$-\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$
$$-]$$

$$= E_t[\mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f - \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f]$$

$$= \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f - E_t\left[\left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f\right]$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$= \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f - \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right) E_t\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\right] \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$

$$= \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f - \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\boldsymbol{\Lambda} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$

$$= \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) - \boldsymbol{\Lambda}\right) \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f$$

Note $A_1 \equiv E_t\left[\sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f - \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h_x}^{l}\mathbf{x}_t^f\right]$

Therefore we immediately see from the structure of the terms that
$A_2 \equiv \mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h_x}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

$$= E_t[\mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} - \mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) - \mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)]$$

$$= \mathbf{h_x}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h_x}^{l-1} \otimes \mathbf{h_x}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) - \boldsymbol{\Lambda}\right)$$

For the third term:

$$A_3 \equiv \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} - \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} - \boldsymbol{\Gamma}(l)$$

The only element which is not directly computable is $\Gamma \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$ which must be computed element by element. Hence, consider

$$\boldsymbol{\Gamma}(l) \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right]$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\textstyle\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\textstyle\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\textstyle\sum_{\phi_2=1}^{n_e}\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$= \left\{\mathbf{h}_{\mathbf{x}}^{l-j}\left(\gamma_{3,:}\right)\sigma\textstyle\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1) \times \left\{\mathbf{h}_{\mathbf{x}}^{l}(\gamma_2,:)\,\mathbf{x}_t^f \times \left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

because the innovations are indepdendent. This expression is directly implementable.

$$A_4 = E_t\left[\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$+ \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$- \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$- \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$

$$
\begin{aligned}
&+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\
&-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\
&-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{0}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\left(\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\\
&+\left(\mathbf{0}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\right)\\
&+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{0}\right)\\
&-(\mathbf{0}+\mathbf{0})\\
&-(0+0)\\
&-\left(0+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+0\right)\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&+(0)]
\end{aligned}
$$

$$
\begin{aligned}
&=\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]\\
&+E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]\\
&+E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]\\
&-E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]
\end{aligned}
$$

How consider the following terms:

$$A_{4,1}\equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]$$

$$
\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-3}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&\quad+\ldots\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]
\end{aligned}
$$

$$
\begin{aligned}
&=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{h}_{\mathbf{x}}^{l-2}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-3}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{h}_{\mathbf{x}}^{l-3}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3}\\
&\quad+\ldots\\
&\quad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}\otimes\mathbf{h}_{\mathbf{x}}^{l-l}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+l}]
\end{aligned}
$$

because the innovations are independent across time

$$=\sum_{j=2}^{l}E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$=\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\right]$$

$$=\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)E_t\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\right]$$

because the innovations are identical distributed across time

$$=\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\Lambda$$

We therefore immediately have for the second term:

$$A_{4,2}\equiv E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]=\sum_{j=2}^{l}\Lambda\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

For the third term (when using the results from above)

$$A_{4,3} \equiv E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right]$$

$$=\sum_{j=2}^{l}E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$=\sum_{j=2}^{l}\boldsymbol{\Omega}_j$$

where $\boldsymbol{\Omega}_j \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. So

$$\boldsymbol{\Omega}_j \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right]$$

$$=E_t\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

$$=E_t\left\{\begin{array}{c}\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_2=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1)\\ \times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\end{array}\right\}_{\gamma_3=1}^{n_x}$$

$$=E_t\left\{\begin{array}{c}\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_2=1}^{n_e}\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1)\\ \times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\end{array}\right\}_{\gamma_3=1}^{n_x}$$

$$=\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_{3,:})\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\nu}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-j}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}$$

because the innovations are indepdendent. This expression is directly implementable.

For the fourth term must be computed element by element

$$A_{4,4} \equiv E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

$$=E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right]$$

$$=E_t\left[\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_{3,:})\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_1,:)\,\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}_{\gamma_1=1}^{n_x}\right\}_{\gamma_2=1}^{n_x}\right\}_{\gamma_3=1}^{n_x}\right]$$

$$=E_t[\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_{3,:})\,\sigma\sum_{\phi_3=1}^{n_e}\boldsymbol{\eta}(:,\phi_3)\,\boldsymbol{\epsilon}_{t+1}(\phi_3,1)\times$$
$$\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\sum_{\phi_3=1}^{n_e}\boldsymbol{\eta}(:,\phi_2)\,\boldsymbol{\epsilon}_{t+1}(\phi_2,1)\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_1,:)\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\}_{\gamma_2=1}^{n_x}\}_{\gamma_3=1}^{n_x}]$$

$$=E_t[\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_{3,:})\,\sigma\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\times$$
$$\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_1,:)\,\sigma\boldsymbol{\eta}(:,\phi_1)\,\boldsymbol{\epsilon}_{t+1}(\phi_1,1)\right\}_{\gamma_1=1}^{n_x}\}_{\gamma_2=1}^{n_x}\}_{\gamma_3=1}^{n_x}]$$

$$=\{\sigma^3\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_{3,:})\sum_{\phi_1=1}^{n_e}\boldsymbol{\eta}(:,\phi_1)\,E_t\left[\boldsymbol{\epsilon}_{t+1}^3(\phi_1,1)\right]\times\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_2,:)\,\boldsymbol{\eta}(:,\phi_1)\times\left\{\mathbf{h}_{\mathbf{x}}^{l-1}(\gamma_1,:)\,\boldsymbol{\eta}(:,\phi_1)\right\}_{\gamma_1=1}^{n_x}\}_{\gamma_2=1}^{n_x}\}_{\gamma_3=1}^{n_x}]$$

Thus, we finally have:

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]$$

$$= \mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$A_1)$ $$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)-\boldsymbol{\Lambda}\right)\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f$$

$A_2)$ $$+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)-\boldsymbol{\Lambda}\right)$$

$A_3)$ $$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}-\boldsymbol{\Gamma}(l)$$

$A_4)$ $$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$A_{4,1})$ $$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\Lambda$$

$A_{4,2})$ $$+\sum_{j=2}^{l}\Lambda\left(\mathbf{h}_{\mathbf{x}}^{l-j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$A_{4,3})$ $$+\sum_{j=2}^{l}\boldsymbol{\Omega}_j$$

$A_{4,4})$ $$+E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

11.3.2 For $\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)$

Recall from above that

$$\mathbf{x}^f_{t+1}\otimes\mathbf{x}^s_{t+1}=\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$

Therefore:

$$\mathbf{x}^f_{t+2}\otimes\mathbf{x}^s_{t+2}=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left[\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f\right.$$
$$\left.+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\right]$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^2\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^s\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

and

$$\mathbf{x}^f_{t+3}\otimes\mathbf{x}^s_{t+3}=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^s\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+2}^f$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)[\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t}^{f}$$

$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1}$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+1}^{f}$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+2}]$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+2}^{f}$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{3}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t}^{f}$$

$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1}$$

$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+1}^{f}$$

$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{s}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+2}$$

$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+2}^{f}$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{s}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+3}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{3}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$

$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+i}^{f}$$

$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1+i}$$

$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{s}\right)$$

$$+\sum_{i=0}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$

And in general

$$\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{s}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$

$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\mathbf{x}_{t+i}^{f}$$

$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\boldsymbol{\epsilon}_{t+1+i}$$

$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{s}\right)$$

$$+\sum_{i=0}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^{f}\otimes\mathbf{x}_{t+i}^{f}\right)$$

for $l=0,1,2,3....$ Note for $l=0$ we have $\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}=\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)$ as desired.

We therefore have

$$\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^s = (\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\tilde{\mathbf{x}}_t^f \otimes \tilde{\mathbf{x}}_t^s\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \tilde{\mathbf{x}}_{t+i}^f$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1+i}$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}_{t+i}^s\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f\right)$$

Thus
$E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^s - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^s\right]$

$$= E_t[(\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\tilde{\mathbf{x}}_t^f \otimes \tilde{\mathbf{x}}_t^s\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \tilde{\mathbf{x}}_{t+i}^f$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1+i}$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}_{t+i}^s\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f\right)$$

$$-\{(\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \mathbf{x}_{t+i}^f$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+1+i}$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}_{t+i}^s\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right)\}$$

$$]$$

$$= E_t[\sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\boldsymbol{\delta}_{t+1+i} - \boldsymbol{\epsilon}_{t+1+i}\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}_{t+i}^s - \boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}_{t+i}^s\right)$$

$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right)]$$

because the shock hits in period $t+1$, meaing that $\mathbf{x}_t^f = \tilde{\mathbf{x}}_t^f$ and similar for $\tilde{\mathbf{x}}_t^s$

$$= E_t[\sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1}$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\delta}_{t+1} \otimes \tilde{\mathbf{x}}^s_t - \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}^s_t)$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^s_{t+i} - \boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^s_{t+i}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1} \otimes \tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^f_t - \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)]$$

$$= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1}$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\delta}_{t+1} \otimes \tilde{\mathbf{x}}^s_t - \mathbf{0})$$
$$+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\mathbf{0} - \mathbf{0})$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1} \otimes \tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^f_t - \mathbf{0}\right)$$
$$+ \sum_{i=2}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) (\mathbf{0} - \mathbf{0})$$

because $\mathbf{x}^s_{t+i}$ is a function of $\mathbf{x}^f_{t+i}$ which is a function of $\boldsymbol{\epsilon}_{t+i}$. The zero-mean iid innovations therefore implies that, $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^s_{t+i}\right)\right] = \mathbf{0}$ and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i} \otimes \tilde{\mathbf{x}}^s_{t+i}\right)\right] = \mathbf{0}$
The same argument implies that $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)\right] = \mathbf{0}$
and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)\right] = \mathbf{0}$

$$= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\nu}$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\nu} \otimes \mathbf{x}^s_t)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\nu} \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)$$

the shock hitting in period $t+1 \Longrightarrow \tilde{\mathbf{x}}^f_t = \mathbf{x}^f_t$ and $\tilde{\mathbf{x}}^s_t = \mathbf{x}^s_t$

$$= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) (\boldsymbol{\nu} \otimes 1)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) (\boldsymbol{\nu} \otimes \mathbf{x}^s_t)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\nu} \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}_t^s\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta\nu} \otimes \mathbf{h_x}\mathbf{x}_t^s + \sigma\boldsymbol{\eta\nu} \otimes \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

as the shock hittings in period $t+1$

To derive a recursive version for this sum, we let

$$
\begin{aligned}
X_l &= \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=1}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1} \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

so

$$
X_1 = \sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)
$$

$$
\begin{aligned}
X_2 &= \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f \otimes \tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f \otimes \mathbf{x}_{t+i}^f\right) \\
&+ \sum_{i=1}^{1} (\mathbf{h_x} \otimes \mathbf{h_x})^{1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+i}^f - \mathbf{x}_{t+i}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f\right) \\
&+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathbf{h_x} \otimes \mathbf{h_x}) X_1 + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f \otimes \tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}_{t+1}^f - \mathbf{x}_{t+1}^f\right)
\end{aligned}
$$

$$X_3 = \sum_{i=1}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=1}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+i} - \mathbf{x}^f_{t+i}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^2 \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} \otimes \tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right)$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x}) \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+1} - \mathbf{x}^f_{t+1}\right)$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2}\right)$$
$$+ (\mathbf{h_x} \otimes \mathbf{h_x})^2 \left(\sigma\boldsymbol{\eta\nu} \otimes \left(\mathbf{h_x}\mathbf{x}^s_t + \tfrac{1}{2}\mathbf{H_{xx}} \left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x}) X_2 + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} \otimes \tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right)$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+2} - \mathbf{x}^f_{t+2}\right)$$

Hence, in general

$$X_k = (\mathbf{h_x} \otimes \mathbf{h_x}) X_{k-1} + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+k-1} \otimes \tilde{\mathbf{x}}^f_{t+k-1} \otimes \tilde{\mathbf{x}}^f_{t+k-1} - \mathbf{x}^f_{t+k-1} \otimes \mathbf{x}^f_{t+k-1} \otimes \mathbf{x}^f_{t+k-1}\right)$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \left(\tilde{\mathbf{x}}^f_{t+k-1} - \mathbf{x}^f_{t+k-1}\right)$$

11.3.3 Summarizing

At third order, the total effect on the state variables is:

$$E_t\left[\tilde{\mathbf{x}}_{t+l} - \mathbf{x}_{t+l}\right] = E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^{rd}_{t+l} - \mathbf{x}^{rd}_{t+l}\right]$$

For the control variables:

$$\mathbf{y}^{rd}_{t+l} = \mathbf{g_x}\left(\mathbf{x}^f_{t+l} + \mathbf{x}^s_{t+l} + \mathbf{x}^{rd}_{t+l}\right) + \tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right) + 2\left(\mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l}\right)\right)$$
$$+ \tfrac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+l} + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$
$$\tilde{\mathbf{y}}^{rd}_{t+l} = \mathbf{g_x}\left(\tilde{\mathbf{x}}^f_{t+l} + \tilde{\mathbf{x}}^s_{t+l} + \tilde{\mathbf{x}}^{rd}_{t+l}\right) + \tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l}\right) + 2\left(\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l}\right)\right)$$
$$+ \tfrac{1}{6}\mathbf{G_{xxx}}\left(\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l}\right) + \tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}^f_{t+l} + \tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3$$

So:

$$E_t\left[\tilde{\mathbf{y}}^{rd}_{t+l} - \mathbf{y}^{rd}_{t+l}\right]$$
$$= \mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^{rd}_{t+l} - \mathbf{x}^{rd}_{t+l}\right]\right)$$
$$+ \tfrac{1}{2}\mathbf{G_{xx}}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right] + 2E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l}\right]\right)$$
$$+ \tfrac{1}{6}\mathbf{G_{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right] + \tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]$$

12 Impulse response functions - GIRF version 2

In our derivations of the GIRF's above, we condition on the entire vector $\boldsymbol{\nu}$ even though only one or two elements in $\boldsymbol{\nu}$ may differ from zero. This interpretation of the definition of GIRF's has some unfortunate properties - for instance, following a disturbance to shock i we may actually see changes in shock j if shock j follows a non-linear law of motion! To avoid this odd behaviour we will now adopt another interpretation of the definition of a GIRF where we only condition on the one-zero shocks hitting the economy.

Hence, this section derives closed-form solutions for the generalized impulse response function in non-linear DSGE models defined as

$$GIRF_{\mathbf{var}}(l, \nu_i, \mathbf{w}_t) = E_t[\mathbf{var}_{t+l}|\nu_i] - E_t[\mathbf{var}_{t+l}]$$

for a disturbance to innovation i. To reduce the notational burden in the derivations below, we adopt the parsimonious notation

$$IRF_{\mathbf{var}}(l, \nu_i, \mathbf{w}_t) = E_t[\widetilde{\mathbf{var}}_{t+l}] - E_t[\mathbf{var}_{t+l}]$$

in relation to the conditional expectation operators. The formulas we derive below also apply if we want to explore the joint effects of more than one shock - for instance simultaneous shocking disturbances i and j, i.e. $GIRF_{\mathbf{var}}(l, \nu_i, \nu_j, \mathbf{w}_t) = E_t[\mathbf{var}_{t+l}|\nu_i, \nu_j] - E_t[\mathbf{var}_{t+l}]$.

12.1 The model for the conditional information

This subsection explains how we will compute conditional expectations by conditioning on ν_i - and possible more disturbances. Let $\mathbf{S}$ be $n_e \times n_e$ diagonal selection matrix with either 1 or zeros on the diagonal, and let the shock sizes appear in the vector $\boldsymbol{\nu}$ of dimension $n_e \times 1$. For shocks which are not hit by a disturbance, we simply put them to zero.

As an example, consider an economy with three shocks and we want to condition our expectations on the first shock. Hence, we need the vector

$$\begin{bmatrix} \nu_1 \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{bmatrix}$$

We can form this vector by letting

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ 0 \\ 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} \mathbf{S}\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S})\boldsymbol{\epsilon}_{t+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nu_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{bmatrix} \\ &= \begin{bmatrix} \nu_1 \\ \epsilon_{2,t+1} \\ \epsilon_{3,t+1} \end{bmatrix} \end{aligned}$$

Similarly, if we want to condition on the first two shocks, then we let

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{\nu} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ 0 \end{bmatrix}.$$

meaning that

$$\mathbf{S}\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S})\,\boldsymbol{\epsilon}_{t+1} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \epsilon_{3,t+1} \end{bmatrix}$$

12.2 At first order

Recall that we have:

$$\mathbf{x}^f_{t+1} = \mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

and

$$\begin{aligned}\mathbf{x}^f_{t+2} &= \mathbf{h_x}\mathbf{x}^f_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^f_t + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} \\ &= \mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}^f_{t+3} &= \mathbf{h_x}\mathbf{x}^f_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h_x}\left(\mathbf{h}^2_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2}\right) + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^f_t + \mathbf{h}^2_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h_x}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+2} + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+3} \\ &= \mathbf{h}^3_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{3}\mathbf{h}^{3-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\end{aligned}$$

In general

$$\mathbf{x}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

With a shock of $\boldsymbol{\nu}$ in period $t+1$, we have

$$\tilde{\mathbf{x}}^f_{t+l} = \mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t + \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

where we define $\boldsymbol{\delta}_t$ such that

$$\boldsymbol{\delta}_{t+j} = \begin{cases} \boldsymbol{\nu} + (\mathbf{I} - \mathbf{S})\,\boldsymbol{\epsilon}_{t+1} & \text{for } j = 1 \\ \boldsymbol{\epsilon}_{t+j} & \text{for } j \neq 1 \end{cases}$$

Agents know the size of the shock $\boldsymbol{\nu}$ at time $t+1$, and it is therefore in agents' information set. I.e. $\boldsymbol{\nu}$ is non-stochastic. So

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] = E_t\left[\sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l}\mathbf{h}^{l-j}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= E_t\left[\mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\right]$$

$$= \mathbf{h}^{l-1}_{\mathbf{x}}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

and

$$E_t\left[\tilde{\mathbf{y}}^f_{t+l} - \mathbf{y}^f_{t+l}\right] = \mathbf{g_x}E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]$$

12.3 At second order

We need to consider:

$$\mathbf{x}_{t+1}^{s}=\mathbf{h}_{\mathbf{x}}\mathbf{x}_{t}^{s}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$$

$$\begin{aligned}\mathbf{x}_{t+2}^{s}&=\mathbf{h}_{\mathbf{x}}\mathbf{x}_{t+1}^{s}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&=\mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}\mathbf{x}_{t}^{s}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&=\mathbf{h}_{\mathbf{x}}^{2}\mathbf{x}_{t}^{s}+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\end{aligned}$$

$$\begin{aligned}\mathbf{x}_{t+3}^{s}&=\mathbf{h}_{\mathbf{x}}\mathbf{x}_{t+2}^{s}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&=\mathbf{h}_{\mathbf{x}}\left(\mathbf{h}_{\mathbf{x}}^{2}\mathbf{x}_{t}^{s}+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\right)\\&\quad+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&=\mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{s}+\mathbf{h}_{\mathbf{x}}^{2}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{h}_{\mathbf{x}}^{2}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&\quad+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&=\mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{s}+\mathbf{h}_{\mathbf{x}}^{2}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)\\&\quad+\mathbf{h}_{\mathbf{x}}^{2}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\\&=\mathbf{h}_{\mathbf{x}}^{3}\mathbf{x}_{t}^{s}+\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\left(\sum_{j=0}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}\end{aligned}$$

and in general

$$\mathbf{x}_{t+l}^{s}=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{s}+\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\left(\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\right)\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}$$

for $l=1,2,3,...$

Thus, to compute $E_t\left[\tilde{\mathbf{x}}_{t+l}^{s}-\mathbf{x}_{t+l}^{s}\right]$, we need to find $E_t\left[\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{f}-\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{f}\right]$. Hence, consider:

$$\begin{aligned}\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{f}&=\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\&=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{f}&=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\\&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_{t}^{f}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\end{aligned}$$

where we define $\boldsymbol{\delta}_t$ such that:

$$\boldsymbol{\delta}_{t+j}=\begin{cases}\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} & \text{for } j=1\\ \boldsymbol{\epsilon}_{t+j} & \text{for } j\neq 1\end{cases}$$

This means that:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^{f}\otimes\tilde{\mathbf{x}}_{t+l}^{f}-\mathbf{x}_{t+l}^{f}\otimes\mathbf{x}_{t+l}^{f}\right]$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$+ \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$- \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f - \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}$$
$$- \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f - \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1}$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f + \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$- \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f - 0$$
$$-0 - \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}]$$

using $E_t\left[\boldsymbol{\epsilon}_{t+j}\right] = \mathbf{0}$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\delta}_{t+j}$$
$$- \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}]$$

cancelling $\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$
$$- \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$
$$+ \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$
$$- \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$
$$- \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f \otimes \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} + \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_\mathbf{x}^l \mathbf{x}_t^f$$
$$+ \mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \left(\mathbf{h}_\mathbf{x}^{l-1} \sigma \boldsymbol{\eta} \left(\boldsymbol{\nu} + (\mathbf{I} - \mathbf{S}) \boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l} \mathbf{h}_\mathbf{x}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j}\right)$$

$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{0}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+0$$
$$+0$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-0$$
$$-0$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

using $E_t\left[\boldsymbol{\epsilon}_{t+j}\right]=\mathbf{0}$ and $\boldsymbol{\epsilon}_{t+j}$ are uncorrelated across time

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

canceling terms $\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}-\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)]$$

$$=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(E_t\left[\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]-E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]\right)$$

Hence, we only need to compute $E_t\left[\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$ and $E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$. We know $E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]=E_t\left[\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]=\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}\right)$
using $\left(\mathbf{A}\otimes\mathbf{B}\right)\left(\mathbf{C}\otimes\mathbf{D}\right)=\mathbf{AC}\otimes\mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

and

$$E_t\left[\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]=E_t\left[\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\right]=\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)vec\left(\mathbf{I}\right)$$

Thus, we have

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)vec\left(\mathbf{I}\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}\right)\right)$$

$$=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\right)vec\left(\mathbf{I}\right)$$

$$=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)+\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)vec\left(\mathbf{I}\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}\right)\right]$$

$$=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)\left[\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\boldsymbol{\Lambda}\right]$$

where

$$\boldsymbol{\Lambda}\equiv\left(\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\right)vec\left(\mathbf{I}\right)$$

Or (using another index)

$$E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]=\mathbf{h}_{\mathbf{x}}^j\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^j\mathbf{x}_t^f+\left(\mathbf{h}_{\mathbf{x}}^{j-1}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\right)\left[\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\boldsymbol{\Lambda}\right]$$

for $j=1,2,3,...$

Thus, we have in general

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^s\right]=E_t\left[\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\right)-\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)\right]$$

$$=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$

the shock hits in period $t+1$, so $\left(\tilde{\mathbf{x}}_t^f\otimes\tilde{\mathbf{x}}_t^f\right)=\mathbf{x}_t^f\otimes\mathbf{x}_t^f$

$$=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{h}_{\mathbf{x}}^j\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{j-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^j\mathbf{x}_t^f+\left(\mathbf{h}_{\mathbf{x}}^{j-1}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\right)\left[\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\boldsymbol{\Lambda}\right]\right)$$

When implementing the GIRF, it may be useful to have a recursive expression. Here, it is must convenient to use the general expression

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^s-\mathbf{x}_{t+l}^s\right]=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$

So

$$E_t\left[\tilde{\mathbf{x}}_{t+1}^s-\mathbf{x}_{t+1}^s\right]=0$$

$$E_t\left[\tilde{\mathbf{x}}_{t+2}^s-\mathbf{x}_{t+2}^s\right]=\sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$
$$=\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right]$$

$$E_t\left[\tilde{\mathbf{x}}_{t+3}^s-\mathbf{x}_{t+3}^s\right]=\sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f-\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right]$$
$$=\mathbf{h}_{\mathbf{x}}\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right]$$

$$+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$
$$=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^s_{t+2}-\mathbf{x}^s_{t+2}\right]+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+2}\otimes\tilde{\mathbf{x}}^f_{t+2}-\mathbf{x}^f_{t+2}\otimes\mathbf{x}^f_{t+2}\right]$$

So in general

$$E_t\left[\tilde{\mathbf{x}}^s_{t+k}-\mathbf{x}^s_{t+k}\right]=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^s_{t+k-1}-\mathbf{x}^s_{t+k-1}\right]+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+k-1}\otimes\tilde{\mathbf{x}}^f_{t+k-1}-\mathbf{x}^f_{t+k-1}\otimes\mathbf{x}^f_{t+k-1}\right]$$

For the total state variable:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}-\mathbf{x}_{t+l}\right]=E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]$$

For the control variables:

$$\mathbf{y}^s_{t+l}=\mathbf{g}_{\mathbf{x}}\left(\mathbf{x}^f_{t+l}+\mathbf{x}^s_{t+l}\right)+\tfrac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right)+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$
$$\tilde{\mathbf{y}}^s_{t+l}=\mathbf{g}_{\mathbf{x}}\left(\tilde{\mathbf{x}}^f_{t+l}+\tilde{\mathbf{x}}^s_{t+l}\right)+\tfrac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\right)+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

$$E_t\left[\tilde{\mathbf{y}}^s_{t+l}-\mathbf{y}^s_{t+l}\right]=\mathbf{g}_{\mathbf{x}}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]\right)+\tfrac{1}{2}\mathbf{G}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]$$

12.4 Second order: at the steady state with shock size of unity

If we restrict the focus and do the GIRF's at the unconditional mean of $\mathbf{x}^f_t=\mathbf{0}$, then we get

$$E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\left(\mathbf{h}_{\mathbf{x}}^{j-1}\otimes\mathbf{h}_{\mathbf{x}}^{j-1}\right)\left[\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\boldsymbol{\Lambda}\right]\right)$$

If we further assume that there is only one shock to the ith innovation and that the size of this shock is one, i.e. $\nu(i,1)=\pm1$ and $\nu(j,1)=0$ for $i\neq j$. In this case we have

$$\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\boldsymbol{\Lambda}$$
$$=\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\left(\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\right)vec\left(\mathbf{I}\right)$$

$$=\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left(\boldsymbol{\nu}\otimes\boldsymbol{\nu}\right)+\left(\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left(\left(\mathbf{I}-\mathbf{S}\right)\otimes\left(\mathbf{I}-\mathbf{S}\right)\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\right)vec\left(\mathbf{I}\right)$$

$$=\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{S}\otimes\mathbf{S}\right)vec\left(\mathbf{I}\right)+\left(\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left(\left(\mathbf{I}-\mathbf{S}\right)\otimes\left(\mathbf{I}-\mathbf{S}\right)\right)-\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\right)vec\left(\mathbf{I}\right)$$

because $\boldsymbol{\nu}\otimes\boldsymbol{\nu}=\left(\mathbf{S}\otimes\mathbf{S}\right)vec\left(\mathbf{I}\right)$

$$=\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left\{\mathbf{S}\otimes\mathbf{S}+\left(\left(\mathbf{I}-\mathbf{S}\right)\otimes\left(\mathbf{I}-\mathbf{S}\right)\right)-\mathbf{I}\otimes\mathbf{I}\right\}vec\left(\mathbf{I}\right)$$

because $\mathbf{I}_{n_e^2}=\mathbf{I}\otimes\mathbf{I}$ where $\mathbf{I}$ has dimension $n_e\times n_e$

$$=\left(\sigma\boldsymbol{\eta}\otimes\sigma\boldsymbol{\eta}\right)\left\{2\left(\mathbf{S}\otimes\mathbf{S}\right)-\mathbf{I}\otimes\mathbf{S}-\mathbf{S}\otimes\mathbf{I}\right\}vec\left(\mathbf{I}\right)$$

We now only need to realize that the term in the curly bracket is zero. To do so, let us introduce the matrix $\mathbf{D}_i$, defined as $\mathbf{D}_i(i,i)=1$ with all remaining elements being zero. Hence, $\mathbf{I}$ with dimension $n_\varepsilon\times n_\varepsilon$ can be written as $\mathbf{I}=\sum_{j=1}^{n_\varepsilon}\mathbf{D}_j$. Furthermore, $\mathbf{S}=\mathbf{D}_i$ by assumption. Thus, we have for the expression in the curly bracket:

$$2\left(\mathbf{S}\otimes\mathbf{S}\right)-\mathbf{I}\otimes\mathbf{S}-\mathbf{S}\otimes\mathbf{I}$$

$$=2\left(\mathbf{D}_i\otimes\mathbf{D}_i\right)-\left(\sum_{j=1}^{n_\varepsilon}\mathbf{D}_j\right)\otimes\mathbf{D}_i-\mathbf{D}_i\otimes\left(\sum_{j=1}^{n_\varepsilon}\mathbf{D}_j\right)$$

$$=2\left(\mathbf{D}_i\otimes\mathbf{D}_i\right)-\left(\sum_{j=1}^{n_\varepsilon}\mathbf{D}_j\otimes\mathbf{D}_i\right)-\left(\sum_{j=1}^{n_\varepsilon}\mathbf{D}_i\otimes\mathbf{D}_j\right)$$

$$=2\left(\mathbf{D}_i\otimes\mathbf{D}_i\right)-\left(\sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\mathbf{D}_j\otimes\mathbf{D}_i+\mathbf{D}_i\otimes\mathbf{D}_i\right)-\left(\sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\mathbf{D}_i\otimes\mathbf{D}_j+\mathbf{D}_i\otimes\mathbf{D}_i\right)$$

$$= -\sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \mathbf{D}_j \otimes \mathbf{D}_i - \sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \mathbf{D}_i \otimes \mathbf{D}_j$$

Hence,

$$\sigma\boldsymbol{\eta\nu} \otimes \sigma\boldsymbol{\eta\nu} + \boldsymbol{\Lambda} = (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}) \{2(\mathbf{S} \otimes \mathbf{S}) - \mathbf{I} \otimes \mathbf{S} - \mathbf{S} \otimes \mathbf{I}\} \, vec(\mathbf{I})$$

$$= (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}) \left\{ -\sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \mathbf{D}_j \otimes \mathbf{D}_i - \sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \mathbf{D}_i \otimes \mathbf{D}_j \right\} vec\left(\sum_{k=1}^{n_\varepsilon} \mathbf{D}_k\right)$$

$$= (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}) \left\{ -\sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \mathbf{D}_j \otimes \mathbf{D}_i - \sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \mathbf{D}_i \otimes \mathbf{D}_j \right\} \left(\sum_{k=1}^{n_\varepsilon} vec(\mathbf{D}_k)\right)$$

$$= (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}) \left\{ -\sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \sum_{k=1}^{n_\varepsilon} (\mathbf{D}_j \otimes \mathbf{D}_i) \, vec(\mathbf{D}_k) - \sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \sum_{k=1}^{n_\varepsilon} (\mathbf{D}_i \otimes \mathbf{D}_j) \, vec(\mathbf{D}_k) \right\}$$

$$= (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}) \left\{ -\sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \sum_{k=1}^{n_\varepsilon} vec(\mathbf{D}_i\mathbf{D}_k\mathbf{D}_j) - \sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \sum_{k=1}^{n_\varepsilon} vec(\mathbf{D}_j\mathbf{D}_k\mathbf{D}_i) \right\}$$

because $vec(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A}) \, vec(\mathbf{B})$ and $\mathbf{D}_i$ is symmetri

$$= (\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}) \left\{ -\sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \sum_{k=1}^{n_\varepsilon} vec(\mathbf{0}) - \sum_{\substack{j=1\\i\neq j}}^{n_\varepsilon} \sum_{k=1}^{n_\varepsilon} vec(\mathbf{0}) \right\} = \mathbf{0}$$

because $\mathbf{D}_i\mathbf{D}_k\mathbf{D}_j$ is only different from the zero matrix when $i = k = j$, but we have $i \neq j$. As a result, $E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right] = \mathbf{0}$ and from above, this also implies $E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right]$. Thus, we have that

For the states:

$$E_t\left[\tilde{\mathbf{x}}_{t+l} - \mathbf{x}_{t+l}\right] = E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right]$$

$$= E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]$$

For the control variables:

$$E_t\left[\tilde{\mathbf{y}}^s_{t+l} - \mathbf{y}^s_{t+l}\right] = \mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right] + E_t\left[\tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^s_{t+l}\right]\right) + \tfrac{1}{2}\mathbf{G_{xx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^f_{t+l}\right]$$

$$= \mathbf{g_x}E_t\left[\tilde{\mathbf{x}}^f_{t+l} - \mathbf{x}^f_{t+l}\right]$$

which are precisely the impulse response functions at first order.

12.5 At third order

At third order, we additionally need to consider:

$$\mathbf{x}^{rd}_{t+1} = \mathbf{h_x}\mathbf{x}^{rd}_t + \mathbf{H_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^s_t\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

and

$$\mathbf{x}^{rd}_{t+2} = \mathbf{h_x}\mathbf{x}^{rd}_{t+1} + \mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^s_{t+1}\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1} + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h_x}\left(\mathbf{h_x}\mathbf{x}^{rd}_t + \mathbf{H_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^s_t\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$
$$+\mathbf{H_{xx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^s_{t+1}\right) + \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1} \otimes \mathbf{x}^f_{t+1}\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+1} + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$= \mathbf{h}^2_\mathbf{x}\mathbf{x}^{rd}_t + \mathbf{h_x}\mathbf{H_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^s_t\right) + \mathbf{h_x}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \mathbf{h_x}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_t + \mathbf{h_x}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

$$+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+1}^{f}+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}$$

and

$$\mathbf{x}_{t+3}^{rd}=\mathbf{h_{x}}\mathbf{x}_{t+2}^{rd}+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+2}^{f}+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}$$

$$\begin{aligned}
&=\mathbf{h_{x}}[\mathbf{h_{x}^{2}}\mathbf{x}_{t}^{rd}+\mathbf{h_{x}}\mathbf{H_{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\mathbf{h_{x}}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{h_{x}}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t}^{f}+\mathbf{h_{x}}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}\\
&+\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+1}^{f}+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}]\\
&+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+2}^{f}+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}
\end{aligned}$$

$$\begin{aligned}
&=\mathbf{h_{x}^{3}}\mathbf{x}_{t}^{rd}+\mathbf{h_{x}^{2}}\mathbf{H_{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{s}\right)+\mathbf{h_{x}^{2}}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\mathbf{h_{x}^{2}}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t}^{f}+\mathbf{h_{x}^{2}}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}\\
&+\mathbf{h_{x}}\mathbf{H_{xx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{s}\right)+\mathbf{h_{x}}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\right)+\mathbf{h_{x}}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+1}^{f}+\mathbf{h_{x}}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}\\
&+\mathbf{H_{xx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\otimes\mathbf{x}_{t+2}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+2}^{f}+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}
\end{aligned}$$

$$\begin{aligned}
&=\mathbf{h_{x}^{3}}\mathbf{x}_{t}^{rd}+\sum_{j=0}^{2}\mathbf{h_{x}}^{2-j}\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)\\
&+\sum_{j=0}^{2}\mathbf{h_{x}}^{2-j}\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)\\
&+\sum_{j=0}^{2}\mathbf{h_{x}}^{2-j}\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+j}^{f}\\
&+\sum_{j=0}^{2}\mathbf{h_{x}}^{2-j}\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}
\end{aligned}$$

$$\begin{aligned}
&=\mathbf{h_{x}^{3}}\mathbf{x}_{t}^{rd}+\sum_{j=0}^{2}\mathbf{h_{x}}^{2-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+j}^{f}\right]\\
&+\left(\sum_{j=0}^{2}\mathbf{h_{x}}^{2-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}
\end{aligned}$$

In general

$$\begin{aligned}
\mathbf{x}_{t+l}^{rd}&=\mathbf{h_{x}}^{l}\mathbf{x}_{t}^{rd}+\sum_{j=0}^{l-1}\mathbf{h_{x}}^{l-1-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+j}^{f}\right]\\
&+\left(\sum_{j=0}^{l-1}\mathbf{h_{x}}^{l-1-j}\right)\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^{3}
\end{aligned}$$

Thus

$$\begin{aligned}
E_{t}\left[\tilde{\mathbf{x}}_{t+l}^{rd}-\mathbf{x}_{t+l}^{rd}\right]&=E_{t}[\sum_{j=0}^{l-1}\mathbf{h_{x}}^{l-1-j}\left[\mathbf{H_{xx}}\left(\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\tilde{\mathbf{x}}_{t+j}^{f}\right]\\
&\quad-\{\sum_{j=0}^{l-1}\mathbf{h_{x}}^{l-1-j}\left[\mathbf{H_{xx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right)+\tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right)+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}\mathbf{x}_{t+j}^{f}\right]\}]
\end{aligned}$$

$$\begin{aligned}
&=\sum_{j=1}^{l-1}\mathbf{h_{x}}^{l-1-j}\left(\mathbf{H_{xx}}E_{t}\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{s}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{s}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^{2}E_{t}\left[\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\right]\right)\\
&+\sum_{j=1}^{l-1}\mathbf{h_{x}}^{l-1-j}\tfrac{1}{6}\mathbf{H_{xxx}}E_{t}\left[\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}\otimes\tilde{\mathbf{x}}_{t+j}^{f}-\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\otimes\mathbf{x}_{t+j}^{f}\right]
\end{aligned}$$

as the shock hits the economy in period $t+1$

A recursive version:
$E_t\left[\tilde{\mathbf{x}}^{rd}_{t+1}-\mathbf{x}^{rd}_{t+1}\right]=0$

$$\begin{aligned}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+2}-\mathbf{x}^{rd}_{t+2}\right] &= \sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{s}_{t+j}-\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{s}_{t+j}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}-\mathbf{x}^{f}_{t+j}\right]\right)\\
&\quad+\sum_{j=1}^{1}\mathbf{h}_{\mathbf{x}}^{1-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{f}_{t+j}-\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{f}_{t+j}\right]\\
&=\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+1}\otimes\tilde{\mathbf{x}}^{s}_{t+1}-\mathbf{x}^{f}_{t+1}\otimes\mathbf{x}^{s}_{t+1}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+1}-\mathbf{x}^{f}_{t+1}\right]\\
&\quad+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+1}\otimes\tilde{\mathbf{x}}^{f}_{t+1}\otimes\tilde{\mathbf{x}}^{f}_{t+1}-\mathbf{x}^{f}_{t+1}\otimes\mathbf{x}^{f}_{t+1}\otimes\mathbf{x}^{f}_{t+1}\right]\end{aligned}$$

$$\begin{aligned}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+3}-\mathbf{x}^{rd}_{t+3}\right] &= \sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{s}_{t+j}-\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{s}_{t+j}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}-\mathbf{x}^{f}_{t+j}\right]\right)\\
&\quad+\sum_{j=1}^{2}\mathbf{h}_{\mathbf{x}}^{2-j}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{f}_{t+j}-\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{f}_{t+j}\right]\\
&=\mathbf{h}_{\mathbf{x}}\left(\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+1}\otimes\tilde{\mathbf{x}}^{s}_{t+1}-\mathbf{x}^{f}_{t+1}\otimes\mathbf{x}^{s}_{t+1}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+1}-\mathbf{x}^{f}_{t+1}\right]\right)\\
&\quad+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+2}\otimes\tilde{\mathbf{x}}^{s}_{t+2}-\mathbf{x}^{f}_{t+2}\otimes\mathbf{x}^{s}_{t+2}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+2}-\mathbf{x}^{f}_{t+2}\right]\\
&\quad+\mathbf{h}_{\mathbf{x}}\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+1}\otimes\tilde{\mathbf{x}}^{f}_{t+1}\otimes\tilde{\mathbf{x}}^{f}_{t+1}-\mathbf{x}^{f}_{t+1}\otimes\mathbf{x}^{f}_{t+1}\otimes\mathbf{x}^{f}_{t+1}\right]\\
&\quad+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+2}\otimes\tilde{\mathbf{x}}^{f}_{t+2}\otimes\tilde{\mathbf{x}}^{f}_{t+2}-\mathbf{x}^{f}_{t+2}\otimes\mathbf{x}^{f}_{t+2}\otimes\mathbf{x}^{f}_{t+2}\right]\\
&=\mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+2}-\mathbf{x}^{rd}_{t+2}\right]\\
&\quad+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+2}\otimes\tilde{\mathbf{x}}^{s}_{t+2}-\mathbf{x}^{f}_{t+2}\otimes\mathbf{x}^{s}_{t+2}\right]+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+2}-\mathbf{x}^{f}_{t+2}\right]\\
&\quad+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+2}\otimes\tilde{\mathbf{x}}^{f}_{t+2}\otimes\tilde{\mathbf{x}}^{f}_{t+2}-\mathbf{x}^{f}_{t+2}\otimes\mathbf{x}^{f}_{t+2}\otimes\mathbf{x}^{f}_{t+2}\right]\end{aligned}$$

So in general

$$\begin{aligned}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+k}-\mathbf{x}^{rd}_{t+k}\right] &= \mathbf{h}_{\mathbf{x}}E_t\left[\tilde{\mathbf{x}}^{rd}_{t+k-1}-\mathbf{x}^{rd}_{t+k-1}\right]+\mathbf{H}_{\mathbf{xx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+k-1}\otimes\tilde{\mathbf{x}}^{s}_{t+k-1}-\mathbf{x}^{f}_{t+k-1}\otimes\mathbf{x}^{s}_{t+k-1}\right]\\
&\quad+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^{f}_{t+k-1}-\mathbf{x}^{f}_{t+k-1}\right]\\
&\quad+\tfrac{1}{6}\mathbf{H}_{\mathbf{xxx}}E_t\left[\tilde{\mathbf{x}}^{f}_{t+k-1}\otimes\tilde{\mathbf{x}}^{f}_{t+k-1}\otimes\tilde{\mathbf{x}}^{f}_{t+k-1}-\mathbf{x}^{f}_{t+k-1}\otimes\mathbf{x}^{f}_{t+k-1}\otimes\mathbf{x}^{f}_{t+k-1}\right]\end{aligned}$$

Thus, we know $E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}-\mathbf{x}^{f}_{t+j}\right]$. So we only need to compute $E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{f}_{t+j}-\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{f}_{t+j}\right]$ and $E_t\left[\tilde{\mathbf{x}}^{f}_{t+j}\otimes\tilde{\mathbf{x}}^{s}_{t+j}-\mathbf{x}^{f}_{t+j}\otimes\mathbf{x}^{s}_{t+j}\right]$. This is done in the next two subsections. For these derivations recall that we define $\boldsymbol{\delta}_t$ as above, i.e.

$$\boldsymbol{\delta}_{t+j}=\begin{cases}\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} & \text{for } j=1\\ \boldsymbol{\epsilon}_{t+j} & \text{for } j\neq 1\end{cases}.$$

12.5.1 For $\left(\mathbf{x}^f_t\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$

Consider

$$\begin{aligned}\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l} &= \left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\\
&=(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\end{aligned}$$

$$\otimes\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$

$$=(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j})\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

And we therefore have

$$\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f=\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

This means that:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f-\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\right]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[0+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[+\mathbf{0}+\mathbf{0}$$
$$+\mathbf{0}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

using $E_t\left[\boldsymbol{\epsilon}_{t+i}\right]=0$ and cancelling the term $\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$

$$-E_t[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\left(\mathbf{h}_{\mathbf{x}}^{l}\otimes\mathbf{h}_{\mathbf{x}}^{l}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)+\left(\left(\mathbf{h}_{\mathbf{x}}^{l}\otimes\mathbf{h}_{\mathbf{x}}^{l}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

using $\boldsymbol{\delta}_{t+1}=\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}$ and $E_t\left[\boldsymbol{\epsilon}_{t+1}\right]=0$. We now evaluate the expressions on each of the four last lines:

$$A_1\equiv E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f-\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

using $\boldsymbol{\delta}_{t+j}=\begin{cases}\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} & \text{for } j=1\\ \boldsymbol{\epsilon}_{t+j} & \text{for } j\neq 1\end{cases}$.

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}+\mathbf{0}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+\left(0+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$

$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{0}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- \left(0 + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

using $\boldsymbol{\epsilon}_{t+1}$ has mean zero and is independent across time

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+0$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- (0) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

cancelling $\left(\sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$+ \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$
$$- \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f]$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right) E_t[\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) + \left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right) - \left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)] \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$

From the GIRF's at second order we have

$E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$

$= E_t\left[vec\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}'\right)\right)\right]$

using $a \otimes b = vec\left(ba'\right)$

$= E_t\left[vec\left(\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\boldsymbol{\eta}'\sigma\right)\right]$

$= vec\left(E_t\left[\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\boldsymbol{\epsilon}_{t+1}'\sigma\boldsymbol{\eta}'\right]\right)$

$= vec\left(\sigma\boldsymbol{\eta}\mathbf{I}\sigma\boldsymbol{\eta}'\right)$

$= \left(\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)$

using $vec\left(ABC\right) = \left(C' \otimes A\right) vec\left(B\right)$

and

$E_t\left[\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right] = E_t\left[\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right] = \left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) vec\left(\mathbf{I}\right)$

and defined

$\boldsymbol{\Lambda} \equiv \left(\left(\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) - \left(\sigma\boldsymbol{\eta} \otimes \sigma\boldsymbol{\eta}\right)\right) vec\left(\mathbf{I}\right)$

Thus

$$A_1 \equiv E_t\left[\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right]$$
$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) + \boldsymbol{\Lambda}\right] \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f$$

Therefore we immediately see from the structure of the terms that

$$A_2 \equiv \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left[\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) + \boldsymbol{\Lambda}\right]$$

For the third term:

$$A_3 \equiv E_t\left[\sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=1}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

using $\boldsymbol{\delta}_{t+j} = \begin{cases} \boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1} & \text{for } j=1 \\ \boldsymbol{\epsilon}_{t+j} & \text{for } j \neq 1 \end{cases}$.

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\mathbf{0}$$
$$+\left(0 + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{0}\right)$$
$$-\left(0 + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

using $\boldsymbol{\epsilon}_{t+1}$ has mean zero and indepdendent across time

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu} + (\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}\right)$$
$$+(0)$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-(0)]$$

cancelling the term $\sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

So we only need to compute the two terms involving $\boldsymbol{\epsilon}_{t+1}$. We next consider:
$E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$

$$= E_t\left[vec\left(\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)'\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$$
using $a \otimes b = vec\left(ba'\right)$

$$= E_t\left[vec\left(\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1}'\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)'\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$$

$$= E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\right) vec\left(\boldsymbol{\epsilon}_{t+1}'\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$$
using $vec\left(ABC\right) = \left(C' \otimes A\right) vec\left(B\right)$

$$= E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right]$$
using $vec\left(\boldsymbol{\epsilon}_{t+1}'\right) = \boldsymbol{\epsilon}_{t+1}$

$$= E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) vec\left(\mathbf{I}\right)$$

and

$E_t\left[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$

$$= E_t\left[vec\left(\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)'\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$
using $a \otimes b = vec\left(ba'\right)$

$$= E_t\left[vec\left(\left(\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\left(\boldsymbol{\epsilon}_{t+1}\right)'\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\right)'\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$

$$= E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right) vec\left(\boldsymbol{\epsilon}_{t+1}'\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$
using $vec\left(ABC\right) = \left(C' \otimes A\right) vec\left(B\right)$

$$= E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right]$$
using $vec\left(\boldsymbol{\epsilon}_{t+1}'\right) = \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$

$$= E_t\left[\left(\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\right]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\right) E_t\left[\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)$$

Thus,

$$A_3 = E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right) vec\left(\mathbf{I}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} + \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right) - \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\right) vec\left(\mathbf{I}\right)$$

Finally

$$A_4 = E_t\left[\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} - \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j} \otimes \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}$$

$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\delta}_{t+j}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

using $\boldsymbol{\delta}_{t+j}=\left\{\begin{array}{ll}\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1} & \text{for } j=1\\ \boldsymbol{\epsilon}_{t+j} & \text{for } j\neq 1\end{array}\right.$.

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+0\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+0\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$

$-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\mathbf{0}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$
-0
-0
$-\left(0+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$
using $\boldsymbol{\epsilon}_{t+j}$ has mean zero and is independent across time

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)$$
$$\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+0\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$-0]$$
cancelling $\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$

$$=E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$=E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\boldsymbol{\nu}+0\right)$$
$$+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}$$
$$-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$$

$$
\begin{aligned}
&= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\\
&+0\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+0\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]
\end{aligned}
$$

$$
\begin{aligned}
&= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]
\end{aligned}
$$

$$
\begin{aligned}
&= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}+\mathbf{0}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\left(\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&+\left(\mathbf{0}+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]
\end{aligned}
$$

$$
\begin{aligned}
&\qquad= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&A_{4,1})\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&A_{4,2})\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&A_{4,3})\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&A_{4,4})\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\boldsymbol{\epsilon}_{t+1}\\
&A_{4,5})\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\\
&A_{4,6})\qquad+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\\
&A_{4,7})\qquad+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}
\end{aligned}
$$

$A_{4,8}) \qquad - \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}]$

We can directly compute the first term in this expression. As for all remaining terms, we provide formulas below:

$$A_{4,1} = E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \left(\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})\right)]$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right) vec\,(\mathbf{I})$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \times 1 \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right) vec\,(\mathbf{I})$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right)(1 \otimes vec\,(\mathbf{I}))$$

using $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right) vec\,(\mathbf{I})$$

$$A_{4,2} = E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}]$$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right)(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right) vec\,(\mathbf{I}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) vec\,(\mathbf{I})$$

$$A_{4,3} = E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\left\{\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}\right)'\right\} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

using $a \otimes b = vec\,(ba')$

$$= E_t[\left\{\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\boldsymbol{\epsilon}_{t+1}'\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right)'\right\} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\left\{\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) vec\left(\boldsymbol{\epsilon}_{t+1}'\right)\right\} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

using $vec\,(ABC) = (C' \otimes A)\,vec\,(B)$

$$= E_t[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\left\{\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right\}(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right) vec\,(\mathbf{I})$$

$$A_{4,4} = E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\left\{\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right\}(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\,\boldsymbol{\epsilon}_{t+1}]$$

$$= E_t[\left\{\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S}) \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\,(\mathbf{I}-\mathbf{S})\right\}(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1})]$$

$$= \left\{ \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\left(\mathbf{I}-\mathbf{S}\right)\right\} \mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1},\boldsymbol{\epsilon}_{t+1},\boldsymbol{\epsilon}_{t+1}\right)$$

where $\mathbf{m}^{3}\left(\boldsymbol{\epsilon}_{t+1},\boldsymbol{\epsilon}_{t+1},\boldsymbol{\epsilon}_{t+1}\right)$ has dimension $n_e^3 \times 1$ and contains all the third moments of $\boldsymbol{\epsilon}_{t+1}$.

$$A_{4,5} = E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\sum_{j=2}^{l}\sum_{k=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-k}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+k}\right)\right)]$$

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)]$$

because $\boldsymbol{\epsilon}_{t+j}$ are independent across time

$$= E_t[\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\right)\left(\boldsymbol{\epsilon}_{t+j}\otimes\boldsymbol{\epsilon}_{t+j}\right)]$$

$$= \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}\right)$$

$$= \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}\right)$$

$$A_{4,6} = E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right]$$

$$= \sum_{j=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\right)vec\left(\mathbf{I}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}$$

$$A_{4,7} = E_t\left[\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= \sum_{j=2}^{l}E_t\left[\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

because $\boldsymbol{\epsilon}_{t+j}$ is indepdent across time

$$= \sum_{j=2}^{l}E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right)'\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

using $a\otimes b = vec\left(ba'\right)$

$$= \sum_{j=2}^{l}E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)\boldsymbol{\epsilon}_{t+j}'\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\right)'\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

$$= \sum_{j=2}^{l}E_t\left[\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\boldsymbol{\eta}\boldsymbol{\nu}\right)vec\left(\boldsymbol{\epsilon}_{t+j}'\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+j}\right]$$

using $vec\left(ABC\right) = \left(C'\otimes A\right)vec\left(B\right)$

$$= \sum_{j=2}^{l} E_t \left[\left(\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \right) \boldsymbol{\epsilon}_{t+j} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+j} \right]$$

$$= \sum_{j=2}^{l} E_t \left[\left(\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \right) \left(\boldsymbol{\epsilon}_{t+j} \otimes \boldsymbol{\epsilon}_{t+j} \right) \right]$$

$$= \sum_{j=2}^{l} \left(\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \right) vec\left(\mathbf{I} \right)$$

$$A_{4,8} = E_t \left[\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right]$$

$$= E_t \left[\left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\epsilon}_{t+1} \right]$$

$$= E_t \left[\left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \right) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \right) \right]$$

$$= \left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \right) \mathbf{m}^3 \left(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1} \right)$$

where $\mathbf{m}^3 \left(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1} \right)$ has dimension $n_e^3 \times 1$ and contains all the third moments of $\boldsymbol{\epsilon}_{t+1}$.

Thus, we finally have

$$
\begin{aligned}
& E_t \left[\tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f \otimes \tilde{\mathbf{x}}_{t+l}^f - \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \otimes \mathbf{x}_{t+l}^f \right] = \\
& \quad + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \\
& \quad + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \left(\left(\mathbf{h}_{\mathbf{x}}^l \otimes \mathbf{h}_{\mathbf{x}}^l \right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) \right) \\
& \quad + \left(\left(\mathbf{h}_{\mathbf{x}}^l \otimes \mathbf{h}_{\mathbf{x}}^l \right) \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \right) \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \\
& A_1) + \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \right) \left[\left(\sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\nu} \right) + \boldsymbol{\Lambda} \right] \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \\
& A_2) + \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \right) \left[\left(\sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \sigma \boldsymbol{\eta} \boldsymbol{\nu} \right) + \boldsymbol{\Lambda} \right] \\
& A_3) + \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \\
& \quad + \left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) - \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^l \mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \right) vec\left(\mathbf{I} \right) \\
& A_{4,1}) + \left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \right) vec\left(\mathbf{I} \right) \\
& A_{4,2}) + \left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \right) vec\left(\mathbf{I} \right) \\
& A_{4,3}) + \left(\left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \right) vec\left(\mathbf{I} \right) \\
& A_{4,4}) + \left\{ \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \left(\mathbf{I} - \mathbf{S} \right) \right\} \mathbf{m}^3 \left(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1} \right) \\
& A_{4,5}) + \sum_{j=2}^{l} \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \right) vec\left(\mathbf{I} \right) \\
& A_{4,6}) + \sum_{j=2}^{l} \left(\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \right) vec\left(\mathbf{I} \right) \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \\
& A_{4,7}) + \sum_{j=2}^{l} \left(\mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \boldsymbol{\nu} \otimes \mathbf{h}_{\mathbf{x}}^{l-j} \sigma \boldsymbol{\eta} \right) vec\left(\mathbf{I} \right) \\
& A_{4,8}) - \left(\mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}^{l-1} \sigma \boldsymbol{\eta} \right) \mathbf{m}^3 \left(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}_{t+1} \right)
\end{aligned}
$$

12.5.2 For $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)$

Recall from above that

$$\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s = \left(\mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right) \otimes \left(\mathbf{h_x}\mathbf{x}_t^s + \tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$

Therefore:

$$\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s = \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)[\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

and

$$\mathbf{x}_{t+3}^f \otimes \mathbf{x}_{t+3}^s = \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^s\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+2}^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3} \otimes \mathbf{x}_{t+2}^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3} \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}$$
$$= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)[\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}]$$
$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+2}^f$$
$$+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+3} \otimes \mathbf{x}_{t+2}^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+3} \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+3}$$

$$= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^3\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f$$
$$+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}$$
$$+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+1}^f$$
$$+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^s\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+2} \otimes \mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+2}$$

$$+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f \otimes \mathbf{x}_{t+2}^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+2}^f$$

$$+ (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+3} \otimes \mathbf{x}^s_{t+2}\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\epsilon}_{t+3} \otimes \mathbf{x}^f_{t+2} \otimes \mathbf{x}^f_{t+2}\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+3}$$

$$= (\mathbf{h_x} \otimes \mathbf{h_x})^3 \left(\mathbf{x}^f_t \otimes \mathbf{x}^s_t\right) + \sum_{i=0}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \mathbf{x}^f_{t+i}$$
$$+ \sum_{i=0}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+1+i}$$
$$+ \sum_{i=0}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{2} (\mathbf{h_x} \otimes \mathbf{h_x})^{2-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$

And in general

$$\mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l} = (\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\mathbf{x}^f_t \otimes \mathbf{x}^s_t\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \mathbf{x}^f_{t+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\epsilon}_{t+1+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\epsilon}_{t+1+i} \otimes \mathbf{x}^f_{t+i} \otimes \mathbf{x}^f_{t+i}\right)$$

for $l = 1, 2, 3....$

We therefore have

$$\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l} = (\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^s_t\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \tilde{\mathbf{x}}^f_{t+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} (\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\boldsymbol{\delta}_{t+1+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$

Thus

$$E_t\left[\tilde{\mathbf{x}}^f_{t+l} \otimes \tilde{\mathbf{x}}^s_{t+l} - \mathbf{x}^f_{t+l} \otimes \mathbf{x}^s_{t+l}\right]$$

$$= E_t[(\mathbf{h_x} \otimes \mathbf{h_x})^l \left(\tilde{\mathbf{x}}^f_t \otimes \tilde{\mathbf{x}}^s_t\right) + \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right) \left(\tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i} \otimes \tilde{\mathbf{x}}^f_{t+i}\right)$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \tilde{\mathbf{x}}^f_{t+i}$$
$$+ \sum_{i=0}^{l-1} (\mathbf{h_x} \otimes \mathbf{h_x})^{l-1-i} \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right) \boldsymbol{\delta}_{t+1+i}$$

$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\right)$$

$$-\{(\mathbf{h_x}\otimes\mathbf{h_x})^{l}\left(\mathbf{x}^f_t\otimes\mathbf{x}^s_t\right)+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}^f_{t+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1+i}$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)\}$$
$$]$$

$$=E_t[\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(\boldsymbol{\delta}_{t+1+i}-\boldsymbol{\epsilon}_{t+1+i})$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}-\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\delta}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)]$$

$$=E_t[\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1}-\boldsymbol{\epsilon}_{t+1})$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})((\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1})\otimes\tilde{\mathbf{x}}^s_t-\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^s_t)$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^s_{t+i}-\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^s_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left((\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1})\otimes\tilde{\mathbf{x}}^f_t\otimes\tilde{\mathbf{x}}^f_t-\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}^f_t\otimes\mathbf{x}^f_t\right)$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)]$$
using $\boldsymbol{\delta}_{t+j}=\begin{cases}\boldsymbol{\nu}+(\mathbf{I}-\mathbf{S})\boldsymbol{\epsilon}_{t+1} & \text{for } j=1\\ \boldsymbol{\epsilon}_{t+j} & \text{for } j\neq 1\end{cases}$.

$$=E_t[\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}\otimes\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\otimes\mathbf{x}^f_{t+i}\right)$$
$$+\sum_{i=0}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}^f_{t+i}-\mathbf{x}^f_{t+i}\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)(\boldsymbol{\nu}+0-\mathbf{0})$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})((\boldsymbol{\nu}+0)\otimes\tilde{\mathbf{x}}^s_t-\mathbf{0})$$

$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\mathbf{0}-\mathbf{0})$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left((\boldsymbol{\nu}+0)\otimes\tilde{\mathbf{x}}_t^f\otimes\tilde{\mathbf{x}}_t^f-\mathbf{0}\right)$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)(\mathbf{0}-\mathbf{0})]$$

using $\mathbf{x}_{t+i}^s$ is a function of $\mathbf{x}_{t+i}^f$ which is a function of $\boldsymbol{\epsilon}_{t+i}$. The zero-mean iid innovations therefore implies that, $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^s\right)\right]=\mathbf{0}$ and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}_{t+i}^s\right)\right]=\mathbf{0}$
The same argument implies that $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)\right]=\mathbf{0}$
and $E_t\left[\left(\boldsymbol{\epsilon}_{t+1+i}\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\right)\right]=\mathbf{0}$

$$=E_t[\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\nu}\otimes\mathbf{x}_t^s)$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)]$$

because the shock hits in period $t+1$, meaing that $\mathbf{x}_t^f=\tilde{\mathbf{x}}_t^f$ and similar for $\tilde{\mathbf{x}}_t^s$

$$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)\right]$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)\right]$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left\{\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\nu}+(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x})(\boldsymbol{\nu}\otimes\mathbf{x}_t^s)+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\nu}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right\}$$

$$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)\right]$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)\right]$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left\{\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\mathbf{h_x}\mathbf{x}_t^s+\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right\}$$

using $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\mathbf{AC}\otimes\mathbf{BD}$ if $\mathbf{AC}$ and $\mathbf{BD}$ are defined

$$=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)E_t\left[\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)\right]$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)E_t\left[\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)\right]$$
$$+(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1}\left\{\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right\}$$

To derive a recursive version for this sum, we let

$$X_l=\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)$$
$$+\sum_{i=1}^{l-1}(\mathbf{h_x}\otimes\mathbf{h_x})^{l-1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)$$

$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1}\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

so

$$X_1=\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$\begin{aligned}
X_2&=\sum_{i=1}^{1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)\\
&\quad+\sum_{i=1}^{1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{1-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)\\
&=\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)\\
&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)X_1+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right)
\end{aligned}$$

$$\begin{aligned}
X_3&=\sum_{i=1}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f\otimes\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+i}^f\right)\\
&\quad+\sum_{i=1}^{2}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{2-i}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+i}^f-\mathbf{x}_{t+i}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)\\
&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f\otimes\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\otimes\mathbf{x}_{t+1}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+1}^f-\mathbf{x}_{t+1}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^2\left(\sigma\boldsymbol{\eta}\boldsymbol{\nu}\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)\\
&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)X_2+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f\otimes\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\otimes\mathbf{x}_{t+2}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+2}^f-\mathbf{x}_{t+2}^f\right)
\end{aligned}$$

Hence, in general

$$\begin{aligned}
X_k&=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)X_{k-1}+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\tilde{\mathbf{x}}_{t+k-1}^f\otimes\tilde{\mathbf{x}}_{t+k-1}^f\otimes\tilde{\mathbf{x}}_{t+k-1}^f-\mathbf{x}_{t+k-1}^f\otimes\mathbf{x}_{t+k-1}^f\otimes\mathbf{x}_{t+k-1}^f\right)\\
&\quad+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\left(\tilde{\mathbf{x}}_{t+k-1}^f-\mathbf{x}_{t+k-1}^f\right)
\end{aligned}$$

12.5.3 Summarizing

At third order, the total effect on the state variables is:

$$E_t\left[\tilde{\mathbf{x}}_{t+l}-\mathbf{x}_{t+l}\right]=E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^{rd}_{t+l}-\mathbf{x}^{rd}_{t+l}\right]$$

For the control variables:

$$\begin{aligned}\mathbf{y}^{rd}_{t+l}&=\mathbf{g_x}\left(\mathbf{x}^f_{t+l}+\mathbf{x}^s_{t+l}+\mathbf{x}^{rd}_{t+l}\right)+\tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right)+2\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^s_{t+l}\right)\right)\\&\quad+\tfrac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right)+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}^f_{t+l}+\tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3\\\tilde{\mathbf{y}}^{rd}_{t+l}&=\mathbf{g_x}\left(\tilde{\mathbf{x}}^f_{t+l}+\tilde{\mathbf{x}}^s_{t+l}+\tilde{\mathbf{x}}^{rd}_{t+l}\right)+\tfrac{1}{2}\mathbf{G_{xx}}\left(\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\right)+2\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^s_{t+l}\right)\right)\\&\quad+\tfrac{1}{6}\mathbf{G_{xxx}}\left(\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\right)+\tfrac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\tilde{\mathbf{x}}^f_{t+l}+\tfrac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3\end{aligned}$$

So:

$$\begin{aligned}&E_t\left[\tilde{\mathbf{y}}^{rd}_{t+l}-\mathbf{y}^{rd}_{t+l}\right]\\&\quad=\mathbf{g_x}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^s_{t+l}\right]+E_t\left[\tilde{\mathbf{x}}^{rd}_{t+l}-\mathbf{x}^{rd}_{t+l}\right]\right)\\&\quad+\tfrac{1}{2}\mathbf{G_{xx}}\left(E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]+2E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^s_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^s_{t+l}\right]\right)\\&\quad+\tfrac{1}{6}\mathbf{G_{xxx}}E_t\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}\right]+\tfrac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2E_t\left[\tilde{\mathbf{x}}^f_{t+l}-\mathbf{x}^f_{t+l}\right]\end{aligned}$$

13 Alternative notation with σ in the state vector

When deriving the perturbation approximation, the perturbation parameter σ is treated as a variable. It may therefore be natural to consider σ as a part of the state vector when constructing the state space system for the approximated model. We therefore define $\tilde{\mathbf{x}}^f_t=\begin{bmatrix}\left(\mathbf{x}^f_t\right)' & \sigma\end{bmatrix}'$, $\tilde{\mathbf{x}}^s_t=\begin{bmatrix}\left(\mathbf{x}^s_t\right)' & \sigma\end{bmatrix}'$, and $\tilde{\mathbf{x}}^{rd}_t=\begin{bmatrix}\left(\mathbf{x}^{rd}_t\right)' & \sigma\end{bmatrix}'$.

At first-order:

$$\mathbf{y}^f_t=\begin{bmatrix}\mathbf{g_x} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}^f_t\\\sigma\end{bmatrix}$$

$\Updownarrow$

$$\mathbf{y}^f_t=\tilde{\mathbf{g}}_{\tilde{\mathbf{x}}}\tilde{\mathbf{x}}^f_t$$

$$\begin{bmatrix}\mathbf{x}^f_{t+1}\\\sigma\end{bmatrix}=\begin{bmatrix}\mathbf{h_x} & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{x}^f_t\\\sigma\end{bmatrix}+\begin{bmatrix}\sigma\boldsymbol{\eta}\\\mathbf{0}\end{bmatrix}\boldsymbol{\epsilon}_{t+1}$$

$\Updownarrow$

$$\tilde{\mathbf{x}}^f_{t+1}=\tilde{\mathbf{h}}_{\tilde{\mathbf{x}}}\tilde{\mathbf{x}}^f_t+\begin{bmatrix}\sigma\boldsymbol{\eta}\\\mathbf{0}\end{bmatrix}\boldsymbol{\epsilon}_{t+1}$$

At second order:

$$y^s_t(i)=\begin{bmatrix}\mathbf{g_x}(i,:) & 0\end{bmatrix}\left(\begin{bmatrix}\mathbf{x}^f_t\\\sigma\end{bmatrix}+\begin{bmatrix}\mathbf{x}^s_t\\\sigma\end{bmatrix}\right)+\frac{1}{2}\begin{bmatrix}\left(\mathbf{x}^f_t\right)' & \sigma\end{bmatrix}\begin{bmatrix}\mathbf{g_{xx}}(i,:,:) & 0\\\mathbf{0} & g_{\sigma\sigma}(i,1)\end{bmatrix}\begin{bmatrix}\mathbf{x}^f_t\\\sigma\end{bmatrix}$$

$\Updownarrow$

$$y^s_t(i)=\tilde{\mathbf{g}}_{\tilde{\mathbf{x}}}(i,:)\left(\tilde{\mathbf{x}}^f_t+\tilde{\mathbf{x}}^s_t\right)+\frac{1}{2}\left(\tilde{\mathbf{x}}^f_t\right)'\tilde{\mathbf{g}}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}(i,:,:)\tilde{\mathbf{x}}^f_t$$

$$x^s_t(i)=\begin{bmatrix}\mathbf{h_x}(i,:) & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}^s_t\\\sigma\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\left(\mathbf{x}^f_t\right)' & \sigma\end{bmatrix}\begin{bmatrix}\mathbf{h_{xx}}(i,:,:) & 0\\\mathbf{0} & h_{\sigma\sigma}(i,1)\end{bmatrix}\begin{bmatrix}\mathbf{x}^f_t\\\sigma\end{bmatrix}$$

$\Updownarrow$

$$x_t^s(i) = \tilde{\mathbf{h}}_{\tilde{\mathbf{x}}}(i,:)\,\tilde{\mathbf{x}}_t^s + \frac{1}{2}\left(\tilde{\mathbf{x}}_t^f\right)'\tilde{\mathbf{h}}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}(i,:,:)\,\tilde{\mathbf{x}}_t^f$$

for $i = 1, 2, ..., n_x$.

At third order:

$$\begin{aligned} y_t^{rd}(i) &= \begin{bmatrix} \mathbf{g}_{\mathbf{x}}(i,:) & 0 \end{bmatrix}\left(\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t^s \\ \sigma \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t^{rd} \\ \sigma \end{bmatrix}\right) \\ &+\frac{1}{2}\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xx}}(i,:,:) & 0 \\ \mathbf{0} & g_{\sigma\sigma} \end{bmatrix}\left(\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} + 2\begin{bmatrix} \mathbf{x}_t^s \\ \sigma \end{bmatrix}\right) \\ &+\frac{1}{6}\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,1,:,:) & 0 \\ \mathbf{0} & 3g(i,1)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,2,:,:) & 0 \\ \mathbf{0} & 3g(i,2)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ ... \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) & 0 \\ \mathbf{0} & 3g(i,n_x)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{0} & g(i,1)_{\sigma\sigma\sigma} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \end{bmatrix} \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} y_t^{rd}(i) &= \tilde{\mathbf{g}}_{\tilde{\mathbf{x}}}(i,:)\left(\tilde{\mathbf{x}}_t^f + \tilde{\mathbf{x}}_t^s + \tilde{\mathbf{x}}_t^{rd}\right) \\ &+\frac{1}{2}\left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix} \mathbf{g}_{\mathbf{xx}}(i,:,:) & 0 \\ \mathbf{0} & g_{\sigma\sigma} \end{bmatrix}\left(\tilde{\mathbf{x}}_t^f + 2\tilde{\mathbf{x}}_t^s\right) \\ &+\frac{1}{6}\left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix} \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,1,:,:) & 0 \\ \mathbf{0} & 3g(i,1)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\tilde{\mathbf{x}}_t^f \\ \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,2,:,:) & 0 \\ \mathbf{0} & 3g(i,2)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\tilde{\mathbf{x}}_t^f \\ ... \\ \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) & 0 \\ \mathbf{0} & 3g(i,n_x)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\tilde{\mathbf{x}}_t^f \\ \left(\tilde{\mathbf{x}}_t^f\right)'\begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{0} & g(i,1)_{\sigma\sigma\sigma} \end{bmatrix}\tilde{\mathbf{x}}_t^f \end{bmatrix} \end{aligned}$$

Notice that

$$\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,1,:,:) & 0 \\ \mathbf{0} & 3g(i,1)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,2,:,:) & 0 \\ \mathbf{0} & 3g(i,2)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ ... \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{g}_{\mathbf{xxx}}(i,n_x,:,:) & 0 \\ \mathbf{0} & 3g(i,n_x)_{\sigma\sigma\mathbf{x}} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \\ \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \sigma \end{bmatrix}\begin{bmatrix} \mathbf{0} & 0 \\ \mathbf{0} & g(i,1)_{\sigma\sigma\sigma} \end{bmatrix}\begin{bmatrix} \mathbf{x}_t^f \\ \sigma \end{bmatrix} \end{bmatrix}$$

$$=\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{c}\left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,1,:,:)\mathbf{x}_t^f+3g(i,1)_{\sigma\sigma\mathbf{x}}\sigma^2\\ \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,2,:,:)\mathbf{x}_t^f+3g(i,2)_{\sigma\sigma\mathbf{x}}\sigma^2\\ \dots\\ \left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,n_x,:,:)\mathbf{x}_t^f+3g(i,n_x)_{\sigma\sigma\mathbf{x}}\sigma^2\\ g(i,1)_{\sigma\sigma\sigma}\sigma^2\end{array}\right]$$

$$=\sum_{k=1}^{n_x}x_t^f(k)\left(\left(\mathbf{x}_t^f\right)'\mathbf{g_{xxx}}(i,k,:,:)\mathbf{x}_t^f+3g(i,k)_{\sigma\sigma\mathbf{x}}\sigma^2\right)+g(i,1)_{\sigma\sigma\sigma}\sigma^3$$

as desired.

$$\begin{aligned}x_t^{rd}(i) &= \left[\begin{array}{cc}\mathbf{h_x}(i,:) & 0\end{array}\right]\left[\begin{array}{c}\mathbf{x}_t^{rd}\\ \sigma\end{array}\right]\\ &+\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{cc}\mathbf{h_{xx}}(i,:,:) & 0\\ \mathbf{0} & h_{\sigma\sigma}\end{array}\right]\left[\begin{array}{c}\mathbf{x}_t^s\\ \sigma\end{array}\right]\\ &+\frac{1}{6}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{c}\left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{cc}\mathbf{h_{xxx}}(i,1,:,:) & 0\\ \mathbf{0} & 3h(i,1)_{\sigma\sigma\mathbf{x}}\end{array}\right]\left[\begin{array}{c}\mathbf{x}_t^f\\ \sigma\end{array}\right]\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{cc}\mathbf{h_{xxx}}(i,2,:,:) & 0\\ \mathbf{0} & 3h(i,2)_{\sigma\sigma\mathbf{x}}\end{array}\right]\left[\begin{array}{c}\mathbf{x}_t^f\\ \sigma\end{array}\right]\\ \dots\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{cc}\mathbf{h_{xxx}}(i,n_x,:,:) & 0\\ \mathbf{0} & 3h(i,n_x)_{\sigma\sigma\mathbf{x}}\end{array}\right]\left[\begin{array}{c}\mathbf{x}_t^f\\ \sigma\end{array}\right]\\ \left[\begin{array}{cc}\left(\mathbf{x}_t^f\right)' & \sigma\end{array}\right]\left[\begin{array}{cc}\mathbf{0} & 0\\ \mathbf{0} & h(i,1)_{\sigma\sigma\sigma}\end{array}\right]\left[\begin{array}{c}\mathbf{x}_t^f\\ \sigma\end{array}\right]\end{array}\right]\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}x_t^{rd}(i) &= \left[\begin{array}{cc}\mathbf{h_x}(i,:) & 0\end{array}\right]\tilde{\mathbf{x}}_t^{rd}\\ &+\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc}\mathbf{h_{xx}}(i,:,:) & 0\\ \mathbf{0} & h_{\sigma\sigma}\end{array}\right]\tilde{\mathbf{x}}_t^f\\ &+\frac{1}{6}\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{c}\left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc}\mathbf{h_{xxx}}(i,1,:,:) & 0\\ \mathbf{0} & 3h(i,1)_{\sigma\sigma\mathbf{x}}\end{array}\right]\tilde{\mathbf{x}}_t^f\\ \left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc}\mathbf{h_{xxx}}(i,2,:,:) & 0\\ \mathbf{0} & 3h(i,2)_{\sigma\sigma\mathbf{x}}\end{array}\right]\tilde{\mathbf{x}}_t^f\\ \dots\\ \left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc}\mathbf{h_{xxx}}(i,n_x,:,:) & 0\\ \mathbf{0} & 3h(i,n_x)_{\sigma\sigma\mathbf{x}}\end{array}\right]\tilde{\mathbf{x}}_t^f\\ \left(\tilde{\mathbf{x}}_t^f\right)'\left[\begin{array}{cc}\mathbf{0} & 0\\ \mathbf{0} & h(i,1)_{\sigma\sigma\sigma}\end{array}\right]\tilde{\mathbf{x}}_t^f\end{array}\right]\end{aligned}$$

14 Accurcy of the pruned state-space system with $\sigma \longrightarrow 0$

This section shows that the errors induced by the pruned state-space system at second order are $O\left(\sigma^3\right)$ for $\sigma \longrightarrow 0$, and that the errors induced by the state space system at third order are $O\left(\sigma^4\right)$ for $\sigma \longrightarrow 0$. This corresponds to showing that the errors implied by the unpruned and pruned approximations are of the same order for $\sigma \longrightarrow 0$, provided that the unpruned sample path is stable. The procedure for showing these results follow the one applied in the Appendix to Haan & Wind (2012).

14.1 Some auxiliary expressions

Recall that the unpruned third-order approximation of the state equation is given by

$$\begin{aligned}\mathbf{x}_{t+1}^{(3)} &= \left(\mathbf{h}_{\mathbf{x}}+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\\ &\quad+\frac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+O\left(\sigma^4\right),\end{aligned}$$

where $\mathbf{x}_t^{(3)}$ denotes the unpruned third-order approximation. We know that the errors induced by this approximation is of fourth order, i.e. $O\left(\sigma^4\right)$. For the derivations below, we need some auxiliary expressions. We thereefore first note that

$$\begin{aligned}&\mathbf{x}_{t+1}^{(3)}\otimes\mathbf{x}_{t+1}^{(3)}=\\
&\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\otimes\\
&\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&=\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}\right)\otimes\\
&\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\\
&\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\\
&\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\otimes\\
&\quad\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\\
&\quad\left(\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\\
&=\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}\otimes\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\\
&+\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}\otimes\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}\otimes\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\\
&+\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\
&+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\left(\mathbf{h}_{\mathbf{x}}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\\
&+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\\
&+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

$$+\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\left(\mathbf{h_x}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$
$$+\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\otimes\left(\mathbf{h_x}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\otimes\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\otimes\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\otimes\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)$$
$$+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{h_x}+\tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)}+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

Preserving terms up to third order we have

$$\mathbf{x}_{t+1}^{(3)}\otimes\mathbf{x}_{t+1}^{(3)}=\mathbf{h_x}\mathbf{x}_t^{(3)}\otimes\mathbf{h_x}\mathbf{x}_t^{(3)}+\mathbf{h_x}\mathbf{x}_t^{(3)}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\mathbf{h_x}\mathbf{x}_t^{(3)}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\mathbf{h_x}\mathbf{x}_t^{(3)}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\otimes\mathbf{h_x}\mathbf{x}_t^{(3)}$$
$$+\mathbf{h_x}\mathbf{x}_t^{(3)}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$
$$+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h_x}\mathbf{x}_t^{(3)}+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma^2\right)$$
$$+\left(\tfrac{1}{2}\mathbf{H_{xx}}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\mathbf{h_x}\right)\left(\sigma^2\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h_x}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{H_{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^3$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h_x}\right)\left(\sigma\otimes\mathbf{x}_t^{(3)}\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\sigma\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\sigma^3+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^2$$

$$=\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}+\tfrac{1}{2}\mathbf{H_{xx}}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+$$
$$+\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma^2\right)$$
$$+\left(\mathbf{h_x}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{H_{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^3$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h_x}\right)\left(\sigma\otimes\mathbf{x}_t^{(3)}\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\sigma\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\sigma^3+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^2$$

Preserving terms up to second order we have

$$\begin{aligned}\mathbf{x}_{t+1}^{(2)}\otimes\mathbf{x}_{t+1}^{(2)} &= \left(\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(2)}\otimes\mathbf{x}_t^{(2)}\right)+\left(\mathbf{h_x}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(2)}\otimes\sigma\right)\\ &\quad+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h_x}\right)\left(\sigma\otimes\mathbf{x}_t^{(2)}\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^2\end{aligned}$$

where $\mathbf{x}_t^{(2)}$ denotes the unpruned second-order approximation.

Moveover, preserving terms up to third order in $\mathbf{x}_{t+1}^{(3)}\otimes\mathbf{x}_{t+1}^{(3)}\otimes\mathbf{x}_{t+1}^{(3)}$ clearly gives

$$\mathbf{x}_{t+1}^{(3)}\otimes\mathbf{x}_{t+1}^{(3)}\otimes\mathbf{x}_{t+1}^{(3)}=\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^{(3)}\right)+\left(\mathbf{h_x}\otimes\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\mathbf{x}_t^{(3)}\otimes\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h_x}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_t^{(3)}\otimes\boldsymbol{\epsilon}_{t+1}\right)$$

$$+ (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \mathbf{h_x}) \left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^{(3)}\right) + (\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta} \otimes \boldsymbol{\sigma\eta}) (\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}) + O\left(\sigma^4\right)$$

Next, recall that the pruned approximation at third order reads

$$\mathbf{x}_{t+1}^f = \mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

$$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$

$$\mathbf{x}_{t+1}^{rd} = \mathbf{h_x}\mathbf{x}_t^{rd} + \mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3$$

Adding up these terms we have

$$\begin{aligned}\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s + \mathbf{x}_{t+1}^{rd} &= \mathbf{h_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{2}\mathbf{H_{xx}}\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + 2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right) \\ &+ \frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

For the controls we have without pruning that

$$\mathbf{y}_t^{(2)} = \mathbf{g_x}\mathbf{x}_t^{(2)} + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

and

$$\begin{aligned}\mathbf{y}_t^{(3)} &= \mathbf{g_x}\mathbf{x}_t^{(3)} + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \frac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\ &+ \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^{(3)} + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3 + O\left(\sigma^4\right)\end{aligned}$$

in an unpruned second- and third-order approximation, respectively

14.2 Proof for a second order approximation

14.2.1 First order terms

At first order we trivially have

$$\begin{aligned}\mathbf{x}_{t+1}^f - \mathbf{x}_{t+1}^{(2)} &= \mathbf{h_x}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\ &- \left\{\mathbf{h_x^{(2)}}\mathbf{x}_t^{(2)} + \tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + O\left(\sigma^3\right)\right\} \\ &= \mathbf{h_x}\left(\mathbf{x}_t^f - \mathbf{x}_t^{(2)}\right) + O\left(\sigma^2\right)\end{aligned}$$

We therefore see that errors in $\mathbf{x}_t^f$ are of second order, i.e. $\mathbf{x}_t^f - \mathbf{x}_t^{(2)} = O\left(\sigma^2\right)$. Note, that we do not need to require that the initial errors are of second order, i.e. $\mathbf{x}_0^f - \mathbf{x}_0^{(2)} = O\left(\sigma^2\right)$, provided that t is sufficiently far away from zero. This is because any error committed at time 0 will die out because all eigenvalues of $\mathbf{h_x}$ are less than one.

For the controls, the pruned expression is $\mathbf{y}_t^f = \mathbf{g_x}\mathbf{x}_t^f$, so $\mathbf{y}_t^f - \mathbf{y}_t^{(2)} = \mathbf{g_x}\left(\mathbf{x}_t^f - \mathbf{x}_t\right) + O\left(\sigma^2\right)$. Given that $\mathbf{x}_t^f - \mathbf{x}_t^{(2)} = O\left(\sigma^2\right)$, this clearly also holds for the controls, i.e. $\mathbf{y}_t^f - \mathbf{y}_t^{(2)} = O\left(\sigma^2\right)$.

14.2.2 Second order terms

At second order the pruned approximation of the states is given by

$$\mathbf{x}_{t+1}^{f}+\mathbf{x}_{t+1}^{s}=\mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}\right)+\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}$$

Comparing this approximation to a second-order expansion without pruning, we obtain

$$\begin{aligned}&\mathbf{x}_{t+1}^{f}+\mathbf{x}_{t+1}^{s}-\mathbf{x}_{t+1}^{(2)}=\mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}\right)+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\\&\quad-\left\{\mathbf{h}_{\mathbf{x}}\mathbf{x}_{t}^{(2)}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^{2}+\sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}+O\left(\sigma^{3}\right)\right\}\\&\Updownarrow\end{aligned}$$

$$\mathbf{x}_{t+1}^{f}+\mathbf{x}_{t+1}^{s}-\mathbf{x}_{t+1}^{(2)}=\mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}-\mathbf{x}_{t}^{(2)}\right)+\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}\right)+O\left(\sigma^{3}\right)$$

We next want to show that $\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}-\mathbf{x}_{t}^{(2)}=O\left(\sigma^{3}\right)$ and $\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}=O\left(\sigma^{3}\right)$. To do so, we recall that

$$\begin{aligned}\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}\quad=\quad&\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_{t}^{f}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

Hence, we have

$$\begin{aligned}&\mathbf{x}_{t+1}^{f}\otimes\mathbf{x}_{t+1}^{f}-\mathbf{x}_{t+1}^{(2)}\otimes\mathbf{x}_{t+1}^{(2)}=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_{t}^{f}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&\quad+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}\right)+\left(\sigma\boldsymbol{\eta}\otimes\boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\epsilon}_{t+1}\right)\\&\\&\quad-\{\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}\right)+\\&\quad+\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_{t}^{(2)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^{3}\\&\quad+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\otimes\mathbf{x}_{t}^{(2)}\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^{2}+O\left(\sigma^{3}\right)\}\\&\\&\quad=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\right)\left(\mathbf{x}_{t}^{f}\otimes\sigma\boldsymbol{\epsilon}_{t+1}-\mathbf{x}_{t}^{(2)}\otimes\sigma\boldsymbol{\epsilon}_{t+1}\right)\\&\quad+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{f}-\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{x}_{t}^{(2)}\right)+O\left(\sigma^{3}\right)\\&\\&\quad=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}\right)+\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\right)\left(\left(\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\right)\otimes\sigma\boldsymbol{\epsilon}_{t+1}\right)\\&\quad+\left(\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\right)\right)+O\left(\sigma^{3}\right)\end{aligned}$$

We know that $\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}=O\left(\sigma^{2}\right)$. Now recall the meaning of our notation. We say that a function $f\left(\sigma\right)$ is

$$f\left(\sigma\right)=O\left(\sigma^{m}\right)$$

$\Updownarrow$

$$\lim_{\sigma\longrightarrow0}\frac{f\left(\sigma\right)}{\sigma^{m}}<M<\infty$$

This clearly implies that

$$\frac{\sigma f\left(\sigma\right)}{\sigma^{m+1}}=O\left(\sigma^{m+1}\right),$$

or $\sigma f\left(\sigma\right)=O\left(\sigma^{m+1}\right)$. So in our case we have $\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}=O\left(\sigma^{2}\right)$, and therefore $\sigma\left(\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\right)=O\left(\sigma^{3}\right)$. Hence, $\mathbf{x}_{t}^{f}\otimes\mathbf{x}_{t}^{f}-\mathbf{x}_{t}^{(2)}\otimes\mathbf{x}_{t}^{(2)}=O\left(\sigma^{3}\right)$. This in turn means that $\mathbf{x}_{t}^{f}+\mathbf{x}_{t}^{s}-\mathbf{x}_{t}^{(2)}=O\left(\sigma^{3}\right)$ as desired.

For the controls we have in the pruned approximation

$$\mathbf{y}_t^s = \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

So we have

$$\mathbf{y}_t^s - \mathbf{y}_t^{(2)} = \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s - \mathbf{x}_t^{(2)}\right) + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}\right) + O\left(\sigma^3\right)$$

Given that $\mathbf{x}_t^f + \mathbf{x}_t^s - \mathbf{x}_t^{(2)} = O\left(\sigma^3\right)$ and $\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)} = O\left(\sigma^3\right)$, we clearly have $\mathbf{y}_t^s - \mathbf{y}_t^{(2)} = O\left(\sigma^3\right)$ as desired.

14.3 Proof for a third order approximation

14.3.1 First order terms

At first order we trivially have

$$\begin{aligned}
\mathbf{x}_{t+1}^f - \mathbf{x}_{t+1}^{(3)} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^f + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\
&\quad -\{\left(\mathbf{h}_{\mathbf{x}} + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)} + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\
&\qquad +\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + O\left(\sigma^4\right)\} \\
&= \mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_t^f - \mathbf{x}_t^{(3)}\right) + O\left(\sigma^2\right)
\end{aligned}$$

We therefore see that errors in $\mathbf{x}_t^f$ are of second order, i.e. $\mathbf{x}_t^f - \mathbf{x}_t^{(3)} = O\left(\sigma^2\right)$. For the controls, the pruned expression is $\mathbf{y}_t^f = \mathbf{g}_{\mathbf{x}}\mathbf{x}_t^f$, so $\mathbf{y}_t^f - \mathbf{y}_t^{(3)} = \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f - \mathbf{x}_t\right) + O\left(\sigma^2\right)$. Given that $\mathbf{x}_t^f - \mathbf{x}_t^{(3)} = O\left(\sigma^2\right)$, this clearly also holds for the controls, i.e. $\mathbf{y}_t^f - \mathbf{y}_t^{(3)} = O\left(\sigma^2\right)$.

14.3.2 Second order terms

Comparing this approximation to a third-order expansion without pruning, we obtain

$$\begin{aligned}
\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s - \mathbf{x}_{t+1}^{(3)} &= \mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \\
&\quad -\{\left(\mathbf{h}_{\mathbf{x}} + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)} + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \tfrac{1}{6}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\
&\qquad +\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} + O\left(\sigma^4\right)\}
\end{aligned}$$

$$\Updownarrow$$

$$\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s - \mathbf{x}_{t+1}^{(3)} = \mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s - \mathbf{x}_t^{(3)}\right) + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + O\left(\sigma^3\right)$$

We next want to show that $\mathbf{x}_t^f + \mathbf{x}_t^s - \mathbf{x}_t^{(3)} = O\left(\sigma^3\right)$ and $\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} = O\left(\sigma^3\right)$. To do so, we recall that

$$\begin{aligned}
\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f &= \left(\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&\quad + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f - \mathbf{x}_{t+1}^{(3)} \otimes \mathbf{x}_{t+1}^{(3)} &= \left(\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&\quad + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&\quad -\{\left(\mathbf{h}_{\mathbf{x}} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}} + \tfrac{1}{2}\mathbf{H}_{\mathbf{xx}} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \\
&\quad + \left(\mathbf{h}_{\mathbf{x}} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma} + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma} \otimes \mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)} \otimes \sigma^2\right)
\end{aligned}$$

$$
+ \left(\mathbf{h_x} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)} \otimes \sigma\right) + \left(\tfrac{1}{2}\mathbf{H_{xx}} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \sigma\right) + \left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^3
$$

$$
+ \left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{h_x}\right)\left(\sigma \otimes \mathbf{x}_t^{(3)}\right) + \left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\sigma \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\sigma^3 + \left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^2 + O\left(\sigma^4\right)\}
$$

$$
= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \left(\mathbf{h_x} \otimes \boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \sigma\boldsymbol{\epsilon}_{t+1} - \mathbf{x}_t^{(3)} \otimes \sigma\boldsymbol{\epsilon}_{t+1}\right)
$$

$$
+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f - \boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^{(3)}\right) + O\left(\sigma^3\right)
$$

$$
= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \left(\mathbf{h_x} \otimes \boldsymbol{\eta}\right)\left(\left(\mathbf{x}_t^f - \mathbf{x}_t^{(3)}\right) \otimes \sigma\boldsymbol{\epsilon}_{t+1}\right)
$$

$$
+ \left(\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\sigma\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{x}_t^f - \mathbf{x}_t^{(3)}\right)\right) + O\left(\sigma^3\right)
$$

We know that $\mathbf{x}_t^f - \mathbf{x}_t^{(3)} = O\left(\sigma^2\right)$. Now recall the meaning of our notation. We say that a function $f\left(\sigma\right)$ is

$$
f\left(\sigma\right) = O\left(\sigma^m\right)
$$

$\Updownarrow$

$$
\lim_{\sigma \longrightarrow 0} \frac{f\left(\sigma\right)}{\sigma^m} < M < \infty
$$

This clearly implies that

$$
\frac{\sigma f\left(\sigma\right)}{\sigma^{m+1}} = O\left(\sigma^{m+1}\right),
$$

or $\sigma f\left(\sigma\right) = O\left(\sigma^{m+1}\right)$. So in our case we have $\mathbf{x}_t^f - \mathbf{x}_t^{(3)} = O\left(\sigma^2\right)$, and therefore $\sigma\left(\mathbf{x}_t^f - \mathbf{x}_t^{(3)}\right) = O\left(\sigma^3\right)$. Hence, $\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f - \mathbf{x}_{t+1}^{(3)} \otimes \mathbf{x}_{t+1}^{(3)} = O\left(\sigma^3\right)$. This in turn means that $\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s - \mathbf{x}_{t+1}^{(3)} = O\left(\sigma^3\right)$ as desired.

For the controls we have in the pruned approximation

$$
\mathbf{y}_t^s = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2
$$

So we have

$$
\mathbf{y}_t^s - \mathbf{y}_t^{(3)} = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s - \mathbf{x}_t^{(3)}\right) + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + O\left(\sigma^3\right)
$$

Given that $\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s - \mathbf{x}_{t+1}^{(3)} = O\left(\sigma^3\right)$ and $\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f - \mathbf{x}_{t+1}^{(3)} \otimes \mathbf{x}_{t+1}^{(3)} = O\left(\sigma^3\right)$, we clearly have $\mathbf{y}_t^s - \mathbf{y}_t^{(3)} = O\left(\sigma^3\right)$ as desired.

14.3.3 Third order terms

Comparing this approximation to a third-order expansion without pruning, we obtain at third order

$$
\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s + \mathbf{x}_{t+1}^{rd} - \mathbf{x}_{t+1}^{(3)} = \mathbf{h_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{x}_t^s \otimes \mathbf{x}_t^f\right)\right)
$$

$$
+ \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\mathbf{x}_t^f\sigma^2 + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}
$$

$$
- \left\{\left(\mathbf{h_x} + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\right)\mathbf{x}_t^{(3)} + \tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \tfrac{1}{6}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right\}
$$

$$
= \mathbf{h_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd} - \mathbf{x}_t^{(3)}\right) + \tfrac{1}{2}\mathbf{H_{xx}}\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{x}_t^s \otimes \mathbf{x}_t^f\right) - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right)
$$

$$
+ \tfrac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \tfrac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\left(\mathbf{x}_t^f - \mathbf{x}_t^{(3)}\right)
$$

We then need to show that each of the terms are accurate up to $O\left(\sigma^4\right)$. We start by considering the last term. Here, $\mathbf{x}_t^f - \mathbf{x}_t^{(3)} = O\left(\sigma^2\right)$, and therefore $\sigma^2\left(\mathbf{x}_t^f - \mathbf{x}_t^{(3)}\right) = O\left(\sigma^4\right)$, as desired. We clearly also have $\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} = O\left(\sigma^4\right)$. To consider the single remaining term, recall from above that

$$\begin{aligned}\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f &= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) \\ &\quad + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right)\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s &= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \frac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f \\ &\quad + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}_{t+1}^s \otimes \mathbf{x}_{t+1}^f &= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^s \otimes \mathbf{x}_t^f\right) + \left(\frac{1}{2}\mathbf{H_{xx}} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \otimes \mathbf{h_x}\right)\mathbf{x}_t^f \\ &\quad + \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^s \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\frac{1}{2}\mathbf{H_{xx}} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \otimes \sigma\boldsymbol{\eta}\right)\boldsymbol{\epsilon}_{t+1}\end{aligned}$$

Thus

$$\begin{aligned}&\left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^f\right) + \left(\mathbf{x}_{t+1}^f \otimes \mathbf{x}_{t+1}^s\right) + \left(\mathbf{x}_{t+1}^s \otimes \mathbf{x}_{t+1}^f\right) - \left(\mathbf{x}_{t+1}^{(3)} \otimes \mathbf{x}_{t+1}^{(3)}\right) \\
&= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \boldsymbol{\sigma\eta}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_t^f \\
&+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^s\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\sigma\boldsymbol{\eta} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\boldsymbol{\epsilon}_{t+1} \\
&+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^s \otimes \mathbf{x}_t^f\right) + \left(\tfrac{1}{2}\mathbf{H_{xx}} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \otimes \mathbf{h_x}\right)\mathbf{x}_t^f \\
&+ \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^s \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\tfrac{1}{2}\mathbf{H_{xx}} \otimes \sigma\boldsymbol{\eta}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \boldsymbol{\epsilon}_{t+1}\right) + \left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \otimes \sigma\boldsymbol{\eta}\right)\boldsymbol{\epsilon}_{t+1} \\
\\
&-\{\left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}} + \tfrac{1}{2}\mathbf{H_{xx}} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma} + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)} \otimes \sigma^2\right) \\
&+ \left(\mathbf{h_x} \otimes \boldsymbol{\eta\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)} \otimes \sigma\right) + \left(\tfrac{1}{2}\mathbf{H_{xx}} \otimes \boldsymbol{\eta\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \sigma\right) + \left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma} \otimes \boldsymbol{\eta\epsilon}_{t+1}\right)\sigma^3 \\
&+ \left(\boldsymbol{\eta\epsilon}_{t+1} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^{(3)} \otimes \sigma\right) + \left(\boldsymbol{\eta\epsilon}_{t+1} \otimes \tfrac{1}{2}\mathbf{H_{xx}}\right)\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \sigma\right) + \left(\boldsymbol{\eta\epsilon}_{t+1} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\sigma^3 + \left(\boldsymbol{\eta\epsilon}_{t+1} \otimes \boldsymbol{\eta\epsilon}_{t+1}\right)\sigma^2\} \\
\\
&= \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\
&+ \left(\mathbf{h_x} \otimes \sigma\boldsymbol{\eta}\right)\left(\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) \otimes \boldsymbol{\epsilon}_{t+1}\right) \\
&+ \left(\sigma\boldsymbol{\eta} \otimes \mathbf{h_x}\right)\left(\boldsymbol{\epsilon}_{t+1} \otimes \left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)\right) \\
&+ \left(\mathbf{h_x} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s + \mathbf{x}_t^s \otimes \mathbf{x}_t^f\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{H_{xx}} + \tfrac{1}{2}\mathbf{H_{xx}} \otimes \mathbf{h_x}\right)\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) \\
&+ \left(\mathbf{h_x} \otimes \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \otimes \mathbf{h_x}\right)\mathbf{x}_t^f \\
&+ \left(\boldsymbol{\eta\epsilon}_{t+1} \otimes \tfrac{1}{2}\mathbf{H_{xx}} + \tfrac{1}{2}\mathbf{H_{xx}} \otimes \boldsymbol{\eta\epsilon}_{t+1}\right)\left(\sigma \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)\end{aligned}$$

$$+\left(\sigma\boldsymbol{\eta}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\otimes\sigma\boldsymbol{\eta}\right)\boldsymbol{\epsilon}_{t+1}$$

$$-\{\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)+$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma^2\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\sigma^3$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\right)\sigma^3\}$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\sigma\boldsymbol{\eta}\right)\left(\left(\mathbf{x}_t^f+\mathbf{x}_t^s\right)\otimes\boldsymbol{\epsilon}_{t+1}\right)$$
$$+\left(\sigma\boldsymbol{\eta}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\boldsymbol{\epsilon}_{t+1}\otimes\left(\mathbf{x}_t^f+\mathbf{x}_t^s\right)\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s+\mathbf{x}_t^s\otimes\mathbf{x}_t^f\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\mathbf{h}_{\mathbf{x}}\right)\sigma^2\left(\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\right)$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\sigma\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)$$

$$-\{\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\sigma\right)+$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^{(3)}\otimes\sigma\right)+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\sigma\right)\}$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f+\mathbf{x}_t^s-\mathbf{x}_t^{(3)}\right)\sigma\right)$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{\mathbf{t+1}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\left(\mathbf{x}_t^f+\mathbf{x}_t^s-\mathbf{x}_t^{(3)}\right)\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s+\mathbf{x}_t^s\otimes\mathbf{x}_t^f\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\mathbf{h}_{\mathbf{x}}\right)\sigma^2\left(\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\right)$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\sigma\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\right)$$

$$=\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f+\mathbf{x}_t^f\otimes\mathbf{x}_t^s+\mathbf{x}_t^s\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\left(\mathbf{x}_t^f+\mathbf{x}_t^s-\mathbf{x}_t^{(3)}\right)\sigma\right)$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{\mathbf{t+1}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\sigma\left(\mathbf{x}_t^f+\mathbf{x}_t^s-\mathbf{x}_t^{(3)}\right)\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)$$
$$+\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\otimes\mathbf{h}_{\mathbf{x}}\right)\sigma^2\left(\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\right)$$
$$+\left(\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}+\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\otimes\boldsymbol{\eta}\boldsymbol{\epsilon}_{t+1}\right)\left(\sigma\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\right)\right)$$

We know that $\mathbf{x}_t^f+\mathbf{x}_t^s-\mathbf{x}_t^{(3)}=O\left(\sigma^3\right)$, so $\left(\mathbf{x}_t^f+\mathbf{x}_t^s-\mathbf{x}_t^{(3)}\right)\sigma=O\left(\sigma^4\right)$. We clearly also have that $\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}=O\left(\sigma^4\right)$. Similarly, $\mathbf{x}_t^f-\mathbf{x}_t^{(3)}=O\left(\sigma^2\right)$, so $\sigma^2\left(\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\right)=O\left(\sigma^4\right)$. Finally, $\mathbf{x}_t^f\otimes\mathbf{x}_t^f-\mathbf{x}_t^{(3)}\otimes\mathbf{x}_t^{(3)}=$

$O\left(\sigma^3\right)$, so $\sigma\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) = O\left(\sigma^4\right)$. This means that $\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f + \mathbf{x}_t^f \otimes \mathbf{x}_t^s + \mathbf{x}_t^s \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) = O\left(\sigma^4\right)$. In conclusion we have $\mathbf{x}_{t+1}^f + \mathbf{x}_{t+1}^s + \mathbf{x}_{t+1}^{rd} - \mathbf{x}_{t+1}^{(3)} = O\left(\sigma^4\right)$ as desired.

For the controls we have

$$\begin{aligned}\mathbf{y}_t^{rd} &= \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f + \mathbf{x}_t^f \otimes \mathbf{x}_t^s + \mathbf{x}_t^s \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3 \\ &\quad + \frac{1}{6}\mathbf{G}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{y}_t^{rd} - \mathbf{y}_t^{(3)} &= \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd} - \mathbf{x}_t^{(3)}\right) + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f + \mathbf{x}_t^f \otimes \mathbf{x}_t^s + \mathbf{x}_t^s \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\ &\quad + \frac{1}{6}\mathbf{G}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right).\end{aligned}$$

Given that $\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd} - \mathbf{x}_t^{(3)} = O\left(\sigma^4\right)$, $\mathbf{x}_t^f \otimes \mathbf{x}_t^f + \mathbf{x}_t^f \otimes \mathbf{x}_t^s + \mathbf{x}_t^s \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} = O\left(\sigma^4\right)$, and $\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f - \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} = O\left(\sigma^4\right)$, we clearly have $\mathbf{y}_t^{rd} - \mathbf{y}_t^{(3)} = O\left(\sigma^4\right)$ as desired.

15 The pruning schemes in Den Haan and De Wind (2012)

In the paper Haan & Wind (2012) suggest two pruning schemes for perturbation approximations beyond a second order approximation (see page 1490 in their paper). We shortly present each of these suggestions with special focus devoted to a third order approximation.

Throughout this section we adopt the notation in Haan & Wind (2012). That is all endogenous variables in the DSGE model are in $\mathbf{x}_t$ and all exogenous shocks are in $\mathbf{z}_t$. The policy function is denoted by $\mathbf{f}\left(\right)$. Furthermore, we use the notation that $\mathbf{f}_{n^{th}}^{(j)}$ is the j'th order term in a n'th order Taylor series expansion. Finally, let $\mathbf{x}_{stoch}$ denote the stochastic steady state, i.e. $\mathbf{x}_{stoch} = \mathbf{h}_{n^{th}}\left(\mathbf{x}_{stoch}, \sigma = 1\right)$ where $\mathbf{h}_{n^{th}}\left(\cdot\right)$ is the n'th order Taylor series expansion

15.1 First proposal

Let $\mathbf{x}_{stoch}$ denote the stochastic steady state, i.e. $\mathbf{x}_{stoch} = \mathbf{h}_{n^{th}}\left(\mathbf{x}_{stoch}, \sigma = 1\right)$ where $\mathbf{h}_{n^{th}}\left(\cdot\right)$ is the n'th order Taylor series expansion. The pruning scheme is as follows

$$\mathbf{x}_t^{(j)} - \mathbf{x}_{stoch} = \sum_{i=1}^{j} \mathbf{f}_{n^{th}}^{(i)}\left(\mathbf{x}_{t-1}^{(j-i+1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$$

for $j = 1, 2, ..., n$.

Thus for a third order approximation, we have

$$\mathbf{x}_t^{(1)} - \mathbf{x}_{stoch} = \mathbf{f}_{3^{rd}}^{(1)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$$

$$\begin{aligned}\mathbf{x}_t^{(2)} - \mathbf{x}_{stoch} &= \mathbf{f}_{3^{rd}}^{(1)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \\ &\quad + \mathbf{f}_{3^{rd}}^{(2)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)\end{aligned}$$

$$\begin{aligned}\mathbf{x}_t^{(3)} - \mathbf{x}_{stoch} &= \mathbf{f}_{3^{rd}}^{(1)}\left(\mathbf{x}_{t-1}^{(3)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \\ &\quad + \mathbf{f}_{3^{rd}}^{(2)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \\ &\quad + \mathbf{f}_{3^{rd}}^{(3)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)\end{aligned}$$

Thus, we see that in the expression for $\mathbf{x}_t^{(3)}$, we are squaring some second-order terms in $\mathbf{f}_{3^{rd}}^{(2)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$, which therefore is of fourth order. This holds even when σ is considered as a constant.

15.2 Second proposal

Let $\mathbf{x}_{stoch}$ denote the stochastic steady state, i.e. $\mathbf{x}_{stoch} = \mathbf{h}_{n^{th}}(\mathbf{x}_{stoch}, \sigma = 1)$ where $\mathbf{h}_{n^{th}}(\cdot)$ is the n'th order Taylor series expansion. The pruning scheme is as follows:

$$\mathbf{x}_t^{(1)} - \mathbf{x}_{stoch} = \mathbf{f}_{2^{nd}}^{(1)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$$

$$\mathbf{x}_t^{(2)} - \mathbf{x}_{stoch} = \mathbf{f}_{2^{nd}}^{(1)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) + \mathbf{f}_{2^{nd}}^{(2)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$$

$$\mathbf{x}_t^{(j)} - \mathbf{x}_{stoch} = \mathbf{f}_{n^{th}}^{(1)}\left(\mathbf{x}_{t-1}^{(j)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) + \sum_{i=2}^{j} \mathbf{f}_{n^{th}}^{(i)}\left(\mathbf{x}_{t-1}^{(j-1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$$

for $j = 3, 4, .., n$. Thus for a third order approximation, we have

$$\mathbf{x}_t^{(1)} - \mathbf{x}_{stoch} = \mathbf{f}_{3^{rd}}^{(1)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$$

$$\begin{aligned} \mathbf{x}_t^{(2)} - \mathbf{x}_{stoch} &= \mathbf{f}_{3^{rd}}^{(1)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \\ &\quad + \mathbf{f}_{3^{rd}}^{(2)}\left(\mathbf{x}_{t-1}^{(1)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \end{aligned}$$

$$\begin{aligned} \mathbf{x}_t^{(3)} - \mathbf{x}_{stoch} &= \mathbf{f}_{3^{rd}}^{(1)}\left(\mathbf{x}_{t-1}^{(3)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \\ &\quad + \mathbf{f}_{3^{rd}}^{(2)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \\ &\quad + \mathbf{f}_{3^{rd}}^{(3)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right) \end{aligned}$$

Thus, in the last term $\mathbf{f}_{3^{rd}}^{(3)}\left(\mathbf{x}_{t-1}^{(2)} - \mathbf{x}_{stoch}, \mathbf{z}_t - \mathbf{z}, \sigma\right)$, we are taking the third power of all the second order effects, thus including terms up to sixth order. Again, this holds even when σ is considered as a constant.

16 A New Keynesian Model

16.1 Households

The dynamic optimization problem faced by the representative household is of the form:

$$\underset{c_t,b_t,h_t,k_{t+1},i_t\ \forall t\geq 0}{Max} \quad V_t = u\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}, 1-h_t\right) - \beta\left(E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

$$\text{St.}\quad k_{t+1} = (1-\delta)\,k_t + i_t - i_t\frac{\kappa_1}{2}\left(\frac{i_t}{\Upsilon_t z_t^* I_{SS}} - 1\right)^2 - k_t\frac{\kappa_2}{2}\left(\frac{i_t}{k_t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$$

$$b_t + c_t + \frac{i_t}{\Upsilon_t} = \frac{b_{t-1}\exp\left\{r_{t-1}^b\right\}}{\pi_t} + h_t w_t + r_t^k k_t + div_t^h$$

a no-Ponzi-game condition

$c_t, h_t, k_{t+1}, i_t \geq 0\ \forall t \geq 0$

As for the notation:

- c_t =consumption
- h_t =hours
- z_t^* = the deterministic trend in technology and hence consumption
- V_t = the value function
- k_t = the capital stock
- i_t = investments
- $D_{t,t+1}$ = the nominal stochastic discount factor
- b_t = deposit in the financial intermediary in time period t
- r_t^b = the continuously compounded net rate on deposits offered by the financial intermediary
- Υ_t^{-1} = deterministic trend in the relative price of investments when measured in terms of consumption good units
- π_t = gross inflation of consumption good prices
- w_t = the wage level measured in consumption good units
- r_t^k = the rental rate for capital services sold to firms as measured in consumption good units
- div_t^h = net transfers of profit from firms to households as measured in consumption good units

We follow Altig, Christiano, Eichenbaum & Linde (2011) and define $z_t^* \equiv \Upsilon_t^{\frac{\theta}{1-\theta}} z_t$, meaning that $\mu_{z^*,t} \equiv \mu_{\Upsilon,t}^{\theta/(1-\theta)}\mu_{z,t}$. The process for Υ_t is assumed to evolve according to a deterministic tend and so does z_t. That is

$$z_{t+1} \equiv z_t\mu_{z,t+1}$$

where $\log\mu_{z,t} = \log\mu_{z,ss}$, and

$$\Upsilon_{t+1} \equiv \Upsilon_t\mu_{\Upsilon,t+1}$$

where $\log\mu_{\Upsilon,t} = \log\mu_{\Upsilon,ss}$.

We allow for habit formation in consumption via $b \in [0,1]$. In the derivation, we allow this habit formation to be external or internal via the indicator function $1_{[ha\_in]}$, where

$$1_{[ha\_in]} = \begin{cases} 1 & \text{for internal habits} \\ 0 & \text{for external habits} \end{cases}.$$

To simplify the presentation we follow Rudebusch & Swanson (2012) and consider a finite number of states in each period.[1] As in Rudebusch & Swanson (2012), the optimization problem is formulated as a Lagrange problem maximizing V_t. The Lagranian is therefore given by:

$$\mathcal{L} = V_t + \sum_{l=0}^{\infty} E_t \beta^l m_{t+l} \left[u\left(\frac{c_{t+l} - bc_{t-1+l}}{(z_{t+l}^*)^{\phi_4}}, 1 - h_{t+l} \right) - \beta \left(E_t \left[(-V_{t+l+1})^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}} - V_{t+l} \right]$$
$$+ E_t \sum_{l=0}^{\infty} \beta^l \lambda_{t+l} [\frac{b_{t-1+l} \exp\{r_{t-1+l}^b\}}{\pi_{t+l}} + h_{t+l} w_{t+l} + r_{t+l}^k k_{t+l} + div_{t+l} - b_{t+l} - c_{t+l} - \frac{i_{t+l}}{\Upsilon_{t+l}}]$$
$$+ E_t \sum_{l=0}^{\infty} \beta^l q_{t+l} \lambda_{t+l} \left[(1-\delta) k_{t+l} + i_{t+l} - i_{t+l} \frac{\kappa_1}{2} \left(\frac{i_{t+l}}{\Upsilon_{t+l} z_{t+l}^* I_{SS}} - 1 \right)^2 - k_{t+l} \frac{\kappa_2}{2} \left(\frac{i_{t+l}}{k_{t+l}} - \frac{I_{ss}}{K_{ss}} \mu_{\Upsilon,ss} \mu_{z^*,ss} \right)^2 - k_{t+1+l} \right]$$

That is, we introduce three lagrange multipliers:

- m_t = for the constraint on the utility function
- λ_t = for the budget constraint
- $q_t \lambda_t$ = for the capital accumulation equation

FOC (Kuhn-Tucker conditions)
We look for a solution in the interior, i.e. $c_t, b_t, h_t, k_{t+1}, i_t, V_{t+1} > 0 \; \forall t \geq 0$

1) Consumption, c_t **:**

$$\frac{\partial \mathcal{L}}{\partial c_t} = m_t u_c \left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}, 1 - h_t \right) \frac{1}{(z_t^*)^{\phi_4}} - 1_{[ha\_in]} b\beta E_t m_{t+1} u_c \left(\frac{c_{t+1} - bc_t}{(z_{t+1}^*)^{\phi_4}}, 1 - h_{t+1} \right) \frac{1}{(z_{t+1}^*)^{\phi_4}} - \lambda_t = 0$$
$$\Updownarrow$$
$$\lambda_t = m_t u_c \left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}, 1 - h_t \right) \frac{1}{(z_t^*)^{\phi_4}} - 1_{[ha\_in]} b\beta E_t m_{t+1} u_c \left(\frac{c_{t+1} - bc_t}{(z_{t+1}^*)^{\phi_4}}, 1 - h_{t+1} \right) \frac{1}{(z_{t+1}^*)^{\phi_4}}$$

2) Deposits, b_t **:**

$$\frac{\partial \mathcal{L}}{\partial b_t} = E_t \left[\beta \lambda_{t+1} \frac{\exp\{r_t^b\}}{\pi_{t+1}} - \lambda_t \right] = 0$$
$$\Downarrow$$
$$\exp\{r_t^b\} E_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} \right] = 1$$

3) The labor supply, h_t

$$\frac{\partial \mathcal{L}}{\partial h_t} = -m_t u_{1-h} \left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}, 1 - h_t \right) + \lambda_t w_t = 0$$
$$\Updownarrow$$
$$m_t u_{1-h} \left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}, 1 - h_t \right) = \lambda_t w_t$$

4) The physical capital stock, k_{t+1} **:**

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \lambda_t q_t (-1) + E_t \beta \lambda_{t+1} [r_{t+1}^k + q_{t+1} (1-\delta)$$

[1] This assumption is without loss of generality as shown by Epstein & Zin (1989).

$$-q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2+q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{i+1}}{k_{t+1}^2}k_{t+1}]=0$$

$$\Updownarrow$$

$$q_t\lambda_t=E_t\beta\lambda_{t+1}\left[r_{t+1}^k+q_{t+1}(1-\delta)-q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2+q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{i+1}}{k_{t+1}}\right]$$

5) Investments, i_t **:**

$$\frac{\partial\mathcal{L}}{\partial i_t}=-\Upsilon_t^{-1}\lambda_t+q_t\lambda_t\left(1-\frac{\kappa_1}{2}\left(\frac{i_t}{\Upsilon_t z_t^* I_{SS}}-1\right)^2-\frac{i_t}{\Upsilon_t z_t^* i_{SS}}\kappa_1\left(\frac{i_t}{\Upsilon_t z_t^* I_{SS}}-1\right)-\kappa_2\left(\frac{i_t}{k_t}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)=0$$

$$\Updownarrow$$

$$1=q_t\Upsilon_t\left(1-\frac{\kappa_1}{2}\left(\frac{i_t}{\Upsilon_t z_t^* I_{SS}}-1\right)^2-\frac{i_t}{\Upsilon_t z_t^* I_{SS}}\kappa_1\left(\frac{i_t}{\Upsilon_t z_t^* I_{SS}}-1\right)-\kappa_2\left(\frac{i_t}{k_t}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$$

6) The value function

$$\frac{\partial\mathcal{L}}{\partial V_{t+1}(s)}=m_t(s)\left(-\beta\frac{1}{1-\phi_3}\left(E_t\left[\left(-V_{t+1}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}-\frac{1-\phi_3}{1-\phi_3}}(1-\phi_3)\left(-V_{t+1}(s)\right)^{-\phi_3}(-1)\,prob_t(s)\right)$$
$$-m_{t+1}(s)\,prob_t(s)\,\beta=0$$

$$\Updownarrow$$

$$m_{t+1}(s)=m_t(s)\left(E_t\left[\left(-V_{t+1}\right)^{1-\phi_3}\right]\right)^{\frac{\phi_3}{1-\phi_3}}\left(-V_{t+1}(s)\right)^{-\phi_3}\quad \text{for all states}$$

16.2 Firms

16.2.1 Final Good producers

The representative competitive consumption good producer chooses $y_{i,t}$ for $i\in[0,1]$ to solve:

$$\max_{y_{t,i}}\quad P_t y_t-\int_0^1 P_{i,t}y_{i,t}di$$
$$s.t.$$
$$y_t=\left(\int_0^1\left(y_{i,t}\right)^{\frac{\eta-1}{\eta}}di\right)^{\frac{\eta}{\eta-1}}.$$

As for the notation:

- $y_{i,t}$ = denotes the output from firm i at time t
- $P_{t,i}$ = the price of $y_{i,t}$.
- y_t = output from the final good producers
- P_t = price of y_t

The first order condition to the problem is given by

$$y_{i,t}=\left(\frac{P_{i,t}}{P_t}\right)^{-\eta}y_t.$$

To find the expression for the aggregate price level we use the zero-profit condition, i.e.

$$P_t y_t = \int_0^1 P_{i,t} y_{i,t} di$$

$$= \int_0^1 P_{i,t} \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t di$$

$$= y_t P_t^{\eta} \int_0^1 (P_{i,t})^{1-\eta} di$$

$\Updownarrow$

$$P_t^{1-\eta} = \int_0^1 (P_{i,t})^{1-\eta} di$$

$\Updownarrow$

$$P_t = \left[\int_0^1 (P_{i,t})^{1-\eta} di\right]^{\frac{1}{1-\eta}}$$

16.2.2 Intermediate Good Producer

This section derives the first-order-conditions for the ith firm's optimization problem. We start by deriving the equation for the real dividend payments $div_{i,t}$:

$$div_{i,t} \equiv \left[\left(\frac{P_{i,t}}{P_t}\right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t}\right].$$

So the problem is

$$\underset{h_{i,t},k_{i,t},P_{i,t}\ \forall t\geq 0}{Max} \quad E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} div_{i,t+l}$$

St. $\phi_{i,t} \equiv \left(\frac{P_{i,t}}{P_t}\right)^{1-\eta} y_t - r_t^k k_{i,t} - w_t h_{i,t}$

$a_t k_{i,t}^{\theta} (z_t h_{i,t})^{1-\theta} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$

a no-Ponzi-game condition

Here, we use the following notation:

- r_t^k = the rental rate for capital services sold to firms as measured in consumption good units
- $k_{i,t}$ = the capital stock used by firm i at time t
- w_t = the wage level measured in consumption good units
- $h_{i,t}$ =hours used by firm i at time t
- a_t = stationary technology shocks
- z_t = non-stationary technology shocks
- $D_{t,t+l}$ = the nominal stochastic discount factor, i.e. $D_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}}$

The law of motion for a_t is given by

$$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$$

where $\epsilon_{a,t+1} \sim \mathcal{NID}(0,1)$.

The lagrangian is

$$\mathcal{L} = E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} [\left(\frac{P_{i,t+l}}{P_{t+l}}\right)^{1-\eta} y_{t+l} - r^k_{t+l} k_{i,t+l} - w_{t+l} h_{i,t+l}]$$
$$+ E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} mc_{i,t+l} \left[a_{t+l} k^{\theta}_{i,t+l} \left(z_{t+l} h_{i,t+l}\right)^{1-\theta} - \left(\frac{P_{i,t+l}}{P_{t+l}}\right)^{-\eta} y_{t+l} \right]$$

FOC (Kuhn-Tucker conditions)

We look for a solution in the interior, i.e. $h_{i,t}, k_{i,t}, P_{i,t} > 0 \ \forall \ t \geq 0$.

1) Demand for labor, $h_{i,t}$:

$$\frac{\partial \mathcal{L}}{\partial h_{i,t}} = P_t \left(-w_t + mc_{i,t} z_t a_t \left(1-\theta\right) k^{\theta}_{i,t} \left(z_t h_{i,t}\right)^{-\theta}\right)$$

Since $P_t > 0$ we get

$$mc_{i,t} z_t a_t \left(1-\theta\right) k^{\theta}_{i,t} \left(z_t h_{i,t}\right)^{-\theta} = w_t$$

2) Demand for capital, k$_{i,t}$:

$$\frac{\partial \mathcal{L}}{\partial k_{i.t}} = P_t \left(-r^k_t + mc_{i,t} \theta a_t k^{\theta-1}_{i,t} \left(z_t h_{i,t}\right)^{1-\theta}\right)$$

Since $P_t > 0$ we get

$$r^k_t = \theta a_t k^{\theta-1}_{i,t} \left(z_t h_{i,t}\right)^{1-\theta}$$

3) The optimal price, P$_{i,t}$:

We assume Calvo-pricing determined by α, giving the probability of a firm not being allowed to change its price in a given period. Notice, that all the re-optimizing firms face the same problem, hence they all set the same price. We denote this price by $\tilde{P}_t$. The non-optimizing firms let $P_{i,t} = P_{i,t-1} \pi^{\chi}_{t-1}$. That is, we have

$$\frac{P_{i,t+l}}{P_{t+l}} = \frac{\tilde{P}_t \prod_{i=1}^{l} \pi^{\chi}_{t+i-1}}{P_t \frac{P_{t+l}}{P_t}} = \frac{\tilde{P}_t \prod_{i=1}^{l} \frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}}{P_t} \qquad \text{, where } \pi_{t+l} \equiv \frac{P_{t+l}}{P_{t+l-1}}.$$

If we only write the probability that the new price last forever, the Lagrangian reads

$$\mathcal{L} = E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l [\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{1-\eta} y_{t+l} - r^k_{t+l} k_{i,t+l} - w_{t+l} h_{i,t+l}]$$
$$+ E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l mc_{i,t+l} \left[a_{t+l} k^{\theta}_{i,t+l} \left(z_{t+l} h_{i,t+l}\right)^{1-\theta} - \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \right]$$

The first-order-condition is:

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l [\left(1-\eta\right) \tilde{P}_t^{-\eta} P_t^{\eta-1} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{1-\eta} y_{t+l}$$
$$+ mc_{i,t+l} \eta \tilde{P}_t^{-\eta-1} P_t^{\eta} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{-\eta} y_{t+l}]$$

$$= E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \left[\left(1-\eta\right) P_t^{-1} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right) + \eta \frac{mc_{i,t+l}}{\tilde{P}_t} \right]$$

$$= E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \left[\frac{(\eta-1)}{\eta} \frac{\tilde{P}_t}{P_t} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right) - mc_{i,t+l}\right] \frac{-\eta}{\tilde{P}_t}$$

since $\eta > 1$ and $\tilde{P}_t > 0$, $\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = 0$ implies

$$E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \left[\frac{(\eta-1)}{\eta} \frac{\tilde{P}_t}{P_t} \prod_{i=1}^{l} \left(\frac{\pi^{\chi}_{t+i-1}}{\pi_{t+i}}\right) - mc_{i,t+l}\right] = 0$$

16.2.3 Marginal costs

We next show that marginal costs are identical across all firms, i.e. $mc_{i,t} = mc_t$ for all i. Note from the first order conditions for $k_{i,t}$ and $h_{i,t}$ that

$$\frac{mc_{i,t} a_t z_t (1-\theta) k^{\theta}_{i,t} (z_t h_{i,t})^{-\theta}}{mc_{i,t} a_t \theta k^{\theta-1}_{i,t} (z_t h_{i,t})^{1-\theta}} = \frac{w_t}{r^k_t}$$

$$\Updownarrow$$

$$\frac{z_t (1-\theta)}{\theta k^{-1}_{i,t} (z_t h_{i,t})} = \frac{w_t}{r^k_t}$$

$$\Updownarrow$$

$$\frac{1-\theta}{\theta} \frac{z_t k_{i,t}}{h_{i,t}} = \frac{w_t}{r^k_t}$$

implying that $\frac{k_{i,t}}{h_{i,t}}$ must be constant with respect to i. I.e.

$$\frac{k_{i,t}}{h_{i,t}} = cons_t$$

$$\Updownarrow$$

$$k_{i,t} = h_{i,t} cons_t$$

$$\Downarrow$$

$$\int_0^1 k_{i,t} di = \int_0^1 h_{i,t} cons_t \ di$$

$$\Updownarrow$$

$$k_t = h_t cons_t$$

$$\Updownarrow$$

$$cons_t = k_t / h_t$$

Hence,

$$mc_{i,t} a_t \theta k^{\theta-1}_{i,t} (z_t h_{i,t})^{1-\theta} = r^k_t$$

$$\Updownarrow$$

$$mc_{i,t} a_t \theta \left(\frac{k_{i,t}}{h_{i,t}}\right)^{\theta-1} (z_t)^{1-\theta} = r^k_t$$

$$\Updownarrow$$

$$mc_{i,t} a_t \theta \left(\frac{k_t}{h_t}\right)^{\theta-1} (z_t)^{1-\theta} = r^k_t$$

This shows that $mc_{i,t} = mc_t$.

Hence, we can wright the first order condition for $h_{i,t}$ and $k_{i,t}$ as

$$z_t a_t (1-\theta) \left(z_t \frac{h_t}{k_t}\right)^{-\theta} mc_t = w_t$$

$$r^k_t = \theta a_t \left(z_t \frac{h_t}{k_t}\right)^{1-\theta}$$

16.2.4 The recursive representation of the price relation

We start by defining

$$x_t^1 \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} mc_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$
$$x_t^2 \equiv E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

This implies that the first-order-condition for the price relation can be expressed as

$$E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \left[\frac{\eta-1}{\eta} \frac{\tilde{P}_t}{P_t} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right) - mc_{t+l}\right] = 0$$
$$\Updownarrow$$
$$E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} \frac{(\eta-1)}{\eta} \frac{\tilde{P}_t}{P_t} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right)$$
$$-E_t \sum_{l=0}^{\infty} D_{t,t+l} P_{t+l} \alpha^l \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right)^{-\eta} y_{t+l} mc_{t+l} = 0$$
$$\Updownarrow$$
$$E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l \tilde{P}_t \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta} y_{t+l} \frac{(\eta-1)}{\eta}$$
$$-E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l \tilde{P}_t \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta} y_{t+l} mc_{t+l} = 0 \quad \text{see below}$$
$$\Updownarrow$$

since $\tilde{P}_t > 0$

$$\underbrace{E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}}_{x_t^2} \frac{(\eta-1)}{\eta}$$

$$-\underbrace{E_t \sum_{l=0}^{\infty} D_{t,t+l} \alpha^l y_{t+l} mc_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}}_{x_t^1} = 0$$

$$\Updownarrow$$

$$x_t^2 \frac{(\eta-1)}{\eta} - x_t^1 = 0$$
$$\Updownarrow$$

$$\eta x_t^1 + (1-\eta) x_t^2 = 0$$

We used the fact that

$$P_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right)^{1-\eta} = \tilde{P}_t^{1-\eta} \prod_{i=1}^{l} \left(\pi_{t+i-1}^{\chi}\right)^{1-\eta} P_{t+l}^{\eta} = \tilde{P}_t \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

and

$$P_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}}\right)^{-\eta} = \tilde{P}_t^{-\eta} \prod_{i=1}^{l} \left(\pi_{t+i-1}^{\chi}\right)^{-\eta} P_{t+l}^{1+\eta} = \tilde{P}_t \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

The dynamic process for x_t^1

We define $\frac{\tilde{P}_t}{P_t} = \tilde{p}_t$. Therefore

$$\begin{aligned}
x_t^1 &= E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l y_{t+l} mc_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta} \\
&= y_t mc_{i,t} \tilde{p}_t^{-\eta-1} + E_t \sum_{l=1}^{\infty} D_{t,t+l}\alpha^l y_{t+l} mc_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta} \\
&= y_t mc_t \tilde{p}_t^{-\eta-1} + \tilde{p}_t^{-\eta-1} E_t \left[\alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^{\eta+1} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta} x_{t+1}^1\right] \quad \text{, see below} \\
&= y_t mc_t \tilde{p}_t^{-\eta-1} + E_t \left[\alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta} x_{t+1}^1\right]
\end{aligned}$$

At the third equatility sign we use the fact that

$$x_{t+1}^1 = E_{t+1} \sum_{l=0}^{\infty} D_{t+1,t+1+l}\alpha^l y_{t+1+l} mc_{t+1+l} \left(\frac{\tilde{P}_{t+1}}{P_{t+1}}\right)^{-\eta-1} \prod_{i=1}^{l} \left(\frac{\pi_{t+i}^{\chi}}{\pi_{t+1+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$change of index, $j = l+1$

$$x_{t+1}^1 = \tilde{p}_{t+1}^{-\eta-1} E_{t+1} \sum_{j=1}^{\infty} D_{t+1,t+j}\alpha^{j-1} y_{t+j} mc_{t+j} \prod_{i=1}^{j-1} \left(\frac{\pi_{t+i}^{\chi}}{\pi_{t+1+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha D_{t,t+1} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^j y_{t+j} mc_{t+j} \prod_{i=2}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha D_{t,t+1} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}^{(1+\eta)/\eta}}\right)^{-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^j y_{t+j} mc_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Updownarrow$

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}^{(1+\eta)/\eta}}\right)^{-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^j y_{t+j} mc_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

since $D_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}}$

$\Updownarrow$

$$x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^j y_{t+j} mc_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

$\Downarrow$

$$E_t \left[x_{t+1}^1 \tilde{p}_{t+1}^{1+\eta} \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta}\right] = E_t \sum_{j=1}^{\infty} D_{t,t+j}\alpha^j y_{t+j} mc_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{(1+\eta)/\eta}}\right)^{-\eta}$$

due to the law of iterated expectations, $E_t E_{t+1}[\cdot] = E_t[\cdot]$.

The dynamic process for x_t^2

$x_t^2 = E_t \sum_{l=0}^{\infty} D_{t,t+l}\alpha^l y_{t+l} \left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$

$$= y_t \tilde{p}_t^{-\eta} + \tilde{p}_t^{-\eta} E_t \sum_{l=1}^{\infty} D_{t,t+l}\alpha^l y_{t+l} \prod_{i=1}^{l} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta} \qquad \text{since } \tilde{p}_t = \frac{\tilde{P}_t}{P_t}$$

$$= y_t \tilde{p}_t^{-\eta} + \tilde{p}_t^{-\eta} E_t \left[\alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^{\eta} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta} x_{t+1}^2\right] \qquad \text{, see below}$$

$$= y_t \tilde{p}_t^{-\eta} + E_t \left[\alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta} x_{t+1}^2\right]$$

At the third equatility sign we use the fact that
$x_{t+1}^2 = E_{t+1} \sum_{l=0}^{\infty} D_{t+1,t+1+l}\alpha^l y_{t+1+l} \left(\frac{\tilde{P}_{t+1}}{P_{t+1}}\right)^{-\eta} \prod_{i=1}^{l} \left(\frac{\pi_{t+i}^{\chi}}{\pi_{t+1+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$
$\Updownarrow$change of index $j = 1 + l$

$$x_{t+1}^2 \quad = \tilde{p}_{t+1}^{-\eta} E_{t+1} \sum_{j=1}^{\infty} D_{t+1,t+j}\alpha^{j-1} y_{t+j} \prod_{i=1}^{j-1} \left(\frac{\pi_{t+i}^{\chi}}{\pi_{t+1+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

$\Updownarrow$

$$x_{t+1}^2 \alpha D_{t,t+1} \tilde{p}_{t+1}^{\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^{j} y_{t+j} \prod_{i=2}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

$\Updownarrow$

$$x_{t+1}^2 \alpha D_{t,t+1} \tilde{p}_{t+1}^{\eta} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}^{\eta/(\eta-1)}}\right)^{1-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^{j} y_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

$\Updownarrow$

$$x_{t+1}^2 \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} \tilde{p}_{t+1}^{\eta} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}^{\eta/(\eta-1)}}\right)^{1-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^{j} y_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

since $D_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}}$
$\Updownarrow$

$$x_{t+1}^2 \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^{\eta} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta} = E_{t+1} \sum_{j=1}^{\infty} D_{t,t+j}\alpha^{j} y_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

$\Downarrow$

$$E_t \left[x_{t+1}^2 \alpha\beta \frac{\lambda_{t+1}}{\lambda_t} \tilde{p}_{t+1}^{\eta} \left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta}\right] = E_t \sum_{j=1}^{\infty} D_{t,t+j}\alpha^{j} y_{t+j} \prod_{i=1}^{j} \left(\frac{\pi_{t+i-1}^{\chi}}{\pi_{t+i}^{\eta/(\eta-1)}}\right)^{1-\eta}$$

due to the law of iterated expectations, $E_t E_{t+1}[\cdot] = E_t[\cdot]$.

16.3 Financial intermediary

The financial intermediary is owned by the household. Its is assumed that the financial intermediary invests deposits from households in short- and long-term government bonds and offers the household a non-stochastic return on its deposits of r_t^b. Hence, the financial intermediary carries all the risk from investing in the bond market. The short-term bond is for simplicity assumed to be the one-period bond, whereas the maturity of the long-term bond is denoted by $L > 1$. The net-worth (measured in nominal terms) in period t of the financial intermediary n_t is given by

$$\begin{aligned} n_t &= n_{t-1} + (1-\omega)(1-P_{t,1})\, b_{t,1} + \omega (1-P_{t,L})\, b_{t,L} \\ &\quad + (1-\omega)(1-P_{t,1-1})\, b_{t-1,1} + \omega (1-P_{t,L-1})\, b_{t-1,L} \\ &\quad - \exp\left\{r_{t-1}^b\right\} b_{t-1} + T_t. \end{aligned}$$

where $b_t \equiv (1-\omega)\, b_{t,1} + \omega b_{t,L}$ and $\omega \in [0,1]$ denotes the fraction chosen by the financial intermediary invested in the long-term government bond. The components are:

- n_{t-1} : net worth from previous time period
- $(1-\omega)(1-P_{t,1})\, b_{t,1} + \omega (1-P_{t,L})\, b_{t,L}$: the amount of deposits b_t and the amount $(1-\omega) P_{t,1} b_{t,1} + \omega P_{t,L} b_{t,L}$ spent on buying bonds today
- $(1-\omega) P_{t,1-1} b_{t-1,1} + \omega P_{t,L-1} b_{t-1,L}$: income from selling bonds bought in time period $t-1$.
- $\exp\left\{r_{t-1}^b\right\} b_{t-1}$: paying out returns and repaying deposits from time period $t-1$
- T_t : net lump-sum transfers from the household (in nominal terms)

In relation to the law of motion for n_t we note that in equilibrium, bonds $b_{t,k}$ are in zero net supply, i.e. $b_{t,k} = 0$. This implies that the law of motion for net worth reduces to

$$n_t = n_{t-1} + T_t,$$

and we therefore do not need to include this equation when solving the model. Note that this is exactly the same as in the standad New Keynesian model where households only invest in the central bank account at the rate r_t.

The behavior of the financial intermediary is solely determined by the deposit rate r_t^b. To state its expression, let the ex ante holding period return on the kth bond be

$$hr_{t,k} \equiv \mathbb{E}_t \left[\log P_{t+1,k-1} - \log P_{t,k}\right],$$

where $P_{t,k}$ is the nominal price in period t of a zero-coupon bond maturing in period $t+k$. The excess holding period return is therefore $xhr_{t,k} \equiv hr_{t,k} - r_t$. We then assume that the deposit rate is equal the ex ante holding period return on the invested bond portfolio, i.e.

$$\begin{aligned} r_t^b &\equiv (1-\omega) \times hr_{t,1} + \omega \times hr_{t,L} \\ &= r_t + \omega \times xhr_{t,L} \end{aligned}$$

because $xhr_{t,1} = 0$. Here, $\omega \in [0,1]$ denotes the fraction chosen by the financial intermediary invested in the long-term government bond. In other words, the financial intermediary is simply a mutual fund trading government bonds.

We only need to describe how the financial intermediary prices government bonds. Given that the financial intermediary is owned by the household, we simply use their stochastic discount factor. That is

$$P_{t,1} = \frac{1}{\exp\{r_t\}}$$

and

$$P_{t,k} = E_t \left[\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} P_{t+1,k-1}\right]$$

for $k > 1$. Note then that the yield curve is then given by

$$r_{t,k} = -\frac{1}{k} \log P_{t,k}$$

Note finally that we do not explicitely model a portfolio problem for the financial intermediary but simply assume that it buys these bonds. Although, the considered deposite rate and the policy rate are both risk-free, the deposite rate will on average be higher than the policy rate (because the yield curve is upward sloping) and should therefore be preferred by the households to simply investing in the central bank.

16.4 The central bank

We assume a standard Taylor rule of the form

$$\begin{aligned} \log\left(\frac{\exp\{r_t\}}{\exp\{r_{ss}\}}\right) &= \rho_r \log\left(\frac{\exp\{r_{t-1}\}}{\exp\{r_{ss}\}}\right) \\ &\quad + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right) + \beta_{xhr}\left(xhr_{t,L} - E\left[xhr_{t,L}\right]\right)\right) \end{aligned}$$

$\Updownarrow$

$$\begin{aligned} r_t &= r_{ss}(1-\rho_r) + \rho_r r_{t-1} + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right) \\ &\quad + (1-\rho_r)\beta_{xhr}\left(xhr_{t,L} - E\left[xhr_{t,L}\right]\right) \end{aligned}$$

where r_t is the net one-period risk-free rate. Note that we remove the mean in $xhr_{t,L}$ by substracting $E\left[xhr_{t,L}\right]$. When solving the model by the perturbation method, we approximate $E\left[xhr_{t,L}\right]$ by

$$E\left[xhr_{t,L}\right] \approx E_t\left[(1-\gamma)\sum_{l=0}^{\infty}\gamma^l xhr_{t+l,L}\right]$$

where $\gamma = 0.9999$. This is convenient because we have

$$\begin{aligned} X_{t,L} &\equiv E_t\left[(1-\gamma)\sum_{l=0}^{\infty}\gamma^l xhr_{t+l,L}\right] = (1-\gamma)\, xhr_{t,L} + E_t\left[(1-\gamma)\sum_{l=1}^{\infty}\gamma^l xhr_{t+l,L}\right] \\ &= (1-\gamma)\, xhr_{t,L} + \gamma E_t\left[X_{t+1,L}\right] \end{aligned}$$

as

$$X_{t+1,L} = E_{t+1}\left[(1-\gamma)\sum_{l=0}^{\infty}\gamma^l xhr_{t+1+l,L}\right] = E_{t+1}\left[(1-\gamma)\sum_{l=1}^{\infty}\gamma^{l-1} xhr_{t+l,L}\right]$$

$\Updownarrow$

$$\gamma E_{t+1}\left[X_{t+1,L}\right] = E_{t+1}\left[(1-\gamma)\sum_{l=1}^{\infty}\gamma^l xhr_{t+l,L}\right]$$

$\Downarrow$

$$\gamma E_t\left[E_{t+1}\left[X_{t+1,L}\right]\right] = E_t\left[E_{t+1}\left[(1-\gamma)\sum_{l=1}^{\infty}\gamma^l xhr_{t+l,L}\right]\right]$$

$\Updownarrow$

$$\gamma E_t\left[X_{t+1,L}\right] = E_t\left[(1-\gamma)\sum_{l=1}^{\infty}\gamma^l xhr_{t+l,L}\right]$$

16.5 Aggregation

16.5.1 The goods market: final good producer

From the final good producer we have

$y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$

$\Updownarrow$

$a_t k_{i,t}^{\theta} \left(z_t h_{i,t}\right)^{1-\theta} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$

We next notice that:

1. $\int_0^1 h_{i,t} di \equiv h_t$,
2. $\int_0^1 k_{i,t} di \equiv k_t$
3. $cons_t = k_t / h_t$

Doing the summation with respect to i we get

$\int_0^1 a_t k_{i,t}^{\theta} \left(z_t h_{i,t}\right)^{1-\theta} di = y_t \underbrace{\int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} di}_{s_{t+1}}$

$\Updownarrow$

$\int_0^1 a_t h_{i,t} \left(\frac{k_{i,t}}{h_{i,t}}\right)^{\theta} \left(z_t\right)^{1-\theta} di = y_t s_{t+1}$

$\Updownarrow$

$\int_0^1 a_t h_{i,t} \left(\text{constant}\right)^{\theta} \left(z_t\right)^{1-\theta} di = y_t s_{t+1}$

$\Updownarrow$

$a_t \left(\text{constant}\right)^{\theta} \left(z_t\right)^{1-\theta} \int_0^1 h_{i,t} di = y_t s_{t+1}$

$\Updownarrow$

$a_t \left(\frac{k_t}{h_t}\right)^{\theta} \left(z_t\right)^{1-\theta} h_t = y_t s_{t+1}$

$\Updownarrow$

$a_t \left(k_t\right)^{\theta} \left(z_t\right)^{1-\theta} h_t^{1-\theta} = y_t s_{t+1}$

$\Updownarrow$

$a_t \left(k_t\right)^{\theta} \left(z_t h_t\right)^{1-\theta} = y_t s_{t+1}$

and

$s_{t+1} \equiv \int_0^1 \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} di$

$$= \underbrace{\left(1-\alpha\right)\left(\frac{\tilde{P}_t}{P_t}\right)^{-\eta}}_{\substack{\text{opt. in period } t \\ \text{(0 indexation)}}} + \underbrace{\left(1-\alpha\right)\alpha\left(\frac{\tilde{P}_{t-1}\pi_{t-1}^{\chi}}{P_t}\right)^{-\eta}}_{\substack{\text{opt. in period } t-1 \\ \text{(1 indexation)}}} + \underbrace{\left(1-\alpha\right)\alpha^2\left(\frac{\tilde{P}_{t-2}\pi_{t-1}^{\chi}\pi_{t-2}^{\chi}}{P_t}\right)^{-\eta}}_{\substack{\text{opt. in period } t-2 \\ \text{(2 indexation)}}} + ...$$

$$= (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\left[\frac{\tilde{P}_{t-j}}{P_t}\prod_{s=1}^{j}\pi^{\chi}_{t-j-1+s}\right]^{-\eta}$$

$$= (1-\alpha)\,\tilde{p}_t^{-\eta} + P_t^{\eta}\sum_{j=1}^{\infty}\alpha^j\left[\tilde{P}_{t-j}\prod_{s=1}^{j}\pi^{\chi}_{t-j-1+s}\right]^{-\eta} \qquad \text{where } \tilde{p}_{t-j}\equiv\frac{\tilde{P}_{t-j}}{P_t}$$

$$= (1-\alpha)\,\tilde{p}_t^{-\eta} + P_t^{\eta}\left(s_{t-1}\alpha P_{t-1}^{-\eta}\left(\pi^{\chi}_{t-1}\right)^{-\eta}\right) \qquad \text{,see below}$$

$$= (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\left(\frac{P_t/P_{t-1}}{\pi^{\chi}_{t-1}}\right)^{\eta}s_t$$

$\Updownarrow$

$$s_{t+1} = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\left(\frac{\pi_t}{\pi^{\chi}_{t-1}}\right)^{\eta}s_t$$

We used that

$$s_t = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\left[\frac{\tilde{P}_{t-j-1}}{P_{t-1}}\prod_{s=1}^{j}\pi^{\chi}_{t-j-2+s}\right]^{-\eta}$$

$\Updownarrow$ change of index: $j+1=l$

$$s_t P_{t-1}^{-\eta} = (1-\alpha)\sum_{l=1}^{\infty}\alpha^{l-1}\left[\tilde{P}_{t-l}\prod_{s=1}^{l-1}\pi^{\chi}_{t-l-1+s}\right]^{-\eta}$$

$\Updownarrow$

$$s_t P_{t-1}^{-\eta}\alpha\left(\pi^{\chi}_{t-1}\right)^{-\eta} = (1-\alpha)\sum_{l=1}^{\infty}\alpha^{l}\left[\tilde{P}_{t-l}\prod_{s=1}^{l}\pi^{\chi}_{t-l-1+s}\right]^{-\eta}$$

So the ressource constraint in the goods market is:

1) $a_t k_t^{\theta}\left(z_t h_t\right)^{1-\theta} = y_t s_{t+1}$

2) $s_{t+1} = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\left(\frac{\pi_t}{\pi^{\chi}_{t-1}}\right)^{\eta}s_t$

16.6 The goods market: The relation between the optimale price and the price index

We start by noticing that the number of firms is by construct very large, so there is a fraction of $1-\alpha$ firms reoptimize their prices and the remaining fraction using the indexsation rule. This implies

$$P_t \equiv \left[\int_0^1 P_{i,t}^{1-\eta}di\right]^{\frac{1}{1-\eta}}$$

$\Updownarrow$

$$P_t^{1-\eta} = \int_0^1\left\{(1-\alpha)\,\tilde{P}_t^{1-\eta} + \alpha\left(P_{i,t-1}\pi^{\chi}_{t-1}\right)^{1-\eta}\right\}di$$

$$= (1-\alpha)\,\tilde{P}_t^{1-\eta} + \alpha\int_0^1\left(P_{i,t-1}\pi^{\chi}_{t-1}\right)^{1-\eta}di$$

$$= (1-\alpha)\,\tilde{P}_t^{1-\eta} + \alpha\left(\pi^{\chi}_{t-1}\right)^{1-\eta}\underbrace{\int_0^1 P_{i,t-1}^{1-\eta}di}_{P_{t-1}^{1-\eta}}$$

$\Updownarrow$

$$P_t^{1-\eta} = (1-\alpha)\,\tilde{P}_t^{1-\eta} + \alpha\left(P_{t-1}\pi^{\chi}_{t-1}\right)^{1-\eta}$$

$\Updownarrow$

$1 = (1-\alpha)\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\eta} + \alpha\left(\frac{P_{t-1}}{P_t}\pi_{t-1}^{\chi}\right)^{1-\eta}$

$\Updownarrow$

$1 = (1-\alpha)\,\tilde{p}_t^{1-\eta} + \alpha\left(\frac{\pi_{t-1}^{\chi}}{\pi_t}\right)^{1-\eta}$, since $\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}$

16.7 The resource constraint

Summing the dividend payments from firms we have

$$\begin{aligned} div_t &= \int div_{i,t} di \\ &= \int \left[\left(\frac{P_{i,t}}{P_t}\right) y_{i,t} - r_t^k k_{i,t} - w_t h_{i,t}\right] di \\ &= \int \left(\frac{P_{i,t}}{P_t}\right)^{1-\eta} y_t di - r_t^k \int k_{i,t} di - w_t \int h_{i,t} di \\ &= \frac{1}{P_t^{1-\eta}} y_t \int (P_{i,t})^{1-\eta} di - r_t^k k_t - w_t h_t \\ &= y_t - r_t^k k_t - w_t h_t \end{aligned}$$

because $y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\eta} y_t$ and $P_t = \left[\int_0^1 (P_{i,t})^{1-\eta} di\right]^{\frac{1}{1-\eta}}$. To get the dividends transfered to househould, we need to subtract real governement consumption equal $z_t^* G_t$. That is,

$$\begin{aligned} div_t^h &= div_t - z_t^* G_t \\ &= y_t - z_t^* G_t - r_t^k k_t - w_t h_t. \end{aligned}$$

Inserting this expression in the budget constraint for the households, we get

$$b_t + c_t + \frac{i_t}{\Upsilon_t} = \frac{b_{t-1} \exp\left\{r_{t-1}^b\right\}}{\pi_t} + h_t w_t + r_t^k k_t + div_t^h$$

$\Updownarrow$

$$b_t + c_t + \frac{i_t}{\Upsilon_t} = \frac{b_{t-1} \exp\left\{r_{t-1}^b\right\}}{\pi_t} + y_t - z_t^* G_t$$

Focusing on the case where deposites are zero in equalibrium, we have

$$c_t + \frac{i_t}{\Upsilon_t} + z_t^* G_t = y_t,$$

which constitutes our resource constraint for the economy. We finally assume that g_t is exogenous given and evolves according to

$$\log\left(\frac{G_{t+1}}{G_{ss}}\right) = \rho_G \log\left(\frac{G_t}{G_{ss}}\right) + \sigma_G \epsilon_{G,t+1}$$

with $\epsilon_{G,t+1} \sim \mathcal{NID}(0,1)$.

16.8 The periodic utility function of the representative household

We assume that

$$u\left(\frac{c_t-bc_{t-1}}{z_t^*},1-h_t\right)=\frac{d_t}{1-\phi_2}\left(\left(\frac{c_t-bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{1-\phi_2}-(z_t^*)^{(1-\phi_4)(1-\phi_2)}\right)+d_t\,(z_t^*)^{(1-\phi_4)(1-\phi_2)}\,\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}$$

$$u_c\left(\frac{c_t-bc_{t-1}}{z_t^*},1-h_t\right)=d_t\left[\left(\frac{c_t-bc_{t-1}}{(z_t^*)^{\phi_4}}\right)\right]^{-\phi_2}\frac{1}{(z_t^*)^{\phi_4}}$$

$$u_{1-h}\left(\frac{c_t-bc_{t-1}}{z_t^*},1-h_t\right)=d_t\,(z_t^*)^{(1-\phi_4)(1-\phi_2)}\,\phi_0\,(1-h_t)^{-\phi_1}$$

Note that we introduce a preference shock d_t, having the following law of motion

$$\log d_{t+1}=\rho_d\log d_t+\sigma_d\epsilon_{d,t+1}$$

with $\epsilon_{d,t+1}\sim\mathcal{NID}(0,1)$.

16.9 Summarizing

	The Households
1	$V_t = \frac{d_t}{1-\phi_2}\left(\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{1-\phi_2} - (z_t^*)^{(1-\phi_4)(1-\phi_2)}\right) + (z_t^*)^{(1-\phi_4)(1-\phi_2)} d_t \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1} - \beta\left(E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$
2	$\lambda_t = m_t d_t \left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{-\phi_2} \frac{1}{(z_t^*)^{\phi_4}} - 1_{[ha\_in]} b\beta E_t m_{t+1} d_{t+1} \left(\frac{c_{t+1} - bc_t}{\left(z_{t+1}^*\right)^{\phi_4}}\right)^{-\phi_2} \frac{1}{\left(z_{t+1}^*\right)^{\phi_4}}$
3	$q_t \lambda_t = E_t \beta \lambda_{t+1} [r_{t+1}^k + q_{t+1}(1-\delta) - q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 + q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{t+1}}{k_{t+1}}]$
4	$m_t (z_t^*)^{(1-\phi_4)(1-\phi_2)} d_t \phi_0 (1-h_t)^{-\phi_1} = \lambda_t w_t$
5	$1 = q_t \Upsilon_t \left(1 - \frac{\kappa_1}{2}\left(\frac{i_i}{\Upsilon_t z_t^* I_{SS}} - 1\right)^2 - \frac{i_t}{\Upsilon_t z_t^* i_{SS}}\kappa_1\left(\frac{i_i}{\Upsilon_t z_t^* I_{SS}} - 1\right) - \kappa_2\left(\frac{i_i}{k_t} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$
6	$\lambda_t = \beta \exp\left\{r_t^b\right\} E_t\left[\frac{\lambda_{t+1}}{\pi_{t+1}}\right]$
	The Firms
7	$mc_t a_t z_t (1-\theta)\left(\frac{z_t h_t}{k_t}\right)^{-\theta} = w_t$
8	$a_t mc_t \theta \left(\frac{z_t h_t}{k_t}\right)^{1-\theta} = r_t^k$
9	$\frac{(\eta-1)x_t^2}{\eta} = y_t mc_t \tilde{p}_t^{-\eta-1} + E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{\pi_t^\chi}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)x_{t+1}^2}{\eta}\right]$
10	$x_t^2 = y_t \tilde{p}_t^{-\eta} + E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{\pi_t^\chi}{\pi_{t+1}}\right)^{1-\eta} x_{t+1}^2\right]$
11	$1 = (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{\pi_{t-1}^\chi}{\pi_t}\right)^{1-\eta}$
	The Financial Intermediary
12	$r_t + \omega \times xhr_{t,L}$, where $xhr_{t,L} \equiv \mathbb{E}_t\left[\log\left(P_{t+1,L-1}/P_{t,L}\right)\right] - r_t$
13	$P_{t,1} = \frac{1}{\exp\{r_t\}}$
14	$P_{t,k} = E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}}P_{t+1,k-1}\right]$ for $k = 2, 3, ..., K$
	The Central Bank
15	$r_t = r_{ss}(1-\rho_r) + \rho_r r_{t-1} + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right)$
	$+ (1-\rho_r)\beta_{xhr}(xhr_{t,L} - X_{t,L})$
16	$X_{t,L} = (1-\gamma) xhr_{t,L} + \gamma\mathbb{E}_t\left[X_{t+1,L}\right]$
	Other relations
17	$a_t k_t^\theta (z_t h_t)^{1-\theta} = y_t s_{t+1}$
18	$s_{t+1} = (1-\alpha)\tilde{p}_t^{-\eta} + \alpha\left(\frac{\pi_t}{\pi_{t-1}^\chi}\right)^\eta s_t$
19	$k_{t+1} = (1-\delta)k_t + i_t - i_t\frac{\kappa_1}{2}\left(\frac{i_i}{\Upsilon_t z_t^* I_{SS}} - 1\right)^2 - \frac{\kappa_2}{2}\left(\frac{i_i}{k_t} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 k_t$
20	$y_t = c_t + \Upsilon_t^{-1} i_t + z_t^* G_t$
21	$z_t^* \equiv \Upsilon_t^{\frac{\theta}{1-\theta}} z_t$ and $\mu_{z^*,t} \equiv \mu_{\Upsilon,t}^{\theta/(1-\theta)}\mu_{z,t}$
	Exogenous processes
22	$\log\left(\mu_{z,t}\right) = \log\left(\mu_{z,ss}\right)$ and $z_{t+1} \equiv z_t \mu_{z,t+1}$ (i.e. a deterministic trend)
23	$\log\left(\mu_{\Upsilon,t}\right) = \log \mu_{\Upsilon,ss}$ and $\Upsilon_{t+1} \equiv \Upsilon_t \mu_{\Upsilon,t+1}$ (i.e. a deterministic trend)
24	$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$
25	$\log\left(\frac{G_{t+1}}{G_{ss}}\right) = \rho_G \log\left(\frac{G_t}{G_{ss}}\right) + \sigma_G \epsilon_{G,t+1}$
26	$\log d_{t+1} = \rho_d \log d_t + \sigma_d \epsilon_{d,t+1}$

16.10 A transformation of the DSGE model

Here we seek a transformation of the economy in such a way that the transformed economy is stationary in the sense that it only contains stationary variables.

We propose and verify the following transformation, where capital letters in general are used to denote the transformed variable:

$C_t \equiv \frac{c_t}{z_t^*}$

$R_t^k \equiv \Upsilon_t r_t^k$

$Q_t \equiv \Upsilon_t q_t$

$I_t \equiv \frac{i_t}{\Upsilon_t z_t^*} \quad \Longrightarrow \; i_t = I_t \Upsilon_t \left(\Upsilon_t^{\frac{\theta}{1-\theta}} z_t \right) = I_t \Upsilon_t^{\frac{1-\theta+\theta}{1-\theta}} z_t = I_t \Upsilon_t^{\frac{1}{1-\theta}} z_t$

So, $\mu_{i,t} = \mu_{\Upsilon,t} \mu_{z^*,t} = \mu_{\Upsilon,t}^{\frac{1}{1-\theta}} \mu_{z,t}$

$W_t \equiv \frac{w_t}{z_t^*}$

$Y_t \equiv \frac{y_t}{z_t^*} \quad \Longrightarrow \quad \mu_{y,ss} = \mu_{z^*,ss}$

$K_{t+1} \equiv \frac{k_{t+1}}{\Upsilon_t^{\frac{1}{1-\theta}} z_t} = \frac{k_{t+1}}{\Upsilon_t^{\frac{1}{1-\theta}} \Upsilon_t^{\frac{-\theta}{1-\theta}} z_t^*} = \frac{k_{t+1}}{\Upsilon_t z_t^*} \qquad \text{since } z_t^* \equiv \Upsilon_t^{\frac{\theta}{1-\theta}} z_t$

$\widetilde{V}_t \equiv \frac{V_t}{(z_t^*)^{(1-\phi_4)(1-\phi_2)}}$

$\Lambda_t \equiv \frac{\lambda_t}{m_t (z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}}$

$\Downarrow$

$\mu_{\lambda,t+1} \equiv \frac{\lambda_{t+1}}{\lambda_t}$

$\quad = \frac{\Lambda_{t+1} m_{t+1} \left(z_{t+1}^*\right)^{-\phi_2(1-\phi_4)-\phi_4}}{\Lambda_t m_t (z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}}$

$\Updownarrow$

$\mu_{\lambda,t+1} = \frac{\Lambda_{t+1}}{\Lambda_t} \mu_{z^*,t+1}^{-\phi_2(1-\phi_4)-\phi_4} \left(E_t \left[(-V_{t+1})^{1-\phi_3} \right] \right)^{\frac{\phi_3}{1-\phi_3}} (-V_{t+1})^{-\phi_3}$

$X_t^2 = \frac{x_t^2}{z_t^*}$

All remaining variables are stationary and do not need to be transformed. We now verify that given these variables we can transform our DSGE model from a non-stationary to a stationary economy.

EQ 1

$V_t = \frac{d_t}{1-\phi_2} \left(\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}} \right)^{1-\phi_2} - (z_t^*)^{(1-\phi_4)(1-\phi_2)} \right) + (z_t^*)^{(1-\phi_4)(1-\phi_2)} d_t \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1} - \beta \left(E_t \left[(-V_{t+1})^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}}$

$\Downarrow$

$$\frac{V_t}{(z_t^*)^{(1-\phi_4)(1-\phi_2)}} = \frac{d_t}{1-\phi_2} \left(\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4} (z_t^*)^{(1-\phi_4)}} \right)^{1-\phi_2} - 1 \right) + d_t \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1} - \beta \left(E_t \left[\left(- \frac{V_{t+1}}{(z_t^*)^{(1-\phi_4)(1-\phi_2)}} \frac{\left(z_{t+1}^*\right)^{(1-\phi_4)(1-\phi_2)}}{\left(z_{t+1}^*\right)^{(1-\phi_4)(1-\phi_2)}} \right)^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}}$$

$\Updownarrow$

$$\widetilde{V}_t = \frac{d_t}{1-\phi_2}\left(\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}(z_t^*)^{(1-\phi_4)}}\right)^{1-\phi_2} - 1\right) + d_t\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1} - \beta\left(E_t\left[\left(-\widetilde{V_{t+1}}\mu_{z^*,t+1}^{(1-\phi_4)(1-\phi_2)}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$
$\Updownarrow$

$$\widetilde{V}_t = \left[\frac{d_t}{1-\phi_2}\left(\left(\frac{c_t - bc_{t-1}}{z_t^*}\right)^{1-\phi_2} - 1\right) + d_t\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right] - \beta\left(E_t\left[\left(-\widetilde{V_{t+1}}\mu_{z^*,t+1}^{(1-\phi_4)(1-\phi_2)}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$
$\Updownarrow$

$$\widetilde{V}_t = \left[\frac{d_t}{1-\phi_2}\left(\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{1-\phi_2} - 1\right) + d_t\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right] - \beta\left(E_t\left[\left(-\widetilde{V_{t+1}}\mu_{z^*,t+1}^{(1-\phi_4)(1-\phi_2)}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

We note because z_t^* only has a deterministic trend, then we can simplify the above to:
$$\widetilde{V}_t = \left[\frac{d_t}{1-\phi_2}\left(\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{1-\phi_2} - 1\right) + d_t\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right] - \beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}\left(E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

EQ 2

$$\lambda_t = d_t m_t\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{-\phi_2}\frac{1}{(z_t^*)^{\phi_4}} - 1_{[ha\_in]}b\beta E_t\left[m_{t+1}d_{t+1}\left(\frac{c_{t+1} - bc_t}{\left(z_{t+1}^*\right)^{\phi_4}}\right)^{-\phi_2}\frac{1}{\left(z_{t+1}^*\right)^{\phi_4}}\right]$$
$\Updownarrow$

$$\frac{\lambda_t}{m_t(z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}} = d_t\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{-\phi_2}\frac{1}{(z_t^*)^{\phi_4}}\frac{1}{(z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}} - 1_{[ha\_in]}b\beta E_t\left[\frac{m_{t+1}}{m_t}d_{t+1}\left(\frac{c_{t+1} - bc_t}{\left(z_{t+1}^*\right)^{\phi_4}}\right)^{-\phi_2}\frac{1}{\left(z_{t+1}^*\right)^{\phi_4}}\frac{1}{(z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}}\right]$$
$\Updownarrow$

$$\Lambda_t = d_t\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}(z_t^*)^{(1-\phi_4)}}\right)^{-\phi_2}\frac{1}{(z_t^*)^{\phi_4}}\frac{1}{(z_t^*)^{-\phi_4}} - 1_{[ha\_in]}b\beta E_t[\frac{m_{t+1}}{m_t}d_{t+1}\left(\frac{c_{t+1} - bc_t}{\left(z_{t+1}^*\right)^{\phi_4}}\right)^{-\phi_2}\frac{1}{\left(z_{t+1}^*\right)^{\phi_4}}$$
$$\frac{\left(z_{t+1}^*\right)^{-\phi_2(1-\phi_4)-\phi_4}}{\left(z_{t+1}^*\right)^{-\phi_2(1-\phi_4)-\phi_4}}\frac{1}{(z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}}]$$
$\Updownarrow$

$$\Lambda_t = d_t\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]}b\beta E_t\left[\frac{m_{t+1}}{m_t}d_{t+1}\left(\frac{c_{t+1} - bc_t}{\left(z_{t+1}^*\right)^{\phi_4}}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\frac{1}{\left(z_{t+1}^*\right)^{-\phi_2(1-\phi_4)}}\right]$$
$\Updownarrow$

$$\Lambda_t = d_t\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]}b\beta E_t\left[\frac{m_{t+1}}{m_t}d_{t+1}\left(\frac{c_{t+1} - bc_t}{\left(z_{t+1}^*\right)^{\phi_4}\left(z_{t+1}^*\right)^{(1-\phi_4)}}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\right]$$
$\Updownarrow$

$$\Lambda_t = d_t\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]}b\beta E_t\left[\frac{m_{t+1}}{m_t}d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\right]$$

and using $m_{t+1} = m_t\left(E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{\phi_3}{1-\phi_3}}(-V_{t+1}(s))^{-\phi_3}$
$\Updownarrow$

$$\Lambda_t = d_t \left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]} b\beta E_t\{\left[\left(E_t\left[\left(-V_{t+1}\right)^{1-\phi_3}\right]\right)^{\frac{\phi_3}{1-\phi_3}}\left(-V_{t+1}(s)\right)^{-\phi_3}\right]$$
$$d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\}$$
$\Updownarrow$

$$\Lambda_t = d_t \left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]} b\beta E_t\{\left[\left(\frac{\left[E_t\left[\left(-V_{t+1}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-V_{t+1}(s)}\right)^{\phi_3}\right]$$
$$d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\}$$
$\Updownarrow$

$$\Lambda_t = d_t \left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]} b\beta E_t\{\left[\left(\frac{\left[E_t\left[\left(-V_{t+1}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-V_{t+1}(s)}\frac{\left(z_{t+1}^*\right)^{(1-\phi_4)(1-\phi_2)}}{\left(z_{t+1}^*\right)^{(1-\phi_4)(1-\phi_2)}}\right)^{\phi_3}\right]$$
$$d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\}$$
$\Updownarrow$

$$\Lambda_t = d_t \left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]} b\beta E_t\{\left[\left(\frac{\left[E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-\widetilde{V_{t+1}(s)}}\right)^{\phi_3}\right]$$
$$d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\}$$
because z_t^* is deterministic

EQ 3

$$q_t\lambda_t = E_t\beta\lambda_{t+1}[r_{t+1}^k + q_{t+1}(1-\delta)$$
$$-q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 + q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{t+1}}{k_{t+1}}]$$
$\Updownarrow$

$$\Upsilon_t q_t = E_t\beta\mu_{\lambda,t+1}[\Upsilon_t r_{t+1}^k + \Upsilon_t q_{t+1}(1-\delta)$$
$$-\Upsilon_t q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 + \Upsilon_t q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{t+1}}{k_{t+1}}]$$
$\Updownarrow$

$$Q_t = E_t\beta\mu_{\lambda,t+1}[\Upsilon_t\frac{\Upsilon_{t+1}}{\Upsilon_{t+1}}r_{t+1}^k + \Upsilon_t\frac{\Upsilon_{t+1}}{\Upsilon_{t+1}}q_{t+1}(1-\delta)$$
$$-\Upsilon_t\frac{\Upsilon_{t+1}}{\Upsilon_{t+1}}q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 + \Upsilon_t\frac{\Upsilon_{t+1}}{\Upsilon_{t+1}}q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{t+1}}{k_{t+1}}]$$
$\Updownarrow$

$$Q_t = E_t\beta\mu_{\lambda,t+1}[\frac{\Upsilon_t}{\Upsilon_{t+1}}R_{t+1}^k + \frac{\Upsilon_t}{\Upsilon_{t+1}}Q_{t+1}(1-\delta)$$
$$-\frac{\Upsilon_t}{\Upsilon_{t+1}}Q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 + \frac{\Upsilon_t}{\Upsilon_{t+1}}Q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{t+1}}{k_{t+1}}]$$
$\Updownarrow$

$$Q_t = E_t\beta\mu_{\lambda,t+1}[\mu_{\Upsilon,t+1}^{-1}R_{t+1}^k + \mu_{\Upsilon,t+1}^{-1}Q_{t+1}(1-\delta)$$

$$-\mu_{\Upsilon,t+1}^{-1}Q_{t+1}\frac{\kappa_2}{2}\left(\frac{i_{t+1}}{k_{t+1}}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2+\mu_{\Upsilon,t+1}^{-1}Q_{t+1}\kappa_2\left(\frac{i_{t+1}}{k_{t+1}}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{i_{t+1}}{k_{t+1}}]$$

Now note that

$$\frac{i_{t+1}}{k_{t+1}}=\frac{i_{t+1}}{k_{t+1}}\frac{\Upsilon_t z_t^*}{\Upsilon_t z_t^*}\frac{\Upsilon_{t+1}z_{t+1}^*}{\Upsilon_{t+1}z_{t+1}^*}=\frac{i_t/\left(\Upsilon_{t+1}z_{t+1}^*\right)}{k_{t+1}/(\Upsilon_t z_t^*)}\frac{1}{\Upsilon_t z_t^*}\frac{\Upsilon_{t+1}z_{t+1}^*}{1}=\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}$$

$\Updownarrow$

$$Q_t=E_t\frac{\beta\mu_{\lambda,t+1}}{\mu_{\Upsilon,t+1}}[R_{t+1}^k+Q_{t+1}(1-\delta)$$
$$-Q_{t+1}\frac{\kappa_2}{2}\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2+Q_{t+1}\kappa_2\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}]$$

EQ 4

$$\gamma_t\left(z_t^*\right)^{(1-\phi_4)(1-\phi_2)}d_t\phi_0\left(1-h_t\right)^{-\phi_1}=\lambda_t w_t$$

$\Updownarrow$

$$\left(z_t^*\right)^{(1-\phi_4)(1-\phi_2)}d_t\phi_0\left(1-h_t\right)^{-\phi_1}=\frac{\lambda_t}{\gamma_t(z_t^*)^{-1}}\frac{w_t}{z_t^*}$$

$\Updownarrow$

$$d_t\phi_0\left(1-h_t\right)^{-\phi_1}=\frac{\lambda_t}{(z_t^*)^{(1-\phi_4)(1-\phi_2)-1}\gamma_t}\frac{w_t}{z_t^*}$$

$\Updownarrow$

$$d_t\phi_0\left(1-h_t\right)^{-\phi_1}=\frac{\lambda_t}{(z_t^*)^{1-\phi_4-\phi_2+\phi_2\phi_4-1}\gamma_t}\frac{w_t}{z_t^*}$$

$\Updownarrow$

$$d_t\phi_0\left(1-h_t\right)^{-\phi_1}=\frac{\lambda_t}{(z_t^*)^{-\phi_2(1-\phi_4)-\phi_4}\gamma_t}\frac{w_t}{z_t^*}$$

$\Updownarrow$

$$d_t\phi_0\left(1-h_t\right)^{-\phi_1}=\Lambda_t W_t$$

EQ 5

$$1=q_t\Upsilon_t\left(1-\frac{\kappa_1}{2}\left(\frac{i_t}{\Upsilon_t z_t^* i_{SS}}-1\right)^2-\frac{i_t}{\Upsilon_t z_t^* i_{SS}}\kappa_1\left(\frac{i_t}{\Upsilon_t z_t^* i_{SS}}-1\right)-\kappa_2\left(\frac{i_t}{k_t}-\frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$$

$\Updownarrow$

$$1=Q_t\left(1-\frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}}-1\right)^2-\frac{I_t}{I_{SS}}\kappa_1\left(\frac{I_t}{I_{SS}}-1\right)-\kappa_2\left(\frac{i_t}{k_t}-\frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$$

since $I_t\equiv\frac{i_t}{\Upsilon_t z_t^*}$

$\Updownarrow$

$$1=Q_t\left(1-\frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}}-1\right)^2-\frac{I_t}{I_{SS}}\kappa_1\left(\frac{I_t}{I_{SS}}-1\right)-\kappa_2\left(\frac{i_t}{k_t}\frac{\Upsilon_t z_t^*}{\Upsilon_t z_t^*}\frac{\Upsilon_{t-1}z_{t-1}^*}{\Upsilon_{t-1}z_{t-1}^*}-\frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$$

$\Updownarrow$

$$1=Q_t\left(1-\frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}}-1\right)^2-\frac{I_t}{I_{SS}}\kappa_1\left(\frac{I_t}{I_{SS}}-1\right)-\kappa_2\left(\frac{I_t}{k_t}\frac{\Upsilon_t z_t^*}{1}\frac{1}{\Upsilon_{t-1}z_{t-1}^*}-\frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$$

recall $K_{t+1}\equiv\frac{k_{t+1}}{\Upsilon_t z_t^*}$

$\Updownarrow$

$$1 = Q_t \left(1 - \frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}} - 1\right)^2 - \frac{I_t}{I_{SS}}\kappa_1\left(\frac{I_t}{I_{SS}} - 1\right) - \kappa_2\left(\frac{I_t}{k_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$$

EQ 6

$$1 = E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{\exp\left\{r_t^b\right\}}{\pi_{t+1}}\right]$$

$\Updownarrow$

$$1 = E_t\left[\beta\mu_{\lambda\,t+1}\frac{\exp\left\{r_t^b\right\}}{\pi_{t+1}}\right]$$

EQ 7

$$mc_t\theta z_t^{1-\theta}a_t {k_t}^{\theta-1}{h_t}^{1-\theta} = r_t^k$$

$\Updownarrow$

$$mc_t a_t\theta z_t^{1-\theta}\Upsilon_t\left(h_t\right)^{1-\theta}k_t^{\theta-1} = \Upsilon_t r_t^k$$

$\Updownarrow$

$$mc_t a_t\theta\left(z_t\right)^{1-\theta}\Upsilon_t h_t^{1-\theta}\left(k_t\frac{\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}}{\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}}\right)^{\theta-1} = R_t^k$$

$\Updownarrow$

$$mc_t a_t\theta\left(z_t\right)^{1-\theta}\Upsilon_t h_t^{1-\theta}\left(K_t\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}\right)^{\theta-1} = R_t^k$$

$\Updownarrow$

$$mc_t a_t\theta\Upsilon_t\Upsilon_{t-1}^{-1}\left(z_t\right)^{1-\theta}z_{t-1}^{\theta-1}K_t^{\theta-1}h_t^{1-\theta} = R_t^k$$

$\Updownarrow$

$$mc_t a_t\theta\mu_{\Upsilon,t}\mu_{z,t}^{1-\theta}K_t^{\theta-1}h_t^{1-\theta} = R_t^k$$

EQ 8

$$mc_t\left(1-\theta\right)z_t^{1-\theta}a_t k_t^{\theta}h_t^{-\theta} = w_t$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)\frac{a_t z_t^{1-\theta}}{z_t^*}k_t^{\theta}h_t^{-\theta} = \frac{w_t}{z_t^*}$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)\frac{a_t z_t^{1-\theta}}{z_t^*}\left(k_t\frac{\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}}{\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}}\right)^{\theta}h_t^{-\theta} = W_t$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)\frac{a_t z_t^{1-\theta}}{z_t^*}\left(K_t\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}\right)^{\theta}h_t^{-\theta} = W_t$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)\frac{a_t z_t^{1-\theta}}{\Upsilon_t^{\frac{\theta}{1-\theta}}z_t}\left(K_t\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}\right)^{\theta}h_t^{-\theta}=W_t$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)\frac{a_t z_t^{-\theta}}{\Upsilon_t^{\frac{\theta}{1-\theta}}}\left(\Upsilon_{t-1}^{\frac{1}{1-\theta}}z_{t-1}\right)^{\theta}K_t^{\theta}h_t^{-\theta}=W_t$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)\frac{a_t \Upsilon_{t-1}^{\frac{\theta}{1-\theta}}}{\Upsilon_t^{\frac{\theta}{1-\theta}}}\left(\frac{z_{t-1}}{z_t}\right)^{\theta}K_t^{\theta}h_t^{-\theta}=W_t$$

$\Updownarrow$

$$mc_t\left(1-\theta\right)a_t\mu_{\Upsilon,t}^{\frac{-\theta}{1-\theta}}\mu_{z,t}^{-\theta}K_t^{\theta}h_t^{-\theta}=W_t$$

EQ 9

$$\frac{(\eta-1)x_t^2}{\eta}=y_t mc_t\tilde{p}_t^{-\eta-1}+E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)x_{t+1}^2}{\eta}\right]$$

$\Updownarrow$

$$\frac{(\eta-1)}{\eta}\frac{x_t^2}{z_t^*}=\frac{y_t}{z_t^*}mc_t\tilde{p}_t^{-\eta-1}+E_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)}{\eta}\frac{x_{t+1}^2}{z_t^*}\right]$$

$\Updownarrow$

$$\frac{(\eta-1)}{\eta}X_t^2=Y_t mc_t\tilde{p}_t^{-\eta-1}+E_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)}{\eta}X_{t+1}^2\mu_{z^*,t+1}\right]$$

EQ 10

$$x_t^2=y_t\tilde{p}_t^{-\eta}+E_t\left[\alpha\beta\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta}x_{t+1}^2\right]$$

$\Updownarrow$

$$\frac{x_t^2}{z_t^*}=\frac{y_t}{z_t^*}\tilde{p}_t^{-\eta}+E_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta}\frac{x_{t+1}^2}{z_t^*}\right]$$

$\Updownarrow$

$$X_t^2=Y_t\tilde{p}_t^{-\eta}+E_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta}X_{t+1}^2\mu_{z^*,t+1}\right]$$

EQ 11

$$1=\left(1-\alpha\right)\tilde{p}_t^{1-\eta}+\alpha\left(\frac{\pi_{t-1}^{\chi}}{\pi_t}\right)^{1-\eta}$$

EQ 12

$$r_t^b=r_t+\omega\times xhr_{t,L}$$

EQ 13

$P_{t,1} = \frac{1}{\exp\{r_t\}}$

EQ 14
$P_{t,k} = E_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\pi_{t+1}} P_{t+1,k-1}\right]$
$\Updownarrow$

$P_{t,k} = E_t\left[\beta \mu_{\lambda,t+1} \frac{1}{\pi_{t+1}} P_{t+1,k-1}\right]$

EQ 15
$r_t = r_{ss}\left(1-\rho_r\right) + \rho_r r_{t-1} + \left(1-\rho_r\right)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right)$
$+\left(1-\rho_r\right)\beta_{xhr}\left(xhr_{t,L} - X_{t,L}\right)$
$\Updownarrow$

$r_t = r_{ss}\left(1-\rho_r\right) + \rho_r r_{t-1} + \left(1-\rho_r\right)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{Y_t}{Y_{ss}}\right)\right)$
$+\left(1-\rho_r\right)\beta_{xhr}\left(xhr_{t,L} - X_{t,L}\right)$

EQ 16
$X_{t,L} = \left(1-\gamma\right) xhr_{t,L} + \gamma E_t\left[X_{t+1,L}\right]$

EQ 17
$a_t k_t^\theta \left(z_t h_t\right)^{1-\theta} = y_t s_{t+1}$
$\Updownarrow$

$a_t \frac{k_t^\theta}{z_t^*} \left(z_t h_t\right)^{1-\theta} = \frac{y_t}{z_t^*} s_{t+1}$
$\Updownarrow$

$a_t \frac{1}{z_t^*} \left(k_t \frac{\Upsilon_{t-1}^{\frac{1}{1-\theta}} z_{t-1}}{\Upsilon_{t-1}^{\frac{1}{1-\theta}} z_{t-1}}\right)^\theta \left(z_t h_t\right)^{1-\theta} = Y_t s_{t+1}$
$\Updownarrow$

$a_t \frac{1}{\Upsilon_t^{\frac{\theta}{1-\theta}} z_t} \left(K_t \Upsilon_{t-1}^{\frac{1}{1-\theta}} z_{t-1}\right)^\theta \left(z_t h_t\right)^{1-\theta} = Y_t s_{t+1}$
$\Updownarrow$

$a_t \left(K_t \frac{\Upsilon_{t-1}^{\frac{1}{1-\theta}}}{\Upsilon_t^{\frac{1}{1-\theta}}} z_{t-1}\right)^\theta z_t^{-\theta} h_t^{1-\theta} = Y_t s_{t+1}$
$\Updownarrow$

$a_t \left(K_t \frac{\Upsilon_{t-1}^{\frac{1}{1-\theta}}}{\Upsilon_t^{\frac{1}{1-\theta}}} \frac{z_{t-1}}{z_t}\right)^\theta h_t^{1-\theta} = Y_t s_{t+1}$
$\Updownarrow$

$a_t \left(K_t \mu_{\Upsilon,t}^{\frac{-1}{1-\theta}} \mu_{z,t}^{-1}\right)^\theta h_t^{1-\theta} = Y_t s_{t+1}$

EQ 18

$s_{t+1} = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\left(\frac{\pi_t}{\pi_{t-1}^{\chi}}\right)^{\eta} s_t$

EQ 19

$k_{t+1} = (1-\delta)\,k_t + i_t - i_t\frac{\kappa_1}{2}\left(\frac{i_i}{\Upsilon_t z_t^* I_{SS}} - 1\right)^2 - k_t\frac{\kappa_2}{2}\left(\frac{i_i}{k_t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$

$\Updownarrow$

$\frac{k_{t+1}}{\Upsilon_t z_t^*} = (1-\delta)\,\frac{k_t}{\Upsilon_t z_t^*} + \frac{i_t}{\Upsilon_t z_t^*} - \frac{i_t}{\Upsilon_t z_t^*}\frac{\kappa_1}{2}\left(\frac{i_i}{\Upsilon_t z_t^* I_{SS}} - 1\right)^2 - \frac{k_t}{\Upsilon_t z_t^*}\frac{\kappa_2}{2}\left(\frac{i_i}{k_t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$

$\Updownarrow$

$K_{t+1} = (1-\delta)\,\frac{k_t}{\Upsilon_t z_t^*}\frac{\Upsilon_{t-1}z_{t-1}^*}{\Upsilon_{t-1}z_{t-1}^*} + I_t - I_t\frac{\kappa_1}{2}\left(\frac{I_i}{I_{SS}} - 1\right)^2 - \frac{k_t}{\Upsilon_t z_t^*}\frac{\Upsilon_{t-1}z_{t-1}^*}{\Upsilon_{t-1}z_{t-1}^*}\frac{\kappa_2}{2}\left(\frac{i_i}{k_t}\frac{\Upsilon_t z_t^*}{\Upsilon_t z_t^*}\frac{\Upsilon_{t-1}z_{t-1}^*}{\Upsilon_{t-1}z_{t-1}^*} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$

because $I_t = \frac{i_t}{\Upsilon_t z_t^*}$ and $K_{t+1} = \frac{k_{t+1}}{\Upsilon_t z_t^*}$

$\Updownarrow$

$K_{t+1} = (1-\delta)\,\frac{K_t}{\Upsilon_t z_t^*}\frac{\Upsilon_{t-1}z_{t-1}^*}{1} + I_t - I_t\frac{\kappa_1}{2}\left(\frac{I_i}{I_{SS}} - 1\right)^2 - \frac{K_t}{\Upsilon_t z_t^*}\frac{\Upsilon_{t-1}z_{t-1}^*}{1}\frac{\kappa_2}{2}\left(\frac{I_i}{K_t}\frac{\Upsilon_t z_t^*}{1}\frac{1}{\Upsilon_{t-1}z_{t-1}^*} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$

$\Updownarrow$

$K_{t+1} = (1-\delta)\,K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1} + I_t - I_t\frac{\kappa_1}{2}\left(\frac{I_i}{I_{SS}} - 1\right)^2 - K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1}\frac{\kappa_2}{2}\left(\frac{I_i}{K_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$

EQ 20

$y_t = c_t + \Upsilon_t^{-1}i_t + z_t^* G_t$

$\Updownarrow$

$\frac{y_t}{z_t^*} = \frac{\left(c_t + \Upsilon_t^{-1}i_t\right)}{z_t^*} + G_t$

$\Updownarrow$

$Y_t = C_t + I_t + G_t$

	The Households
1	$\widetilde{V}_t = \left[\frac{d_t}{1-\phi_2}\left(\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{1-\phi_2} - 1\right) + d_t\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right] - \beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}\left(E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$
2	$\Lambda_t = d_t\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]}b\beta E_t\{\left[\left(\frac{\left[E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-\widetilde{V_{t+1}}(s)}\right)^{\phi_3}\right]$
	$\times d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\}$
3	$Q_t = E_t\frac{\beta\mu_{\lambda,t+1}}{\mu_{\Upsilon,t+1}}[R_{t+1}^k + Q_{t+1}(1-\delta) - Q_{t+1}\frac{\kappa_2}{2}\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$
	$+Q_{t+1}\kappa_2\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}]$
4	$d_t\phi_0(1-h_t)^{-\phi_1} = \Lambda_t W_t$
5	$1 = Q_t\left(1 - \frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}} - 1\right)^2 - \frac{I_t}{I_{SS}}\kappa_1\left(\frac{I_t}{I_{SS}} - 1\right) - \kappa_2\left(\frac{I_t}{k_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$
6	$1 = E_t\left[\beta\mu_{\lambda\,t+1}\frac{\exp\{r_t^b\}}{\pi_{t+1}}\right]$
	The Firms
7	$mc_t a_t\theta\mu_{\Upsilon,t}\mu_{z,t}^{1-\theta}K_t^{\theta-1}h_t^{1-\theta} = R_t^k$
8	$mc_t(1-\theta)a_t\mu_{\Upsilon,t}^{\frac{-\theta}{1-\theta}}\mu_{z,t}^{-\theta}K_t^{\theta}h_t^{-\theta} = W_t$
9	$\frac{(\eta-1)}{\eta}X_t^2 = Y_t mc_t\tilde{p}_t^{-\eta-1} + E_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)}{\eta}X_{t+1}^2\mu_{z^*,t+1}\right]$
10	$X_t^2 = Y_t\tilde{p}_t^{-\eta} + E_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{\pi_t^{\chi}}{\pi_{t+1}}\right)^{1-\eta}X_{t+1}^2\mu_{z^*,t+1}\right]$
11	$1 = (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{\pi_{t-1}^{\chi}}{\pi_t}\right)^{1-\eta}$
	The Financial Intermediary
12	$r_t^b = r_t + \omega\times xhr_{t,L}$, where $xhr_{t,L}\equiv\mathbb{E}_t\left[\log\left(P_{t+1,L-1}/P_{t,L}\right)\right] - r_t$
13	$P_{t,1} = \frac{1}{\exp\{r_t\}}$
14	$P_{t,k} = E_t\left[\beta\mu_{\lambda,t+1}\frac{1}{\pi_{t+1}}P_{t+1,k-1}\right]$ for $k = 2, 3, ..., K$
	The Central Bank
15	$r_t = r_{ss}(1-\rho_r) + \rho_r r_{t-1} + (1-\rho_r)\left(\beta_\pi\log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y\log\left(\frac{Y_t}{Y_{ss}}\right)\right)$
	$+(1-\rho_r)\beta_{xhr}(xhr_{t,L} - X_{t,L})$
16	$X_{t,L} = (1-\gamma)xhr_{t,L} + \gamma E_t[X_{t+1,L}]$
	Other relations
17	$a_t\left(K_t\mu_{\Upsilon,t}^{\frac{-1}{1-\theta}}\mu_{z,t}^{-1}\right)^{\theta}h_t^{1-\theta} = Y_t s_{t+1}$
18	$s_{t+1} = (1-\alpha)\tilde{p}_t^{-\eta} + \alpha\left(\frac{\pi_t}{\pi_{t-1}^{\chi}}\right)^{\eta}s_t$
19	$K_{t+1} = (1-\delta)K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1} + I_t - I_t\frac{\kappa_1}{2}\left(\frac{I_i}{I_{SS}} - 1\right)^2 - K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1}\frac{\kappa_2}{2}\left(\frac{I_i}{K_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$
20	$Y_t = C_t + I_t + G_t$
21	$\mu_{z^*,t}\equiv\mu_{\Upsilon,t}^{\theta/(1-\theta)}\mu_{z,t}$
	Exogenous processes
22	$\log\left(\mu_{z,t}\right) = \log\left(\mu_{z,ss}\right)$ and $z_{t+1}\equiv z_t\mu_{z,t+1}$ (i.e. a deterministic trend)
23	$\log\left(\mu_{\Upsilon,t}\right) = \log\mu_{\Upsilon,ss}$ and $\Upsilon_{t+1}\equiv\Upsilon_t\mu_{\Upsilon,t+1}$ (i.e. a deterministic trend)
24	$\log a_{t+1} = \rho_a\log a_t + \sigma_a\epsilon_{a,t+1}$
25	$\log\left(\frac{G_{t+1}}{G_{ss}}\right) = \rho_G\log\left(\frac{G_t}{G_{ss}}\right) + \sigma_G\epsilon_{G,t+1}$
26	$\log d_{t+1} = \rho_d\log d_t + \sigma_d\epsilon_{d,t+1}$

In dealing with the term $\left(E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$ we follow the procedure suggested by Rudebusch & Swanson (2012).

That is we define

$$EV_t = E_t\left[\left(\frac{-V_{t+1}}{AA}\right)^{1-\phi_3}\right] \quad \text{where } EV_{ss} = \left(\frac{-V_{ss}}{AA}\right)^{1-\phi_3} > 0$$

$$PEV_t = AA \times EV_t^{\frac{1}{1-\phi_3}}$$

$$= AA \times \left(E_t\left[\left(\frac{-V_{t+1}}{AA}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

$$= AA \times \left(AA^{-(1-\phi_3)} E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

$$= AA \times AA^{-1}\left(E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

$$= \left(E_t\left[(-V_{t+1})^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

where AA is a scaling constant

16.11 Market completness

This subsection shows that our model implies market completness although it is only the financial intermediary that trades bonds. We first note that the deposit rate r_t^b only enters in the following tree equations within our model:

$$1 = E_t\left[\beta\mu_{\lambda\, t+1}\frac{\exp\{r_t^b\}}{\pi_{t+1}}\right]$$

$$r_t^b = r_t + \omega \times xhr_{t,L}$$

$$r_t = (1-\rho_r)\, r_{ss} + \rho_r r_{t-1} + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right) + (1-\rho_r)\,\beta_{xhr}\,(xhr_{t,L} - X_{t,L})$$

Next, note that $r_t = r_t^b - \omega \times xhr_{t,L}$ and substituted into the Taylor rule implies

$$r_t^b - \omega \times xhr_{t,L} = (1-\rho_r)\, r_{ss} + \rho_r\left(r_{t-1}^b - \omega \times xhr_{t-1,L}\right)$$
$$+ (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right) + (1-\rho_r)\,\beta_{xhr}\,(xhr_{t,L} - X_{t,L})$$

$$\Updownarrow$$

$$r_t^b = (1-\rho_r)\, r_{ss} + \rho_r r_{t-1}^b + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right)$$
$$+\omega \times xhr_{t,L} - \rho_r\omega \times xhr_{t-1,L} + (1-\rho_r)\,\beta_{xhr}\,(xhr_{t,L} - X_{t,L})$$

Hence, our model is equivalent to a standard New Keynesian model with market completness but with a Taylor rule for r_t^b that depends on fast and current values of excess holding period return on the long bond.

16.12 The intertemporal elasticity of substitution (IES)

We want to compute expression for the intertemporal elasticity of substitution (IES) under perfect foresight and then evaluate it at the deterministic steady state. We start by considering the case of external habit formation before considering the case of internal habit formation.

16.12.1 External habit formation

We consider the specification

$$U=\sum_{l=0}^{\infty}\beta^{l}u\left(c_{t+l}-H_{t+l}\right)$$

where we omit leisure (because it does not affect the IES) and denote the external habit level by H_t. We start by computing the intertemporal marginal rate of substitution (IMRS) which equals in this case

$$dU=u_c\left(c_t-H_t\right)dc_t+\beta u_c\left(c_{t+1}-H_{t+1}\right)dc_{t+1}=0$$

$\Updownarrow$

$$\beta u_c\left(c_{t+1}-H_{t+1}\right)dc_{t+1}=-u_c\left(c_t-H_t\right)dc_t$$

$\Updownarrow$

$$IMRS_t:\frac{dc_{t+1}}{dc_t}=-\frac{1}{\beta}\frac{u_c\left(c_t-H_t\right)}{u_c\left(c_{t+1}-H_{t+1}\right)}.$$

The IES is then defined as

$$\begin{aligned}IES_t &= \frac{d\left(\frac{c_{t+1}}{c_t}\right)/\left(\frac{c_{t+1}}{c_t}\right)}{dIMRS_t/IMRS_t}\\ &= \frac{IMRS_t/\left(\frac{c_{t+1}}{c_t}\right)}{dIMRS_t/d\left(\frac{c_{t+1}}{c_t}\right)}\end{aligned}$$

To compute the denominator, we then note that

$$\begin{aligned}IMRS_t &= -\frac{1}{\beta}\frac{u_c\left(c_t-H_t\right)}{u_c\left(c_{t+1}-H_{t+1}\right)}\\ &= -\frac{1}{\beta}\frac{u_c\left(\frac{c_t}{c_t}-\frac{H_t}{c_t}\right)}{u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)}\\ &= -\frac{1}{\beta}u_c\left(1-\frac{H_t}{c_t}\right)\left[u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)\right]^{-1}\end{aligned}$$

provided u_c is homogenous of some order (as assumed for the our considered functional form). Then

$$\frac{dIMRS_t}{d\left(\frac{c_{t+1}}{c_t}\right)}=\frac{1}{\beta}u_c\left(1-\frac{H_t}{c_t}\right)\left[u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)\right]^{-2}u_{cc}\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)$$

Thus, we have

$$\begin{aligned}IES_t &= \frac{-\frac{1}{\beta}u_c\left(1-\frac{H_t}{c_t}\right)\left[u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)\right]^{-1}/\left(\frac{c_{t+1}}{c_t}\right)}{\frac{1}{\beta}u_c\left(1-\frac{H_t}{c_t}\right)\left[u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)\right]^{-2}u_{cc}\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)}\\ &= -\frac{u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)/\left(\frac{c_{t+1}}{c_t}\right)}{u_{cc}\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)}\\ &= -\frac{u_c\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)}{u_{cc}\left(\frac{c_{t+1}}{c_t}-\frac{H_{t+1}}{c_t}\right)}\frac{1}{\frac{c_{t+1}}{c_t}}.\end{aligned}$$

The considered functional form for u_t in our case is $u_t = \frac{1}{1-\phi_2}\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{1-\phi_2} = \frac{(z_t^*)^{(\phi_2-1)\phi_4}}{1-\phi_2}(c_t - bc_{t-1})^{1-\phi_2}$, implying

$$u_c(c_t - bc_{t-1}) = \left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{-\phi_2}\frac{1}{(z_t^*)^{\phi_4}}$$

$$u_{cc}(c_t - bc_{t-1}) = -\phi_2\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{-\phi_2-1}\frac{1}{(z_t^*)^{2\phi_4}}$$

Thus we get

$$\begin{aligned}
IES_t &= -\frac{\left(\frac{c_{t+1}-bc_t}{(z_{t+1}^*)^{\phi_4}c_t}\right)^{-\phi_2}\frac{1}{(z_{t+1}^*)^{\phi_4}}}{\left(-\phi_2\left(\frac{c_{t+1}-bc_t}{(z_{t+1}^*)^{\phi_4}c_t}\right)^{-\phi_2-1}\frac{1}{(z_{t+1}^*)^{2\phi_4}}\right)}\frac{1}{\frac{c_{t+1}}{c_t}} \\
&= \frac{\left(\frac{c_{t+1}-bc_t}{(z_{t+1}^*)^{\phi_4}c_t}\right)}{\phi_2\frac{1}{(z_{t+1}^*)^{\phi_4}}}\frac{1}{\frac{c_{t+1}}{c_t}} \\
&= \frac{1}{\phi_2}\left(\frac{c_{t+1}-bc_t}{c_t}\right)\frac{c_t}{c_{t+1}} \\
&= \frac{1}{\phi_2}\left(\frac{c_{t+1}-bc_t}{c_{t+1}}\right) \\
&= \frac{1}{\phi_2}\left(1 - b\frac{c_t}{c_{t+1}}\frac{z_t^*}{z_t^*}\frac{z_{t+1}^*}{z_{t+1}^*}\right) \\
&= \frac{1}{\phi_2}\left(1 - b\frac{C_t}{C_{t+1}}\frac{z_t^*}{1}\frac{1}{z_{t+1}^*}\right) \\
&= \frac{1}{\phi_2}\left(1 - b\frac{C_t}{C_{t+1}}\mu_{z^*,t}^{-1}\right)
\end{aligned}$$

So in the steady state

$$IES_{ss} = \frac{1}{\phi_2}\left(1 - \frac{b}{\mu_{z^*,ss}}\right).$$

Note that this expression corresponds to the one obtained by using the standard formula $IES_t = -\frac{u_c(t)}{c_t u_{cc}(t)}$ evaluated at the steady state.

16.12.2 Internal habit formation

We consider the general specification

$$U = \sum_{l=0}^{\infty}\beta^l u(c_{t+l} - bc_{t-1+l})$$

Note also that we omit leisure (because it does not affect the IES). We start by computing the intertemporal marginal rate of substitution (IMRS) which equals

$$[u_c(c_t - bc_{t-1}) - \beta b u_c(c_{t+1} - bc_t)]\,dc_t + \left[\beta u_c(c_{t+1} - bc_t) - \beta^2 b u_c(c_{t+2} - bc_{t+1})\right]dc_{t+1} = 0$$

$\Updownarrow$

$$\beta[u_c(c_{t+1} - bc_t) - \beta b u_c(c_{t+2} - bc_{t+1})]\,dc_{t+1} = -[u_c(c_t - bc_{t-1}) - \beta b u_c(c_{t+1} - bc_t)]\,dc_t$$

$\Updownarrow$

$$IMRS_t : \frac{dc_{t+1}}{dc_t} = -\frac{1}{\beta}\frac{u_c\left(c_t - bc_{t-1}\right) - \beta b u_c\left(c_{t+1} - bc_t\right)}{u_c\left(c_{t+1} - bc_t\right) - \beta b u_c\left(c_{t+2} - bc_{t+1}\right)}.$$

The IES is then defined as

$$\begin{aligned} IES_t &= \frac{d\left(\frac{c_{t+1}}{c_t}\right)/\left(\frac{c_{t+1}}{c_t}\right)}{dIMRS_t/IMRS_t} \\ &= \frac{IMRS_t/\left(\frac{c_{t+1}}{c_t}\right)}{dIMRS_t/d\left(\frac{c_{t+1}}{c_t}\right)} \end{aligned}$$

To compute the denominator, we then note that

$$\begin{aligned} IMRS_t &= -\frac{1}{\beta}\frac{u_c\left(c_t - bc_{t-1}\right) - \beta b u_c\left(c_{t+1} - bc_t\right)}{u_c\left(c_{t+1} - bc_t\right) - \beta b u_c\left(c_{t+2} - bc_{t+1}\right)} \\ &= -\frac{1}{\beta}\frac{u_c\left(1 - b\frac{c_{t-1}}{c_t}\right) - \beta b u_c\left(\frac{c_{t+1}}{c_t} - b\right)}{u_c\left(\frac{c_{t+1}}{c_t} - b\right) - \beta b u_c\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)} \\ &= -\frac{1}{\beta}\left[u_c\left(1 - b\frac{c_{t-1}}{c_t}\right) - \beta b u_c\left(\frac{c_{t+1}}{c_t} - b\right)\right] \\ &\quad \times\left[u_c\left(\frac{c_{t+1}}{c_t} - b\right) - \beta b u_c\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)\right]^{-1}. \end{aligned}$$

provided u_c is homogenous of some order (as assumed for the our considered functional form). Then

$$\begin{aligned} \frac{dIMRS_t}{d\left(\frac{c_{t+1}}{c_t}\right)} &= bu_{cc}\left(\frac{c_{t+1}}{c_t} - b\right)\left[u_c\left(\frac{c_{t+1}}{c_t} - b\right) - \beta b u_c\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)\right]^{-1} \\ &\quad +\frac{1}{\beta}\left[u_c\left(1 - b\frac{c_{t-1}}{c_t}\right) - \beta b u_c\left(\frac{c_{t+1}}{c_t} - b\right)\right] \\ &\quad \times\left[u_c\left(\frac{c_{t+1}}{c_t} - b\right) - \beta b u_c\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)\right]^{-2} \\ &\quad \times\left[u_{cc}\left(\frac{c_{t+1}}{c_t} - b\right) + \beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)\right] \\ &= bu_{cc}\left(\frac{c_{t+1}}{c_t} - b\right)\left(-\beta\left[u_c\left(1 - b\frac{c_{t-1}}{c_t}\right) - \beta b u_c\left(\frac{c_{t+1}}{c_t} - b\right)\right]^{-1} IMRS_t\right) \\ &\quad -IMRS_t\left[u_c\left(\frac{c_{t+1}}{c_t} - b\right) - \beta b u_c\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)\right]^{-1} \\ &\quad \times\left[u_{cc}\left(\frac{c_{t+1}}{c_t} - b\right) + \beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)\right] \\ &= -IMRS_t\left\{\frac{b\beta u_{cc}\left(\frac{c_{t+1}}{c_t} - b\right)}{u_c\left(1 - b\frac{c_{t-1}}{c_t}\right) - \beta b u_c\left(\frac{c_{t+1}}{c_t} - b\right)} + \frac{u_{cc}\left(\frac{c_{t+1}}{c_t} - b\right) + \beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)}{u_c\left(\frac{c_{t+1}}{c_t} - b\right) - \beta b u_c\left(\frac{c_{t+2}}{c_t} - b\frac{c_{t+1}}{c_t}\right)}\right\} \end{aligned}$$

Thus, we get
$IES_t = \frac{IMRS_t/\left(\frac{c_{t+1}}{c_t}\right)}{dIMRS_t/d\left(\frac{c_{t+1}}{c_t}\right)}$

$$= \frac{IMRS_t/\left(\frac{c_{t+1}}{c_t}\right)}{-IMRS_t\left\{\frac{b\beta u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)}{u_c\left(1-b\frac{c_{t-1}}{c_t}\right)-\beta b u_c\left(\frac{c_{t+1}}{c_t}-b\right)}+\frac{u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)+\beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}{u_c\left(\frac{c_{t+1}}{c_t}-b\right)-\beta b u_c\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}\right\}}$$

$$= \frac{-1/\left(\frac{c_{t+1}}{c_t}\right)}{\frac{b\beta u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)}{u_c\left(1-b\frac{c_{t-1}}{c_t}\right)-\beta b u_c\left(\frac{c_{t+1}}{c_t}-b\right)}+\frac{u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)+\beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}{u_c\left(\frac{c_{t+1}}{c_t}-b\right)-\beta b u_c\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}}$$

$$= -\left[\frac{b\beta u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)}{u_c\left(1-b\frac{c_{t-1}}{c_t}\right)-\beta b u_c\left(\frac{c_{t+1}}{c_t}-b\right)}+\frac{u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)+\beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}{u_c\left(\frac{c_{t+1}}{c_t}-b\right)-\beta b u_c\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}\right]^{-1}\left[\frac{c_{t+1}}{c_t}\right]^{-1}$$

$$= -\left[\left(\frac{b\beta u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)}{u_c\left(1-b\frac{c_{t-1}}{c_t}\right)-\beta b u_c\left(\frac{c_{t+1}}{c_t}-b\right)}+\frac{u_{cc}\left(\frac{c_{t+1}}{c_t}-b\right)+\beta b^2 u_{cc}\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}{u_c\left(\frac{c_{t+1}}{c_t}-b\right)-\beta b u_c\left(\frac{c_{t+2}}{c_t}-b\frac{c_{t+1}}{c_t}\right)}\right)\frac{c_{t+1}}{c_t}\right]^{-1}$$

We next express these terms for the transformed economy, i.e.

$\frac{c_{t-1}}{c_t} = \frac{C_{t-1}z^*_{t-1}}{C_t z^*_t} = \frac{C_{t-1}}{C_t}\mu^{-1}_{z^*,t}$

$\frac{c_{t+1}}{c_t} = \frac{C_{t+1}z^*_{t+1}}{C_t z^*_t} = \frac{C_{t+1}}{C_t}\mu_{z^*,t+1}$

$\frac{c_{t+2}}{c_t} = \frac{C_{t+2}z^*_{t+2}}{C_t z^*_t}\frac{z^*_{t+1}}{z^*_{t+1}} = \frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}$

Thus,

$$IES_t = -\left[\left(\frac{b\beta u_{cc}\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)}{u_c\left(1-b\frac{C_{t-1}}{C_t}\mu^{-1}_{z^*,t}\right)-\beta b u_c\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)}+\frac{u_{cc}\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)+\beta b^2 u_{cc}\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)}{u_c\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)-\beta b u_c\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)}\right)\mu_{z^*,t+1}\right]^{-1}.$$

The considered functional form for u_t in our case is $u_t = \frac{1}{1-\phi_2}\left(\frac{c_t-bc_{t-1}}{(z^*_t)^{\phi_4}}\right)^{1-\phi_2} = \frac{(z^*_t)^{(\phi_2-1)\phi_4}}{1-\phi_2}\left(c_t-bc_{t-1}\right)^{1-\phi_2}$, implying

$$u_c\left(c_t-bc_{t-1}\right) = \left(c_t-bc_{t-1}\right)^{-\phi_2}\left(z^*_t\right)^{(\phi_2-1)\phi_4}$$

$$u_{cc}\left(c_t-bc_{t-1}\right) = -\phi_2\left(c_t-bc_{t-1}\right)^{-\phi_2-1}\left(z^*_t\right)^{(\phi_2-1)\phi_4}$$

Hence,

$$IES_t = -\left[\left(\frac{b\beta u_{cc}\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)}{u_c\left(1-b\frac{C_{t-1}}{C_t}\mu^{-1}_{z^*,t}\right)-\beta b u_c\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)}+\frac{u_{cc}\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)+\beta b^2 u_{cc}\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)}{u_c\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)-\beta b u_c\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)}\right)\mu_{z^*,t+1}\right]^{-1}$$

$$= -\left[\left(\begin{array}{c}\frac{-\phi_2 b\beta\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2-1}\left(z^*_{t+1}\right)^{(\phi_2-1)\phi_4}}{\left(1-b\frac{C_{t-1}}{C_t}\mu^{-1}_{z^*,t}\right)^{-\phi_2}\left(z^*_t\right)^{(\phi_2-1)\phi_4}-\beta b\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2}\left(z^*_{t+1}\right)^{(\phi_2-1)\phi_4}} \\ +\frac{-\phi_2\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2-1}\left(z^*_{t+1}\right)^{(\phi_2-1)\phi_4}-\phi_2\beta b^2\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)^{-\phi_2-1}\left(z^*_{t+2}\right)^{(\phi_2-1)\phi_4}}{\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2}\left(z^*_{t+1}\right)^{(\phi_2-1)\phi_4}-\beta b\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)^{-\phi_2}\left(z^*_{t+2}\right)^{(\phi_2-1)\phi_4}}\end{array}\right)\mu_{z^*,t+1}\right]^{-1}$$

$$= -\left[\left(\begin{array}{c}\frac{-\phi_2 b\beta\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2-1}}{\left(1-b\frac{C_{t-1}}{C_t}\mu^{-1}_{z^*,t}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-(\phi_2-1)\phi_4}-\beta b\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2}} \\ +\frac{-\phi_2\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2-1}-\phi_2\beta b^2\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)^{-\phi_2-1}\left(\mu_{z^*,t+2}\right)^{(\phi_2-1)\phi_4}}{\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2}-\beta b\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)^{-\phi_2}\left(\mu_{z^*,t+2}\right)^{(\phi_2-1)\phi_4}}\end{array}\right)\mu_{z^*,t+1}\right]^{-1}$$

$$= -\left[\left(\begin{array}{c}\frac{-\phi_2 b\beta\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2-1}}{\left(1-b\frac{C_{t-1}}{C_t}\mu^{-1}_{z^*,t}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-(\phi_2-1)\phi_4}-\beta b\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2}} \\ +\frac{-\phi_2\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2-1}-\phi_2\beta b^2\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)^{-\phi_2-1}\left(\mu_{z^*,t+2}\right)^{(\phi_2-1)\phi_4}}{\left(\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}-b\right)^{-\phi_2}-\beta b\left(\frac{C_{t+2}}{C_t}\mu_{z^*,t+2}\mu_{z^*,t+1}-b\frac{C_{t+1}}{C_t}\mu_{z^*,t+1}\right)^{-\phi_2}\left(\mu_{z^*,t+2}\right)^{(\phi_2-1)\phi_4}}\end{array}\right)\mu_{z^*,t+1}\right]^{-1}$$

Evaluated in the steady state, we get

$$IES_{ss} = -\left[\left(\frac{-\phi_2 b\beta\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}}{\left(1-b\mu_{z^*,ss}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4}-\beta b\left(\mu_{z^*,ss}-b\right)^{-\phi_2}} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}-\phi_2\beta b^2\left(\mu_{z^*,ss}^2-b\mu_{z^*,ss}\right)^{-\phi_2-1}\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4}}{\left(\mu_{z^*,ss}-b\right)^{-\phi_2}-\beta b\left(\mu_{z^*,ss}^2-b\mu_{z^*,ss}\right)^{-\phi_2}\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4}}\right)\mu_{z^*,ss}\right]^{-1}$$

$$= -\left[\left(\frac{-\phi_2 b\beta\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2-1}}{\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4}-\mu_{z^*,ss}^{-\phi_2}\beta b\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}-\phi_2\beta b^2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}}{\left(\mu_{z^*,ss}-b\right)^{-\phi_2}-\beta b\left(\mu_{z^*,ss}-b\right)^{-\phi_2}\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}}\right)\mu_{z^*,ss}\right]^{-1}$$

$$= -\left[\left(\frac{-\phi_2 b\beta\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2-1}}{\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}\left[\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4}-\mu_{z^*,ss}^{-\phi_2}\beta b\right]} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}\right)}{\left(\mu_{z^*,ss}-b\right)^{-\phi_2}\left(1-\beta b\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}\right)}\right)\mu_{z^*,ss}\right]^{-1}$$

$$= -\left[\left(\frac{-\phi_2 b\beta\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}}{\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4}-\mu_{z^*,ss}^{-\phi_2}\beta b} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-1}\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}}\right)\mu_{z^*,ss}\right]^{-1}$$

$$= -\left[\left(\frac{-\phi_2 b\beta\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}}{\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4}-\mu_{z^*,ss}^{-\phi_2}\beta b} + \frac{-\phi_2\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\mu_{z^*,ss}^{-1}\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}}\right)\mu_{z^*,ss}\right]^{-1}$$

$$= \left[\phi_2\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\left(\frac{b\beta\mu_{z^*,ss}^{-\phi_2}}{\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4}-\mu_{z^*,ss}^{-\phi_2}\beta b} + \frac{\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}}\right)\right]^{-1}$$

$$= \left[\phi_2\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\left(\frac{b\beta}{\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4+\phi_2}-\beta b} + \frac{\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}}\right)\right]^{-1}$$

$$= \frac{1}{\phi_2}\left(1-\frac{b}{\mu_{z^*,ss}}\right)\left(\frac{b\beta}{\left(\mu_{z^*,ss}\right)^{-(\phi_2-1)\phi_4+\phi_2}-\beta b} + \frac{\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{(\phi_2-1)\phi_4-\phi_2}}\right)^{-1}$$

Suppose $\phi_4 = 1$, then

$$IES_{ss} = \left[\phi_2\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\left(\frac{b\beta}{\mu_{z^*,ss}-\beta b} + \frac{\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{-2}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{-1}}\right)\right]^{-1}$$

$$= \frac{1}{\phi_2}\left[\frac{1-\frac{b}{\mu_{z^*,ss}}}{\frac{b\beta}{\mu_{z^*,ss}-\beta b}+\frac{1+\beta b^2\left(\mu_{z^*,ss}\right)^{-2}}{1-\beta b\left(\mu_{z^*,ss}\right)^{-1}}}\right]$$

$$= \frac{1}{\phi_2}\left[\frac{1-\frac{b}{\mu_{z^*,ss}}}{\frac{b\beta}{\mu_{z^*,ss}-\beta b}+\frac{\mu_{z^*,ss}+\beta b^2\left(\mu_{z^*,ss}\right)^{-1}}{\mu_{z^*,ss}-\beta b}}\right]$$

$$= \frac{1}{\phi_2}\left[\frac{\left(1-\frac{b}{\mu_{z^*,ss}}\right)\left(\mu_{z^*,ss}-\beta b\right)}{\mu_{z^*,ss}+b\beta+\beta b^2\left(\mu_{z^*,ss}\right)^{-1}}\right]$$

Consider the case $\mu_{z^*,ss} = 1$ and $\beta = 1$, which implies

$IES_{ss} = \frac{1}{\phi_2}\left[\frac{(1-b)(1-b)}{1+b+b^2}\right] = \frac{1}{\phi_2}\frac{(1-b)^2}{1+b+b^2}$

Suppose that $\phi_4 = 0$ then

$IES_t = \left[\phi_2\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\left(\frac{b\beta}{\left(\mu_{z^*,ss}\right)^{\phi_2}-\beta b} + \frac{\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}}\right)\right]^{-1}$

$= \left[\phi_2\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\left(\frac{b\beta\left(\mu_{z^*,ss}\right)^{-\phi_2}}{1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}} + \frac{\left(1+\beta b^2\left(\mu_{z^*,ss}\right)^{-\phi_2-1}\right)}{1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}}\right)\right]^{-1}$

$= \frac{1}{\phi_2}\left[\frac{1-\frac{b}{\mu_{z^*,ss}}}{\frac{b\beta\left(\mu_{z^*,ss}\right)^{-\phi_2}}{1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}}+\frac{1+\beta b^2\left(\mu_{z^*,ss}\right)^{-\phi_2-1}}{1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}}}\right]$

$= \frac{1}{\phi_2}\left[\frac{\frac{1}{\mu_{z^*,ss}}\left(\mu_{z^*,ss}-b\right)\left(1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}\right)}{b\beta\left(\mu_{z^*,ss}\right)^{-\phi_2}+1+\beta b^2\left(\mu_{z^*,ss}\right)^{-\phi_2-1}}\right]$

$= \frac{1}{\phi_2}\left[\frac{\left(\mu_{z^*,ss}-b\right)\left(1-\beta b\left(\mu_{z^*,ss}\right)^{-\phi_2}\right)}{b\beta\left(\mu_{z^*,ss}\right)^{-\phi_2+1}+\mu_{z^*,ss}+\beta b^2\left(\mu_{z^*,ss}\right)^{-\phi_2}}\right]$

Consider the case $\mu_{z^*,ss} = 1$ and $\beta = 1$, which implies

$IES_{ss} = \frac{1}{\phi_2}\left[\frac{(1-b)(1-b)}{b+1+b^2}\right] = \frac{1}{\phi_2}\frac{(1-b)^2}{1+b+b^2}$

Digression

Another approach for the case of $\phi_4 = 0$ to directly obtain IES at the steady state is to evaluate IES_t at this point to get

$$IES_{ss} = -\left[\left(\frac{b\beta u_{cc}\left(\mu_{z^*,ss}-b\right)}{u_c\left(1-\frac{b}{\mu_{z^*,ss}}\right)-\beta b u_c\left(\mu_{z^*,ss}-b\right)} + \frac{u_{cc}\left(\mu_{z^*,ss}-b\right)+\beta b^2 u_{cc}\left(\mu_{z^*,ss}^2-b\mu_{z^*,ss}\right)}{u_c\left(\mu_{z^*,ss}-b\right)-\beta b u_c\left(\mu_{z^*,ss}^2-b\mu_{z^*,ss}\right)}\right)\mu_{z^*,ss}\right]^{-1}.$$

Next let $u(c_{ss}) = \frac{c_{ss}^{1-\phi_2}}{1-\phi_2}$, then $u_c = c_{ss}^{-\phi_2}$ and $u_c = -\phi_2 c_{ss}^{-\phi_2-1}$ so

$$
\begin{aligned}
IES_{ss} &= -\left[\left(\frac{-b\beta\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}}{\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}-\beta b\left(\mu_{z^*,ss}-b\right)^{-\phi_2}} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}-\beta b^2\phi_2\left(\mu_{z^*,ss}^2-b\mu_{z^*,ss}\right)^{-\phi_2-1}}{\left(\mu_{z^*,ss}-b\right)^{-\phi_2}-\beta b\left(\mu_{z^*,ss}^2-b\mu_{z^*,ss}\right)^{-\phi_2}}\right)\mu_{z^*,ss}\right]^{-1} \\
&= -\left[\left(\frac{-b\beta\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}}{\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}-\beta b\mu_{z^*,ss}^{-\phi_2}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}-\beta b^2\mu_{z^*,ss}^{-\phi_2-1}\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}}{\left(\mu_{z^*,ss}-b\right)^{-\phi_2}-\beta b\mu_{z^*,ss}^{-\phi_2}\left(\mu_{z^*,ss}-b\right)^{-\phi_2}}\right)\mu_{z^*,ss}\right]^{-1} \\
&= -\left[\left(\frac{-b\beta\phi_2\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2-1}}{\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-\phi_2}\left[1-\beta b\mu_{z^*,ss}^{-\phi_2}\right]} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-\phi_2-1}\left[1+\beta b^2\mu_{z^*,ss}^{-\phi_2-1}\right]}{\left(\mu_{z^*,ss}-b\right)^{-\phi_2}\left[1-\beta b\mu_{z^*,ss}^{-\phi_2}\right]}\right)\mu_{z^*,ss}\right]^{-1} \\
&= -\left[\left(\frac{-b\beta\phi_2\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}}{\left[1-\beta b\mu_{z^*,ss}^{-\phi_2}\right]} + \frac{-\phi_2\left(\mu_{z^*,ss}-b\right)^{-1}\left[1+\beta b^2\mu_{z^*,ss}^{-\phi_2-1}\right]}{\left[1-\beta b\mu_{z^*,ss}^{-\phi_2}\right]}\right)\mu_{z^*,ss}\right]^{-1} \\
&= \frac{1-\beta b\mu_{z^*,ss}^{-\phi_2}}{\left(\phi_2\left(\mu_{z^*,ss}-b\right)^{-1}\left[1+\beta b^2\mu_{z^*,ss}^{-\phi_2-1}\right]+b\beta\phi_2\mu_{z^*,ss}^{-\phi_2-1}\left(1-\frac{b}{\mu_{z^*,ss}}\right)^{-1}\right)\mu_{z^*,ss}} \\
&= \frac{1-\beta b\mu_{z^*,ss}^{-\phi_2}}{\left(\phi_2\left(\mu_{z^*,ss}-b\right)^{-1}\left[1+\beta b^2\mu_{z^*,ss}^{-\phi_2-1}\right]+b\beta\phi_2\mu_{z^*,ss}^{-\phi_2}\left(\mu_{z^*,ss}-b\right)^{-1}\right)\mu_{z^*,ss}} \\
&= \frac{1-\beta b\mu_{z^*,ss}^{-\phi_2}}{\phi_2\left(\frac{1+\beta b^2\mu_{z^*,ss}^{-\phi_2-1}}{\mu_{z^*,ss}-b}+\frac{b\beta\mu_{z^*,ss}^{-\phi_2}}{\mu_{z^*,ss}-b}\right)\mu_{z^*,ss}} \\
&= \frac{1-\beta b\mu_{z^*,ss}^{-\phi_2}}{\phi_2\left(\frac{\mu_{z^*,ss}+\beta b^2\mu_{z^*,ss}^{-\phi_2}}{\mu_{z^*,ss}-b}+\frac{\mu_{z^*,ss}^{1-\phi_2}b\beta}{\mu_{z^*,ss}-b}\right)}
\end{aligned}
$$

Note that for $b=0$, we get $IES_{ss} = 1/\phi_2$ as desired.

16.12.3 Comparing internal and external habits

Suppose $\mu_{z^*,ss}=1$ and $\beta=1$, then we get

$IES_{ss}^{internal} = \frac{(1-b)^2}{\phi_2(1+b^2+b)}$

whereas we for comparison have $IES_{ss}^{internal} < IES_{ss}^{external} = \frac{1}{\phi_2}\left(1-\frac{b}{\mu_{z^*,ss}}\right) = \frac{1}{\phi_2}(1-b)$ because

$\frac{(1-b)^2}{(1+b^2+b)} < (1-b)$

$\Updownarrow$

$\frac{1-b}{(1+b^2+b)} < 1$

$\Updownarrow$

$1-b < 1+b^2+b$

$\Updownarrow$

$0 < b^2+2b$

which is always true.

Note finaly that if we have $\phi = 0.5$ and $b = 0.7$ then we get

$IES_{ss}^{internal} = \frac{(1-0.7)^2}{0.5(1+0.7^2+0.7)} = 8.219\,2 \times 10^{-2}$
$IES_{ss}^{external} = \frac{(1-0.7)}{0.5} = 0.6$

16.13 The Frisch labor supply elasticity

Recall that this elasticity is given by
$elas_F = \frac{u_h}{h\left(u_{hh} - \frac{u_{ch}}{u_{cc}}\right)}$

In our case
$u_h = -\left(z_t^*\right)^{(1-\phi_4)(1-\phi_2)} \phi_0 \left(1-h_t\right)^{-\phi_1}$
$u_{hh} = -\phi_1 \left(z_t^*\right)^{(1-\phi_4)(1-\phi_2)} \phi_0 \left(1-h_t\right)^{-\phi_1-1}$
$u_{ch} = 0$

So, in the steady state we have
$elas_F = \frac{u_h}{h_{ss}\left(u_{hh} - \frac{u_{ch}}{u_{cc}}\right)}$
$= \frac{-(z_{ss}^*)^{(1-\phi_4)(1-\phi_2)}\phi_0(1-h_{ss})^{-\phi_1}}{h_{ss}\left(-\phi_1(z_{ss}^*)^{(1-\phi_4)(1-\phi_2)}\phi_0(1-h_{ss})^{-\phi_1-1}\right)}$
$= \frac{(1-h_{ss})}{\phi_1 h_{ss}}$

$= \frac{1}{\phi_1}\left(\frac{1}{h_{ss}} - 1\right)$

16.14 Measures of relative risk aversion

We follow Swanson (2012) and compute two measures of relative risk aversion in our model. With recursive Epstein-Zin preferences controlled by ϕ_3, there are two measures of relative risk aversion. The first measure RRA^c applies when there is no upper bound for labor and therefore total household wealth A_t equals the present discounted value of consumption. The other measure RRA^{cl} applies when the upper bound for the household's time endowment is well-specified, meaning that total household wealth $\tilde{A}_t$ equals the present discounted value of leisure plus consumption.

Throughout this section we use the notational convention in Swanson (2012) where a variable in the steady state is denoted without a subscript. For instance c is the steady state value of c_t.

16.14.1 External Habits

The general formulas are (see Swanson (2012), page 24, eq 53 and eq 54)

$$RRA^{cl} = \frac{-u_{11} + \lambda u_{12}}{u_1}\frac{c + w(1-h)}{1+w\lambda} + \phi_3 \frac{(c + w(1-h))\,u_1}{u}$$

$$RRA^c = c\left(\frac{-u_{11} + \lambda u_{12}}{u_1}\frac{1}{1+w\lambda} + \phi_3\frac{u_1}{u}\right)$$

where

$$w = -\frac{u_2}{u_1}$$

$$\lambda = \frac{wu_{11} + u_{12}}{u_{22} + wu_{12}}$$

Note that
$RRA^{cl} = \frac{-u_{11}+\lambda u_{12}}{u_1}\frac{c+w(1-h)}{1+w\lambda} + \phi_3 \frac{(c+w(1-h))u_1}{u}$

$= \frac{c+w(1-h)}{c} c \left[\frac{-u_{11}+\lambda u_{12}}{u_1} \frac{1}{1+w\lambda} + \phi_3 \frac{u_1}{u} \right]$

$\Updownarrow$

$$RRA^{cl} = \left(1 + \frac{w}{c}(1-h)\right) RRA^{c}$$

Here, we use the same notation that

- $u =$ the utility index
- $u_1 =$ the partial derivative of u with respect to consumption
- $u_2 =$ the partial derivative of u with respect to hours worked
- $w =$ the steady state wage level
- $c =$ the steady state consumption level
- $h =$ hours worked

Note that the way Swanson (2012) defines λ is different from the way we define λ_t in our model (i.e. the lagrange multiplier on households' budget restriction). Importantly, Swanson (2012) shows that the above formulas also hold with balanced growth (see Swanson (2012) page 48). Recall that our utility function reads (ignoring d_t as $d_{ss} = 1$)

$u\left(\frac{c_t - bc_{t-1}}{z_t^*}, 1 - h_t\right) = \frac{1}{1-\phi_2}\left(\left(\frac{c_t - bc_{t-1}}{(z_t^*)^{\phi_4}}\right)^{1-\phi_2} - (z_t^*)^{(1-\phi_4)(1-\phi_2)}\right) + (z_t^*)^{(1-\phi_4)(1-\phi_2)} \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}$

$= \frac{1}{1-\phi_2}(z_t^*)^{-\phi_4(1-\phi_2)}\left((c_t - bc_{t-1})^{1-\phi_2} - (z_t^*)^{(1-\phi_2)}\right) + (z_t^*)^{(1-\phi_4)(1-\phi_2)} \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}$

$= (z_t^*)^{-\phi_4(1-\phi_2)} \left\{ \frac{1}{1-\phi_2}(c_t - bc_{t-1})^{1-\phi_2} - \frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)} + (z_t^*)^{(1-\phi_2)} \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1} \right\}$

$= (z_t^*)^{-\phi_4(1-\phi_2)} \left\{ \frac{1}{1-\phi_2}(c_t - bc_{t-1})^{1-\phi_2} - \frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)} + (z_t^*)^{(1-\phi_2)} \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1} \right\}$

From the formulas of risk-aversion we see that scaling u by a constant does not affect the measure of relative risk aversion. Hence, we can without loss of generality consider

$$u\left(\frac{c_t - bc_{t-1}}{z_t^*}, 1 - h_t\right) = \frac{1}{1-\phi_2}(c_t - bc_{t-1})^{1-\phi_2} - \frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)} + (z_t^*)^{(1-\phi_2)} \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}$$

Hence, we have
$u_1 = (c_t - bc_{t-1})^{-\phi_2}$
$u_{11} = -\phi_2 (c_t - bc_{t-1})^{-\phi_2 - 1}$
$u_2 = -(z_t^*)^{(1-\phi_2)} \phi_0 (1-h_t)^{-\phi_1}$
$u_{22} = -(-\phi_1)(z_t^*)^{(1-\phi_2)} \phi_0 (1-h_t)^{-\phi_1 - 1}(-1) = -\phi_1 (z_t^*)^{(1-\phi_2)} \phi_0 (1-h_t)^{-\phi_1 - 1}$
$u_{12} = 0$

Note that in the steady state $c_t - bc_{t-1} = c - bc(\mu_{z^*})^{-1}$. Hence
$u_1 = \left(c - bc\mu_{z^*}^{-1}\right)^{-\phi_2}$
$u_{11} = -\phi_2 \left(c - bc\mu_{z^*}^{-1}\right)^{-\phi_2 - 1}$
$u_2 = -(z^*)^{(1-\phi_2)} \phi_0 (1-h)^{-\phi_1}$

$u_{22} = -\phi_1 (z^*)^{(1-\phi_2)} \phi_0 (1-h)^{-\phi_1 - 1}$
$u_{12} = 0$

Thus
$w = -\frac{u_2}{u_1} = -\frac{-(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}} = \frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}$
$\Updownarrow$
$\phi_0 = \frac{w\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{(z^*)^{(1-\phi_2)}(1-h)^{-\phi_1}}$

And
$$\lambda = \frac{wu_{11}+u_{12}}{u_{22}+wu_{12}} = \frac{\frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_z^{-1}\right)^{-\phi_2}}\left[-\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2-1}\right]}{-\phi_1(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1-1}}$$

$$= \frac{\frac{1}{\left(c-bc\mu_z^{-1}\right)^{-\phi_2}}\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2-1}}{\phi_1(1-h)^{-1}}$$

$$= \frac{\phi_2(1-h)}{\phi_1\left(c-bc\mu_{z^*}^{-1}\right)}$$

Note that
$w\lambda = \frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{\phi_2(1-h)}{\phi_1\left(c-bc\mu_{z^*}^{-1}\right)} = \frac{\phi_2}{\phi_1}\frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}}$

Hence, the first measure of relative risk-aversion is:
$RRA^c = c\left(\frac{u_{11}}{u_1}\frac{1}{1+w\lambda} + \phi_3\frac{u_1}{u}\right)$

$$= c\frac{\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2-1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{1}{1+\frac{\phi_2}{\phi_1}\frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}}}$$
$$+c\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}-\frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+(z^*)^{(1-\phi_2)}\phi_0\frac{(1-h)^{1-\phi_1}}{1-\phi_1}}$$

$$= c\frac{\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-1}}{1}\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}+\frac{\phi_2}{\phi_1}(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}}$$
$$+c\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left\{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}-\frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+(z^*)^{(1-\phi_2)}\phi_0\frac{(1-h)^{1-\phi_1}}{1-\phi_1}\right\}}$$

$$= c\phi_2\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}+\frac{\phi_2}{\phi_1}(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}}$$
$$+c\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left\{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}-\frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+(z^*)^{(1-\phi_2)}\phi_0\frac{(1-h)^{1-\phi_1}}{1-\phi_1}\right\}}$$

$$= c\phi_2\frac{1}{\left(c-bc\mu_{z^*}^{-1}\right)+\frac{\phi_2}{\phi_1}(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}$$
$$+c\phi_3\frac{1}{\left\{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)-\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+(z^*)^{(1-\phi_2)}\phi_0\frac{(1-h)^{1-\phi_1}}{1-\phi_1}\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}\right\}}$$

$$= c\phi_2\frac{1}{\left(c-bc\mu_{z^*}^{-1}\right)+\frac{\phi_2}{\phi_1}(z^*)^{(1-\phi_2)}\frac{w\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{(z^*)^{(1-\phi_2)}(1-h)^{-\phi_1}}(1-h)^{1-\phi_1}\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}$$

$$+c\phi_3 \frac{1}{\left\{ \frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right) - \frac{\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}{1-\phi_2}\left(z_t^*\right)^{(1-\phi_2)} + \left(z^*\right)^{(1-\phi_2)} \frac{w\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left(z^*\right)^{(1-\phi_2)}(1-h)^{-\phi_1}} \frac{(1-h)^{1-\phi_1}}{1-\phi_1}\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2} \right\}}$$

using $\phi_0 = \frac{w\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left(z^*\right)^{(1-\phi_2)}(1-h)^{-\phi_1}}$

$$= c\left(\phi_2 \frac{1}{\left(c-bc\mu_{z^*}^{-1}\right) + \frac{\phi_2}{\phi_1}\frac{w(1-h)}{1}} + \phi_3 \frac{1}{\left\{ \frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right) - \frac{\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}{1-\phi_2}\left(z_t^*\right)^{(1-\phi_2)} + \frac{w}{1}\frac{(1-h)}{1-\phi_1} \right\}} \right)$$

$$= \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{w(1-h)}{c}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}\frac{\left(z_t^*\right)^{(1-\phi_2)}}{c} + \frac{w(1-h)}{c}\frac{1-\phi_2}{1-\phi_1}}$$

We will now express this expression in terms of variables for the transformed economy:
$RRA^c = \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{w(1-h)}{c}\frac{z^*}{z^*}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}\frac{\left(z_t^*\right)^{(1-\phi_2)}}{c} + \frac{w(1-h)}{c}\frac{1-\phi_2}{1-\phi_1}\frac{z^*}{z^*}}$

$$= \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{w(1-h)}{c}\frac{z^*}{z^*}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(\frac{c}{z_t^*} - b\frac{c}{z_t^*}\mu_{z^*}^{-1}\right)^{\phi_2}\frac{1}{c/z_t^*} + \frac{w(1-h)}{c}\frac{1-\phi_2}{1-\phi_1}\frac{z^*}{z^*}}$$

$$= \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{w(1-h)}{c}\frac{z^*}{z^*}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(C-bC\mu_{z^*}^{-1}\right)^{\phi_2}\frac{1}{C} + \frac{w(1-h)}{c}\frac{1-\phi_2}{1-\phi_1}\frac{z^*}{z^*}}$$

$$= \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{W(1-h)}{C}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(C-bC\mu_{z^*}^{-1}\right)^{\phi_2}\frac{1}{C} + \frac{W(1-h)}{C}\frac{1-\phi_2}{1-\phi_1}}$$

$$= \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{W(1-h)}{C}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(1-b\mu_{z^*}^{-1}\right)^{\phi_2}C^{\phi_2-1} + \frac{W(1-h)}{C}\frac{1-\phi_2}{1-\phi_1}}$$

If we follow Swanson (2012) and assume that $C = hW$, then
$RRA^c = \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{W(1-h)}{hW}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(C-bC\mu_{z^*}^{-1}\right)^{\phi_2}\frac{1}{C} + \frac{W(1-h)}{hW}\frac{1-\phi_2}{1-\phi_1}}$

$$= \frac{\phi_2}{1-b\mu_{z^*}^{-1} + \frac{\phi_2}{\phi_1}\frac{(1-h)}{h}} + \phi_3 \frac{1-\phi_2}{1-b\mu_{z^*}^{-1} - \left(C-bC\mu_{z^*}^{-1}\right)^{\phi_2}\frac{1}{C} + \frac{(1-h)}{h}\frac{1-\phi_2}{1-\phi_1}}.$$

And finally:
$RRA^{cl} = \left(1 + \frac{w}{c}(1-h)\right) RRA^c = \left(1 + \frac{W}{C}(1-h)\right) RRA^c$

16.14.2 Internal habits

The general formulas with internal habits are (see Swanson (2012), page 29, eq 72 and eq 73, where the Epstein-Zin coefficient is ϕ_3 and $(ha)_t = \rho\,(ha)_{t-1} + bc_{t-1}$)

$$\begin{aligned} RRA^{cl} &= \frac{-u_{11} + \lambda u_{12}}{u_1}\frac{(1-\beta b)\left(c + w(1-h)\right)}{1 + (1-\beta b)\,w\lambda} + \phi_3 \frac{\left(c + w(1-h)\right)u_1}{u}(1-\beta b) \\ &= (1-\beta b)\left[\frac{-u_{11} + \lambda u_{12}}{u_1}\frac{\left(c + w(1-h)\right)}{1 + (1-\beta b)\,w\lambda} + \phi_3 \frac{\left(c + w(1-h)\right)u_1}{u}\right] \end{aligned}$$

and

$$\begin{aligned} RRA^{c} &= \frac{-u_{11} + \lambda u_{12}}{u_1}\frac{(1-\beta b)\,c}{1 + (1-\beta b)\,w\lambda} + \phi_3 \frac{cu_1}{u}(1-\beta b) \\ &= (1-\beta b)\,c\left[\frac{-u_{11} + \lambda u_{12}}{u_1}\frac{1}{1 + (1-\beta b)\,w\lambda} + \phi_3 \frac{u_1}{u}\right] \end{aligned}$$

As for external habits, we have

$$RRA^{cl} = \left(1 + \frac{w}{c}(1-h)\right) RRA^{c}$$

In the stated expressions, we use the fact that we have $\rho = 0$, so $(ha) = bc$ in the steady state.

We next note that the expressions for w and λ are different with internal habits. In the Appendix in Swanson (2012) (the proof of proposition 15) we have with internal habits that:

$$w = -\frac{u_2}{u_1}\frac{1}{1-\beta b}$$

$$\lambda = \frac{w(1-\beta b)u_{11} + u_{12}}{u_{22} + w(1-\beta b)u_{12}}$$

Recall that we have in the steady state:

$u_1 = \left(c - bc\mu_{z^*}^{-1}\right)^{-\phi_2}$

$u_{11} = -\phi_2\left(c - bc\mu_{z^*}^{-1}\right)^{-\phi_2-1}$

$u_2 = -(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}$

$u_{22} = -\phi_1(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1-1}$

$u_{12} = 0$

Thus

$w = -\frac{u_2}{u_1}\frac{1}{1-\beta b} = -\frac{-(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{1}{1-\beta b} = \frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{1}{1-\beta b}$

$\Updownarrow$

$\phi_0 = \frac{w(1-\beta b)\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{(1-h)^{-\phi_1}(z^*)^{(1-\phi_2)}}$

$\lambda = \frac{w(1-\beta b)u_{11}}{u_{22}} = \frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{(1-\beta b)}{1-\beta b}\frac{-\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2-1}}{-\phi_1(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1-1}}$

$= \frac{(1-\beta b)}{1-\beta b}\frac{\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-1}}{\phi_1(1-h)^{-1}}$

$= \frac{\phi_2(1-h)}{\phi_1\left(c-bc\mu_{z^*}^{-1}\right)}$

Note also that

$w\lambda = \frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{1}{1-\beta b}\frac{1-\beta b}{1-\beta b}\frac{\phi_2(1-h)}{\phi_1\left(c-bc\mu_{z^*}^{-1}\right)}$

$= \frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}}\frac{1}{1-\beta b}\frac{\phi_2}{\phi_1}$

Hence,

$RRA^c = (1-\beta b)\,c\left[\frac{-u_{11}}{u_1}\frac{1}{1+(1-\beta b)w\lambda} + \phi_3\frac{u_1}{u}\right]$

$= (1-\beta b)\,c\frac{\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2-1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}\frac{1}{1+(1-\beta b)\frac{(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}}\frac{1}{1-\beta b}\frac{\phi_2}{\phi_1}}$

$+ (1-\beta b)\,c\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2} - \frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)} + (z^*)^{(1-\phi_2)}\phi_0\frac{(1-h)^{1-\phi_1}}{1-\phi_1}}$

$$= (1-\beta b)\, c\frac{\phi_2\left(c-bc\mu_{z^*}^{-1}\right)^{-1}}{1}\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}+(1-\beta b)(z^*)^{(1-\phi_2)}\phi_0(1-h)^{1-\phi_1}\frac{1}{1-\beta b}\frac{\phi_2}{\phi_1}}$$
$$+(1-\beta b)\, c\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}-\frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+(z^*)^{(1-\phi_2)}\phi_0\frac{(1-h)^{1-\phi_1}}{1-\phi_1}}$$

$$= (1-\beta b)\, c\frac{\phi_2}{1}\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}+(1-\beta b)(z^*)^{(1-\phi_2)}\frac{w(1-\beta b)\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{(1-h)^{-\phi_1}(z^*)^{(1-\phi_2)}}(1-h)^{1-\phi_1}\frac{1}{1-\beta b}\frac{\phi_2}{\phi_1}}+$$
$$+(1-\beta b)\, c\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}-\frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+(z^*)^{(1-\phi_2)}\frac{w(1-\beta b)\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{(1-h)^{-\phi_1}(z^*)^{(1-\phi_2)}}\frac{(1-h)^{1-\phi_1}}{1-\phi_1}}$$

using $\phi_0 = \frac{w(1-\beta b)\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{(1-h)^{-\phi_1}(z^*)^{(1-\phi_2)}}$

$$= (1-\beta b)\, c\left[\frac{\phi_2}{1}\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}+(1-\beta b)\frac{w\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{1}(1-h)\frac{\phi_2}{\phi_1}}+\phi_3\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)^{1-\phi_2}-\frac{1}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+\frac{w(1-\beta b)\left(c-bc\mu_{z^*}^{-1}\right)^{-\phi_2}}{1}\frac{(1-h)}{1-\phi_1}}\right]$$

$$= (1-\beta b)\, c\left[\frac{\phi_2}{1}\frac{1}{\left(c-bc\mu_{z^*}^{-1}\right)+(1-\beta b)w(1-h)\frac{\phi_2}{\phi_1}}+\phi_3\frac{1}{\frac{1}{1-\phi_2}\left(c-bc\mu_{z^*}^{-1}\right)-\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}{1-\phi_2}(z_t^*)^{(1-\phi_2)}+w(1-\beta b)\frac{(1-h)}{1-\phi_1}}\right]$$

$$= \frac{\phi_2}{\frac{1-b\mu_{z^*}^{-1}}{1-\beta b}+\frac{w(1-h)}{c}\frac{\phi_2}{\phi_1}}+\phi_3\frac{1-\phi_2}{\frac{1-b\mu_{z^*}^{-1}}{1-\beta b}-\frac{\left(c-bc\mu_{z^*}^{-1}\right)^{\phi_2}}{1-\beta b}\frac{(z_t^*)^{(1-\phi_2)}}{c}+\frac{w(1-h)}{c}\frac{1-\phi_2}{1-\phi_1}}$$

Expressed in terms of the transformed economy:
$$RRA^c = \frac{\phi_2}{\frac{1-b\mu_{z^*}^{-1}}{1-\beta b}+\frac{W(1-h)}{C}\frac{\phi_2}{\phi_1}}+\phi_3\frac{1-\phi_2}{\frac{1-b\mu_{z^*}^{-1}}{1-\beta b}-\frac{\left(C-bC\mu_{z^*}^{-1}\right)^{\phi_2}}{1-\beta b}\frac{1}{C}+\frac{W(1-h)}{C}\frac{1-\phi_2}{1-\phi_1}}$$

And finally:
$$RRA^{cl} = \left(1+\frac{w}{c}(1-h)\right)RRA^c = \left(1+\frac{W}{C}(1-h)\right)RRA^c$$

16.15 The steady state

This section derives the values for the transformed variables in steady state as a function of the structural parameters. We denote variables in steady state by subscript ss. The steady state value of labor, i.e. h_{ss}, is assumed to be given and we then back out the value of ϕ_0. We also assume that G_{ss}/Y_{ss} is known.

The growth rate in λ_{ss}

$$\mu_{\lambda,t+1} \equiv \frac{\Lambda_{t+1}}{\Lambda_t}\left(\frac{\left[E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-\widetilde{V_{t+1}}}\right)^{\phi_3}\mu_{z^*,t+1}^{-\phi_2(1-\phi_4)-\phi_4}$$
$$\Downarrow$$
$$\mu_{\lambda,ss} \equiv \left(\frac{\left[E_t\left[\left(-\widetilde{V_{ss}}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-\widetilde{V_{ss}}}\right)^{\phi_3}\mu_{z^*,ss}^{-\phi_2(1-\phi_4)-\phi_4}$$
$$\Updownarrow$$
$$\mu_{\lambda,ss} \equiv \mu_{z^*,ss}^{-\phi_2(1-\phi_4)-\phi_4}$$

The optimal relative price, $\tilde{p}_{ss}$.

From EQ 11

$1 = (1-\alpha)\,\tilde{p}_t^{1-\eta} + \alpha\left(\frac{\pi_{t-1}^{\chi}}{\pi_t}\right)^{1-\eta}$

$\Downarrow$

$1 - \alpha\pi_{ss}^{(\chi-1)(1-\eta)} = (1-\alpha)\,\tilde{p}_{ss}^{1-\eta}$

$\Updownarrow$

$\tilde{p}_{ss} = \left[\frac{1-\alpha\pi_{ss}^{(\chi-1)(1-\eta)}}{1-\alpha}\right]^{\frac{1}{1-\eta}}$

The state variable for distortion due to price stickyness, s_{ss}

From EQ 18

$s_{t+1} = (1-\alpha)\,\tilde{p}_t^{-\eta} + \alpha\left(\frac{\pi_t}{\pi_{t-1}^{\chi}}\right)^{\eta} s_t$

$\Downarrow$

$s_{ss} = (1-\alpha)\,\tilde{p}_{ss}^{-\eta} + \alpha\pi_{ss}^{(1-\chi)\eta} s_{ss}$

$\Updownarrow$

$s_{ss}\left(1 - \alpha\pi_{ss}^{(1-\chi)\eta}\right) = (1-\alpha)\,\tilde{p}_{ss}^{-\eta}$

$\Updownarrow$

$s_{ss} = \frac{(1-\alpha)\tilde{p}_{ss}^{-\eta}}{1-\alpha\pi_{ss}^{(1-\chi)\eta}}$

The nominal one period deposite rate, r_{ss}^b

From EQ 6

$1 = E_t\left[\beta\mu_{\lambda\, t+1}\frac{\exp\{r_t^b\}}{\pi_{t+1}}\right]$

$\Downarrow$

$r_{ss}^b = \log\left(\frac{\pi_{ss}}{\beta\mu_{\lambda\, ss}}\right)$

and using $\mu_{\lambda,ss} \equiv \mu_{z^*,ss}^{-\phi_2(1-\phi_4)-\phi_4}$ we get

$r_{ss}^b = \log\left(\frac{\pi_{ss}}{\beta\mu_{z^*,ss}^{-\phi_2(1-\phi_4)-\phi_4}}\right)$

$\quad = \log\left(\frac{\pi_{ss}}{\beta}\right) - \left(-\phi_2\left(1-\phi_4\right) - \phi_4\right)\log\mu_{z^*,ss}$

Note, if $\phi_4 = 0$, then $r_{ss}^b = \log\left(\frac{\pi_{ss}}{\beta}\right) + \phi_2\log\mu_{z^*,ss}$, implying that increasing ϕ_2 also increases the steady state level of the deposite rate as $\mu_{z^*,ss} > 1$. However, when $\phi_4 = 1$, then $r_{ss}^b = \log\left(\frac{\pi_{ss}}{\beta}\right) + \log\mu_{z^*,ss}$ and increasing ϕ_2 no longer increases the steady state level of the deposite rate.

The real price of capital , Q_{ss}

From EQ 5

$1 = Q_t\left(1 - \frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}} - 1\right)^2 - \frac{I_t}{I_{SS}}\kappa_1\left(\frac{I_t}{I_{SS}} - 1\right) - \kappa_2\left(\frac{I_t}{k_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$

We immediately get that $Q_{ss} = 1$

The real price of capital, R_{ss}^k

From EQ 3

$$Q_t = E_t\frac{\beta\mu_{\lambda,t+1}}{\mu_{\Upsilon,t+1}}[R_{t+1}^k + Q_{t+1}(1-\delta)$$
$$-Q_{t+1}\frac{\kappa_2}{2}\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2 + Q_{t+1}\kappa_2\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}]$$

$\Downarrow$
$Q_{ss} = \beta \frac{\mu_{\lambda,ss}}{\mu_{\Upsilon,ss}} \left[R^k_{ss} + Q_{ss} (1 - \delta) \right]$
$\Updownarrow$
$\mu_{\Upsilon,ss} Q_{ss} = \beta \mu_{\lambda,ss} \left[R^k_{ss} + Q_{ss} (1 - \delta) \right]$
$\Updownarrow$
$\frac{\mu_{\Upsilon,ss} Q_{ss}}{\beta \mu_{\lambda,ss}} - Q_{ss} (1 - \delta) = R^k_{ss}$
$\Updownarrow$
$R^k_{ss} = Q_{ss} \left[\frac{\mu_{\Upsilon,ss}}{\beta \mu_{\lambda,ss}} - (1 - \delta) \right]$

The marginal costs in the firms, mc_{ss}
First from EQ 9
$\frac{(\eta-1)}{\eta} X^2_t = Y_t mc_t \tilde{p}_t^{-\eta-1} + E_t \left[\alpha\beta\mu_{\lambda,t+1} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta-1} \left(\frac{\pi^{\chi}_t}{\pi_{t+1}} \right)^{-\eta} \frac{(\eta-1)}{\eta} X^2_{t+1} \mu_{z^*,t+1} \right]$
$\Downarrow$
$\frac{(\eta-1)}{\eta} X^2_{ss} = Y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1} + \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(1-\chi)\eta} \frac{(\eta-1)}{\eta} X^2_{ss} \mu_{z^*,ss}$
$\Downarrow$
$X^2_{ss} \left[\frac{(\eta-1)}{\eta} - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(1-\chi)\eta} \frac{(\eta-1)}{\eta} \mu_{z^*,ss} \right] = Y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1}$
$\Updownarrow$
$X^2_{ss} = \frac{Y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1}}{\frac{(\eta-1)}{\eta} - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(1-\chi)\eta} \frac{(\eta-1)}{\eta} \mu_{z^*,ss}}$

And from EQ 10
$X^2_t = Y_t \tilde{p}_t^{-\eta} + E_t \left[\alpha\beta\mu_{\lambda,t+1} \left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \right)^{-\eta} \left(\frac{\pi^{\chi}_t}{\pi_{t+1}} \right)^{1-\eta} X^2_{t+1} \mu_{z^*,t+1} \right]$
$\Downarrow$
$X^2_{ss} = Y_{ss} \tilde{p}_{ss}^{-\eta} + \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(\chi-1)(1-\eta)} X^2_{ss} \mu_{z^*,ss}$
$\Updownarrow$
$X^2_{ss} \left(1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(\chi-1)(1-\eta)} \mu_{z^*,ss} \right) = Y_{ss} \tilde{p}_{ss}^{-\eta}$
$\Updownarrow$
$X^2_{ss} = \frac{Y_{ss} \tilde{p}_{ss}^{-\eta}}{1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(\chi-1)(1-\eta)} \mu_{z^*,ss}}$

So by setting the two equations equal to one another we get:
$\frac{Y_{ss} mc_{ss} \tilde{p}_{ss}^{-\eta-1}}{\frac{(\eta-1)}{\eta} - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(1-\chi)\eta} \frac{(\eta-1)}{\eta} \mu_{z^*,ss}} = \frac{Y_{ss} \tilde{p}_{ss}^{-\eta}}{1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(\chi-1)(1-\eta)} \mu_{z^*,ss}}$
$\Updownarrow$
$\frac{mc_{ss}}{\left(1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(1-\chi)\eta} \mu_{z^*,ss} \right) \frac{(\eta-1)}{\eta} \tilde{p}_{ss}} = \frac{1}{1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(\chi-1)(1-\eta)} \mu_{z^*,ss}}$
$\Updownarrow$
$mc_{ss} = \frac{\left(1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(1-\chi)\eta} \mu_{z^*,ss} \right) \frac{(\eta-1)}{\eta} \tilde{p}_{ss}}{1 - \alpha\beta\mu_{\lambda,ss} \pi_{ss}^{(\chi-1)(1-\eta)} \mu_{z^*,ss}}$

The Capital level, K_{ss}
From EQ 7
$mc_t a_{ss} \theta \mu_{\Upsilon,t} \mu_{z,t}^{1-\theta} K_t^{\theta-1} h_t^{1-\theta} = R^k_t$
$\Downarrow$
$mc_{ss} \theta \mu_{\Upsilon,ss} \mu_{z,ss}^{1-\theta} K_{ss}^{\theta-1} h_{ss}^{1-\theta} = R^k_{ss}$
since $a_{ss} = 1$
$\Updownarrow$

$\left(\frac{K_{ss}}{h_{ss}}\right)^{\theta-1} = \frac{R_{ss}^{k}}{mc_{ss}\theta\mu_{\Upsilon,ss}\mu_{z,ss}^{1-\theta}}$
$\Updownarrow$
$K_{ss} = h_{ss}\left(\frac{R_{ss}^{k}}{mc_{ss}\theta\mu_{\Upsilon,ss}\mu_{z,ss}^{1-\theta}}\right)^{\frac{1}{\theta-1}}$

The wage level, W_{ss}
From EQ 8
$mc_t\left(1-\theta\right)a_t\mu_{\Upsilon,t}^{\frac{-\theta}{1-\theta}}\mu_{z,t}^{-\theta}K_t^{\theta}h_t^{-\theta} = W_t$
$\Downarrow$
$mc_{ss}\left(1-\theta\right)\mu_{\Upsilon,ss}^{\frac{-\theta}{1-\theta}}\mu_{z,ss}^{-\theta}\left(\frac{K_{ss}}{h_{ss}}\right)^{\theta} = W_{ss}$
since $a_{ss} = 1$
$\Updownarrow$
$W_{ss} = mc_{ss}\left(1-\theta\right)\mu_{\Upsilon,ss}^{\frac{-\theta}{1-\theta}}\mu_{z,ss}^{-\theta}\left(\frac{K_{ss}}{h_{ss}}\right)^{\theta}$

The investment level, I_{ss}
From EQ 19
$K_{t+1} = \left(1-\delta\right)K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1} + I_t - I_t\frac{\kappa_1}{2}\left(\frac{I_t}{I_{SS}}-1\right)^2 - K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1}\frac{\kappa_2}{2}\left(\frac{I_t}{K_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$
$\Downarrow$
$K_{ss} = \left(1-\delta\right)K_t\left(\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^{-1} + I_{ss}$
$\Updownarrow$
$I_{ss} = K_{ss} - \left(1-\delta\right)K_{ss}\mu_{\Upsilon,ss}^{\frac{-1}{1-\theta}}\mu_{z,ss}^{-1}$

The Consumption level, C_{ss}
From 17
$a_t\left(K_t\mu_{\Upsilon,t}^{\frac{-1}{1-\theta}}\mu_{z,t}^{-1}\right)^{\theta}h_t^{1-\theta} = Y_t s_{t+1}$
$\Downarrow$
$\frac{\left(K_{ss}\mu_{\Upsilon,ss}^{\frac{-1}{1-\theta}}\mu_{z,ss}^{-1}\right)^{\theta}h_{ss}^{1-\theta}}{s_{ss}} = Y_{ss}$
Note then that Eq 20 implies
$Y_{ss} = C_{ss} + I_{ss} + G_{ss}$
$\Updownarrow$
$Y_{ss} = C_{ss} + I_{ss} + \frac{G_{ss}}{Y_{ss}}Y_{ss}$
$\Updownarrow$
$Y_{ss}\left(1-\frac{G_{ss}}{Y_{ss}}\right) = C_{ss} + I_{ss}$
$\Updownarrow$
$Y_{ss} = \frac{C_{ss}+I_{ss}}{1-\frac{G_{ss}}{Y_{ss}}}$
Thus, we have
$\frac{\left(K_{ss}\mu_{\Upsilon,ss}^{\frac{-1}{1-\theta}}\mu_{z,ss}^{-1}\right)^{\theta}h_{ss}^{1-\theta}}{s_{ss}} = \frac{C_{ss}+I_{ss}}{1-\frac{G_{ss}}{Y_{ss}}}$
$\Updownarrow$
$\frac{\left(1-\frac{g_{ss}}{Y_{ss}}\right)\left(K_{ss}\mu_{\Upsilon,ss}^{\frac{-1}{1-\theta}}\mu_{z,ss}^{-1}\right)^{\theta}h_{ss}^{1-\theta}}{s_{ss}} - I_{ss} = C_{ss}$

The value of Λ_{ss}
From EQ 2

$$\Lambda_t = d_t \left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]} b\beta E_t\{\left[\left(\frac{\left[E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{-\widetilde{V_{t+1}}(s)}\right)^{\phi_3}\right]$$

$$\times d_{t+1}\left(C_{t+1} - bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-\phi_2(1-\phi_4)-\phi_4}\}$$

$\Downarrow$

$$\Lambda_{ss} = \left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{-\phi_2} - 1_{[ha\_in]} b\beta\left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,ss}\right)^{-\phi_2(1-\phi_4)-\phi_4}$$

The value of ϕ_0
From EQ 4
$d_t\phi_0\left(1-h_t\right)^{-\phi_1} = \Lambda_t W_t$
$\Downarrow$
$\phi_0 = \Lambda_{ss}W_{ss}\left(1-h_{ss}\right)^{\phi_1}$

The level of the value function, V_{ss}
From EQ 1

$$\widetilde{V_t} = \left[\tfrac{d_t}{1-\phi_2}\left(\left(C_t - bC_{t-1}\mu_{z^*,t}^{-1}\right)^{1-\phi_2} - 1\right) + d_t\phi_0\tfrac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right] - \beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}\left(E_t\left[\left(-\widetilde{V_{t+1}}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$$

$\Downarrow$

$$\widetilde{V_{ss}} = \tfrac{1}{1-\phi_2}\left(\left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{1-\phi_2} - 1\right) + \phi_0\tfrac{(1-h_{ss})^{1-\phi_1}}{1-\phi_1} - \beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}\left(-\widetilde{V_{ss}}\right)$$

$\Updownarrow$

$$\widetilde{V_{ss}}\left(1 - \beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}\right) = \tfrac{1}{1-\phi_2}\left(\left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{1-\phi_2} - 1\right) + \phi_0\tfrac{(1-h_{ss})^{1-\phi_1}}{1-\phi_1} \Updownarrow$$

$\Updownarrow$

$$\widetilde{V_{ss}} = \tfrac{1}{1-\beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}}\left[\tfrac{1}{1-\phi_2}\left(\left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{1-\phi_2} - 1\right) + \phi_0\tfrac{(1-h_{ss})^{1-\phi_1}}{1-\phi_1}\right]$$

$\Updownarrow$

$$\widetilde{V_{ss}} = \tfrac{1}{1-\beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}}\left[\tfrac{1}{1-\phi_2}\left(\left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{1-\phi_2} - 1\right) + \phi_0\tfrac{(1-h_{ss})^{1-\phi_1}}{1-\phi_1}\right]$$

If $\widetilde{V_{ss}} < 0$, then we consider

$$-\widetilde{V_{ss}} = \frac{-1}{1-\beta\mu_{z^*,ss}^{(1-\phi_4)(1-\phi_2)}}\left[\frac{1}{1-\phi_2}\left(\left(C_{ss} - bC_{ss}\mu_{z^*,ss}^{-1}\right)^{1-\phi_2} - 1\right) + \phi_0\frac{\left(1-h_{ss}\right)^{1-\phi_1}}{1-\phi_1}\right]$$

and we do the perturbation for $(mV)_t \equiv (-V_t)$ where $(mV)_t > 0$ because $V_t < 0$.

The value of G_{ss}
$G_{ss} = \frac{G_{ss}}{Y_{ss}}Y_{ss}$

16.16 The observables and their moments

16.16.1 Calculating the observables

This section shows how to calculate the considered observables used in the estimation from the approximation of the DSGE model.

The presence of non-stationary shocks (i.e. z_t and Υ_t) imply that variables such as c_t, i_t, and y_t and are non-stationary, and this fact must be taken into account when solving the model. We adopt the standard method to deal with this feature by approximating the model's solution around the economy's balanced growth path. This is done by scaling the non-stationary variables such that they become stationary. For instance, c_t and y_t are scaled by $1/z_t^*$ and i_t is scaled by $1/\Upsilon_t z_t^*$, and this implies that $C_t \equiv c_t/z_t^*$, $Y_t \equiv y_t/z_t^*$, and $I_t \equiv i/\Upsilon_t z_t^*$ are stationary variables.

Applying a log-transformation, the output from approximating the DSGE model is

$$\hat{\mathbf{y}}_t \equiv \mathbf{g}\left(\hat{\mathbf{x}}_t\right)$$

where we use the standard notation that a hat denotes deviation from the deterministic steady state, i.e. $\hat{v}_t = \ln \frac{v_t}{v_{ss}}$. The elements in $\hat{\mathbf{y}}_t$ must be transformed to make them comparable to empirical data series. We now show how this transformation is done.

1. Consumption growth
The expressions for real quarterly consumption growth is given by

$$\begin{aligned}\log \mu_{c,t}^{obs} &\equiv \log \frac{c_t}{c_{t-1}} = \log \frac{C_t z_t^*}{C_{t-1} z_{t-1}^*} = \log \frac{C_t}{C_{ss}} - \log \frac{C_{t-1}}{C_{ss}} + \log \frac{z_t^*}{z_{t-1}^*} \\ &= \hat{C}_t - \hat{C}_{t-1} + \log \mu_{z^*,ss}\end{aligned}$$

2. Investment growth
For the quarterly growth rate in investments

$$\begin{aligned}\log \mu_{i,t}^{obs} &\equiv \log \frac{i_t}{i_{t-1}} = \log \frac{I_t z_t^* \Upsilon_t}{I_{t-1} z_{t-1}^* \Upsilon_{t-1}} = \log \frac{I_t}{I_{ss}} - \log \frac{I_{t-1}}{I_{ss}} + \log \frac{z_t^*}{z_{t-1}^*} + \log \frac{\Upsilon_t}{\Upsilon_{t-1}} \\ &= \hat{I}_t - \hat{I}_{t-1} + \log \mu_{z^*,ss} + \log \mu_{\Upsilon,ss}\end{aligned}$$

3. Inflation
The quarterly inflation rate is given by

$$\log \pi_t = \log \pi_{ss} + \hat{\pi}_t.$$

4. One-period nominal interest rate
The quarterly nominal interest rate is

$$r_t = r_{ss} + \hat{r}_t,$$

5. 40-period nominal interest rate
The quarterly rate is

$$r_{t,40} = r_{ss,40} + \hat{r}_{t,40},$$

6. Excess holding-period return for the 40-period bond
We compute the ex post excess holding period return by
$xhr_{t,40} = \log\left(\frac{P_{t,39}}{P_{t-1,40}}\right) - r_{t-1}$

$= \log\left(P_{t,39}\right) - \log\left(P_{t-1,40}\right) - r_{t-1}$

$$= \log\left(P_{t,39}\right) - \log P_{ss,40} + \log P_{ss,40} - \log\left(P_{t-1,40}\right) - r_{t-1}$$

$$= \hat{P}_{t,39} - \log P_{ss,1} - \hat{P}_{t-1,40} - r_{t-1}$$

$$= \hat{P}_{t,39} - \hat{P}_{t-1,40} - \left(r_{t-1} + \log P_{ss,1}\right)$$

$$= \hat{P}_{t,39} - \hat{P}_{t-1,40} - \left(r_{t-1} - r_{ss}\right)$$

$$xhr_{t,40} = \left(\hat{P}_{t,39} - \hat{P}_{t-1,40}\right) - \hat{r}_{t-1}$$

7. Ratio of government spending to output
We have

$$\begin{aligned} \log\left(\frac{g}{y}\right)_t^{obs} &\equiv \log\left(\frac{z_t^* G_t}{z_t^* Y_t}\right) = \log\left(\frac{G}{Y}\right)_t \\ &= \log\left(\frac{G}{Y}\right)_{ss} + \log\left(\frac{G}{Y}\right)_t - \log\left(\frac{G}{Y}\right)_{ss} \\ &= \log\left(\frac{G}{Y}\right)_{ss} + \widehat{\left(\frac{G}{Y}\right)}_t \end{aligned}$$

8. log of hours
Recall
$h_t = h_{ss} e^{\log h_t - \log h_{ss}} = h_{ss} e^{\hat{h}_t}$
$\Downarrow$

$$\log h_t = \log h_{ss} + \hat{h}_t$$

Note that we scale the empirical measure for average hours per week by 5×24 in order to normalize its value to take values between 0 and 1.

16.16.2 Moments of growth rates and excess holding period return

This section discusses how to compute moments including consumption growth and investment growth. For numerical convenience, we prefer to only have $\mathbf{x}_t$ as the state variable when solving the DSGE model.

1. Consumption growth
$\log \mu_{c,t}^{obs} = \hat{C}_t - \hat{C}_{t-1} + \log \mu_{z^*,ss}$

$$= \left(\mathbf{C}^{(i)}\left(c_t, :\right) \mathbf{z}_t^{(i)} + \mathbf{d}^{(i)}\left(c_t, 1\right)\right) - \left(\mathbf{C}^{(i)}\left(c_t, :\right) \mathbf{z}_{t-1}^{(i)} + \mathbf{d}^{(i)}\left(c_t, 1\right)\right) + \log \mu_{z^*,ss}$$

for approximation order i where we use the compact representation of the pruned state space system
Note that $\mathbf{C}^{(i)}\left(c_t, :\right)$ is the row of matrix $\mathbf{C}^{(i)}$ which relates to consumption. A similar notation is used for $\mathbf{d}^{(i)}\left(c_t, 1\right)$.

$$= \mathbf{C}^{(i)}\left(c_t, :\right) \mathbf{z}_t^{(i)} - \mathbf{C}^{(i)}\left(c_t, :\right) \mathbf{z}_{t-1}^{(i)} + \log \mu_{z^*,ss}$$

$$= \log \mu_{z^*,ss} + \begin{bmatrix} \mathbf{C}^{(i)}\left(c_t, :\right) & -\mathbf{C}^{(i)}\left(c_t, :\right) \end{bmatrix} \begin{bmatrix} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{bmatrix}$$

2. Investment growth

$\log \mu_{i,t}^{obs} = \hat{I}_t - \hat{I}_{t-1} + \log \mu_{z^*,ss} + \log \mu_{\Upsilon,ss}$

$$= \log \mu_{z^*,ss} + \log \mu_{\Upsilon,ss} + \hat{I}_t - \hat{I}_{t-1}$$

$$= \log \mu_{z^*,ss} + \log \mu_{\Upsilon,ss} + \left(\mathbf{C}^{(i)}\left(i_t,:\right)\mathbf{z}_t^{(i)} + \mathbf{d}^{(i)}\left(i_t,1\right)\right) - \left(\mathbf{C}^{(i)}\left(i_t,:\right)\mathbf{z}_{t-1}^{(i)} + \mathbf{d}^{(i)}\left(i_t,1\right)\right)$$

$$= \log \mu_{z^*,ss} + \log \mu_{\Upsilon,ss} + \mathbf{C}^{(i)}\left(i_t,:\right)\mathbf{z}_t^{(i)} - \mathbf{C}^{(i)}\left(i_t,:\right)\mathbf{z}_{t-1}^{(i)}$$

$$= \log \mu_{z^*,ss} + \log \mu_{\Upsilon,ss} + \left[\begin{array}{cc} \mathbf{C}^{(i)}\left(i_t,:\right) & -\mathbf{C}^{(i)}\left(i_t,:\right) \end{array}\right]\left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right]$$

3. Inflation

$\log \pi_t = \log \pi_{ss} + \hat{\pi}_t.$

$$= \log \pi_{ss} + \left[\begin{array}{cc} \mathbf{C}^{(i)}\left(\pi_t,:\right) & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right] + \mathbf{d}^{(i)}\left(\pi_t,1\right)$$

4. One-period nominal interest rate

$r_t = r_{ss} + \hat{r}_t$

$$= r_{ss} + \left[\begin{array}{cc} \mathbf{C}^{(i)}\left(r_t,:\right) & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right] + \mathbf{d}^{(i)}\left(r_t,1\right)$$

5. 40-period nominal interest rate

$r_{t,40} = r_{ss} + \hat{r}_{t,40},$

$$= r_{ss} + \left[\begin{array}{cc} \mathbf{C}^{(i)}\left(r_{t,40},:\right) & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right] + \mathbf{d}^{(i)}\left(r_{t,40},1\right)$$

6. Excess holding-period return for the 40-period bond

We compute the ex post excess holding period return by

$xhr_{t,40} = \left(\hat{P}_{t,39} - \hat{P}_{t-1,40}\right) - \hat{r}_{t-1}$

$$= \left(\mathbf{C}^{(i)}\left(P_{t,39},:\right)\mathbf{z}_t^{(i)} + \mathbf{d}^{(i)}\left(P_{t,39},1\right)\right) - \left(\mathbf{C}^{(i)}\left(P_{t,40},:\right)\mathbf{z}_{t-1}^{(i)} + \mathbf{d}^{(i)}\left(P_{t,40},1\right)\right)$$
$$- \left(\mathbf{C}^{(i)}\left(r_t,:\right)\mathbf{z}_{t-1}^{(i)} + \mathbf{d}^{(i)}\left(r_t,1\right)\right)$$

$$= \mathbf{C}^{(i)}\left(P_{t,39},:\right)\mathbf{z}_t^{(i)} + \mathbf{d}^{(i)}\left(P_{t,39},1\right) - \left(\mathbf{C}^{(i)}\left(P_{t,40},:\right) - \mathbf{C}^{(i)}\left(r_t,:\right)\right)\mathbf{z}_{t-1}^{(i)} - \mathbf{d}^{(i)}\left(P_{t,40},1\right) - \mathbf{d}^{(i)}\left(r_t,1\right)$$

$$= \mathbf{C}^{(i)}\left(P_{t,39},:\right)\mathbf{z}_t^{(i)} - \left(\mathbf{C}^{(i)}\left(P_{t,40},:\right) - \mathbf{C}^{(i)}\left(r_t,:\right)\right)\mathbf{z}_{t-1}^{(i)} + \mathbf{d}^{(i)}\left(P_{t,39},1\right) - \mathbf{d}^{(i)}\left(P_{t,40},1\right) - \mathbf{d}^{(i)}\left(r_t,1\right)$$

7. Ratio of government spending to output

$\log\left(\frac{g}{y}\right)_t^{obs} = \log\left(\frac{G}{Y}\right)_{ss} + \widehat{\left(\frac{G}{Y}\right)}_t$

$$= \log\left(\frac{G}{Y}\right)_{ss} + \mathbf{C}^{(i)}\left(\frac{G}{Y},:\right)\mathbf{z}_t^{(i)} + \mathbf{d}^{(i)}\left(\frac{G}{Y},1\right)$$

$$= \log\left(\frac{G}{Y}\right)_{ss} + \left[\begin{array}{cc} \mathbf{C}^{(i)}\left(\frac{G}{Y},:\right) & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right] + \mathbf{d}^{(i)}\left(\frac{G}{Y},1\right)$$

8. Log of hours

$\log h_t = \log h_{ss} + \hat{h}_t$

$$= \log h_{ss} + \left[\begin{array}{cc} \mathbf{C}^{(i)}(h_t,:) & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right] + \mathbf{d}^{(i)}(h_t,1)$$

To summarize we have

$$\mathbf{y}_t^{obs} \equiv \left[\begin{array}{c} \log\mu_{c,t}^{obs} \\ \log\mu_{i,t}^{obs} \\ \log\pi_t \\ r_t \\ r_{t,40} \\ xhr_{t,40} \\ \log\left(\frac{G}{Y}\right)_t^{obs} \\ \log h_t \end{array}\right] = \left[\begin{array}{c} \log\mu_{z^*,ss} \\ \log\mu_{z^*,ss} + \log\mu_{\Upsilon,ss} \\ \log\pi_{ss} + \mathbf{d}^{(i)}(\pi_t,1) \\ \log R_{ss} + \mathbf{d}^{(i)}(r_t,1) \\ \log R_{ss} + \mathbf{d}^{(i)}(r_{t,40},1) \\ \mathbf{d}^{(i)}(P_{t,39},1) - \mathbf{d}^{(i)}(P_{t,40},1) - \mathbf{d}^{(i)}(r_t,1) \\ \log\left(\frac{G}{Y}\right)_{ss} + \mathbf{d}^{(i)}\left(\frac{G}{Y},1\right) \\ \log h_{ss} + \mathbf{d}^{(i)}(h_t,1) \end{array}\right]$$

$$+ \left[\begin{array}{cc} \mathbf{C}^{(i)}(c_t,:) & -\mathbf{C}^{(i)}(c_t,:) \\ \mathbf{C}^{(i)}(i_t,:) & -\mathbf{C}^{(i)}(i_t,:) \\ \mathbf{C}^{(i)}(\pi_t,:) & \mathbf{0} \\ \mathbf{C}^{(i)}(r_t,:) & \mathbf{0} \\ \mathbf{C}^{(i)}(r_{t,40},:) & \mathbf{0} \\ \mathbf{C}^{(i)}(P_{t,39},:) & -\mathbf{C}^{(i)}(P_{t-1,40},:) - \mathbf{C}^{(i)}(r_t,:) \\ \mathbf{C}^{(i)}\left(\frac{G}{Y},:\right) & \mathbf{0} \\ \mathbf{C}^{(i)}(h_t,:) & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right]$$

$$= \left[\begin{array}{c} \log\mu_{z^*,ss} \\ \log\mu_{z^*,ss} + \log\mu_{\Upsilon,ss} \\ \log\pi_{ss} + \mathbf{d}^{(i)}(\pi_t,1) \\ \log R_{ss} + \mathbf{d}^{(i)}(r_t,1) \\ \log R_{ss} + \mathbf{d}^{(i)}(r_{t,40},1) \\ \mathbf{d}^{(i)}(P_{t,39},1) - \mathbf{d}^{(i)}(P_{t,40},1) - \mathbf{d}^{(i)}(r_t,1) \\ \log\left(\frac{G}{Y}\right)_{ss} + \mathbf{d}^{(i)}\left(\frac{G}{Y},1\right) \\ \log h_{ss} + \mathbf{d}^{(i)}(h_t,1) \end{array}\right] + \left[\begin{array}{cc} \tilde{\mathbf{C}}_{1,1}^{(i)} & \tilde{\mathbf{C}}_{1,2}^{(i)} \\ \tilde{\mathbf{C}}_{2,1}^{(i)} & \tilde{\mathbf{C}}_{2,2}^{(i)} \\ \tilde{\mathbf{C}}_{3,1}^{(i)} & \tilde{\mathbf{C}}_{3,2}^{(i)} \\ \tilde{\mathbf{C}}_{4,1}^{(i)} & \tilde{\mathbf{C}}_{4,2}^{(i)} \\ \tilde{\mathbf{C}}_{5,1}^{(i)} & \tilde{\mathbf{C}}_{5,2}^{(i)} \\ \tilde{\mathbf{C}}_{6,1}^{(i)} & \tilde{\mathbf{C}}_{6,2}^{(i)} \\ \tilde{\mathbf{C}}_{7,1}^{(i)} & \tilde{\mathbf{C}}_{7,2}^{(i)} \\ \tilde{\mathbf{C}}_{8,1}^{(i)} & \tilde{\mathbf{C}}_{8,2}^{(i)} \end{array}\right] \left[\begin{array}{c} \mathbf{z}_t^{(i)} \\ \mathbf{z}_{t-1}^{(i)} \end{array}\right]$$

Computing all mean values are straightforward. It is more difficult to compute all second moments and their formulas are derived below. Ignoring the constant term we have

$$y_{j,t}^{obs} = \left\{\tilde{\mathbf{C}}_{j,1}^{(i)}\mathbf{z}_t^{(i)} + \tilde{\mathbf{C}}_{j,2}^{(i)}\mathbf{z}_{t-1}^{(i)}\right\}_{j=1}^{7}$$

So all contemporary co-variances are given by

$$Cov\left(y_{j,t}^{obs}, y_{k,t}^{obs}\right) = Cov\left(\tilde{\mathbf{C}}_{j,1}^{(i)}\mathbf{z}_t^{(i)} + \tilde{\mathbf{C}}_{j,2}^{(i)}\mathbf{z}_{t-1}^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_t^{(i)} + \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1}^{(i)}\right)$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_t^{(i)} + \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1}^{(i)}\right)$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_t^{(i)} + \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1}^{(i)}\right)$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} \left(Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)' \right)$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} \left(Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)' \right)$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Var\left(\mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Var\left(\mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

for $j, k = 1, 2, ..., 7$

All covariances of the form $Cov\left(y_{j,t}^{obs}, y_{k,t-s}^{obs}\right)$ are given by

$$Cov\left(y_{j,t}^{obs}, y_{k,t-s}^{obs}\right) = Cov\left(\tilde{\mathbf{C}}_{j,1}^{(i)}\mathbf{z}_t^{(i)} + \tilde{\mathbf{C}}_{j,2}^{(i)}\mathbf{z}_{t-1}^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_{t-s}^{(i)} + \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1-s}^{(i)}\right)$$

$$= Cov\left(\tilde{\mathbf{C}}_{j,1}^{(i)}\mathbf{z}_t^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_{t-s}^{(i)} + \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1-s}^{(i)}\right)$$
$$+ Cov\left(\tilde{\mathbf{C}}_{j,2}^{(i)}\mathbf{z}_{t-1}^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_{t-s}^{(i)} + \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1-s}^{(i)}\right)$$

$$= Cov\left(\tilde{\mathbf{C}}_{j,1}^{(i)}\mathbf{z}_t^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_{t-s}^{(i)}\right) + Cov\left(\tilde{\mathbf{C}}_{j,1}^{(i)}\mathbf{z}_t^{(i)}, \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1-s}^{(i)}\right)$$
$$+ Cov\left(\tilde{\mathbf{C}}_{j,2}^{(i)}\mathbf{z}_{t-1}^{(i)}, \tilde{\mathbf{C}}_{k,1}^{(i)}\mathbf{z}_{t-s}^{(i)}\right) + Cov\left(\tilde{\mathbf{C}}_{j,2}^{(i)}\mathbf{z}_{t-1}^{(i)}, \tilde{\mathbf{C}}_{k,2}^{(i)}\mathbf{z}_{t-1-s}^{(i)}\right)$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-s}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1-s}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-s}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-1-s}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

for $j, k = 1, 2, ..., 7$ and for all $s = 1, 2, ...$

So for instance (1 lag):

$$Cov\left(y_{j,t}^{obs}, y_{k,t-1}^{obs}\right) = \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-2}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-2}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-2}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-2}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-2}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Var\left(\mathbf{z}_t^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-1}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

and with 10 lags (for instance)

$$Cov\left(y_{j,t}^{obs}, y_{k,t-10}^{obs}\right) = \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-10}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-11}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+ \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-10}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_{t-1}^{(i)}, \mathbf{z}_{t-11}^{(i)}\right) \left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

$$= \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-10}^{(i)}\right)\left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,1}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-11}^{(i)}\right)\left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$
$$+\tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-9}^{(i)}\right)\left(\tilde{\mathbf{C}}_{k,1}^{(i)}\right)' + \tilde{\mathbf{C}}_{j,2}^{(i)} Cov\left(\mathbf{z}_t^{(i)}, \mathbf{z}_{t-10}^{(i)}\right)\left(\tilde{\mathbf{C}}_{k,2}^{(i)}\right)'$$

16.17 Understanding the dynamics of the price dispersion index

First note that (for $\chi = 0$)

$$1 = (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}$$
$$\Updownarrow$$
$$1 - \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta} = (1-\alpha)\tilde{p}_t^{1-\eta}$$
$$\Updownarrow$$
$$\frac{1-\alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}}{(1-\alpha)} = \tilde{p}_t^{1-\eta}$$
$$\Updownarrow$$
$$\tilde{p}_t = \left[\frac{1-\alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}}{(1-\alpha)}\right]^{\frac{1}{1-\eta}}$$

Then we note that

$$s_{t+1} = (1-\alpha)\tilde{p}_t^{-\eta} + \alpha\pi_t^{\eta} s_t$$
$$= (1-\alpha)\frac{1}{\tilde{p}_t^{\eta}} + \alpha\pi_t^{\eta} s_t$$
$$= (1-\alpha)\frac{1}{\left[\frac{1-\alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}}{1-\alpha}\right]^{\frac{\eta}{1-\eta}}} + \alpha\pi_t^{\eta} s_t$$
$$= (1-\alpha)\left[\frac{1-\alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}}{1-\alpha}\right]^{\frac{\eta}{\eta-1}} + \alpha\pi_t^{\eta} s_t$$
$$= (1-\alpha)\left(\frac{1}{1-\alpha}\right)^{\frac{\eta}{\eta-1}}\left[1-\alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}\right]^{\frac{\eta}{\eta-1}} + \alpha\pi_t^{\eta} s_t$$
$$= (1-\alpha)^{\frac{1-\eta}{1-\eta}}(1-\alpha)^{\frac{\eta}{1-\eta}}\left[1-\alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}\right]^{\frac{\eta}{\eta-1}} + \alpha\pi_t^{\eta} s_t$$
$$= (1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_t^{\eta-1}\right]^{\frac{\eta}{\eta-1}} + \alpha\pi_t^{\eta} s_t$$

Note also that the steady state value is given by

$$s_{ss} = (1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}} + \alpha\pi_{ss}^{\eta} s_{ss}$$
$$\Updownarrow$$
$$s_{ss} = \frac{(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}}{1-\alpha\pi_{ss}^{\eta}}$$

and therefore

$$\frac{\partial s_{ss}}{\partial \pi_{ss}} = \frac{\frac{\eta}{\eta-1}(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}-1}\left(-\alpha(\eta-1)\pi_{ss}^{\eta-2}\right)(1-\alpha\pi_{ss}^{\eta}) - (1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}\left(-\alpha\eta\pi_{ss}^{\eta-1}\right)}{(1-\alpha\pi_{ss}^{\eta})^2}$$
$$= \frac{\frac{-\eta}{\eta-1}(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}-1}\alpha(\eta-1)\pi_{ss}^{\eta-2}(1-\alpha\pi_{ss}^{\eta}) + (1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}\alpha\eta\pi_{ss}^{\eta-1}}{(1-\alpha\pi_{ss}^{\eta})^2}$$

$$= \frac{(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}}{(1-\alpha\pi_{ss}^{\eta})^2}\left[-\eta\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{-1}\alpha\pi_{ss}^{\eta-2}\left(1-\alpha\pi_{ss}^{\eta}\right)+\alpha\eta\pi_{ss}^{\eta-1}\right]$$

$$= \frac{\alpha\eta(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}}{(1-\alpha\pi_{ss}^{\eta})^2}\left[-\frac{1-\alpha\pi_{ss}^{\eta}}{1-\alpha\pi_{ss}^{\eta-1}}\pi_{ss}^{\eta-2}+\pi_{ss}^{\eta-1}\right]$$

$$= \frac{\pi_{ss}^{\eta-2}\alpha\eta(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}}{(1-\alpha\pi_{ss}^{\eta})^2}\left[-\frac{1-\alpha\pi_{ss}^{\eta}}{1-\alpha\pi_{ss}^{\eta-1}}+\pi_{ss}\right]$$

$$= \frac{\pi_{ss}^{\eta-2}\alpha\eta(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}}{(1-\alpha\pi_{ss}^{\eta})^2}\left[\frac{-1+\alpha\pi_{ss}^{\eta}}{1-\alpha\pi_{ss}^{\eta-1}}+\frac{\pi_{ss}-\alpha\pi_{ss}^{\eta}}{1-\alpha\pi_{ss}^{\eta-1}}\right]$$

$$= \frac{\pi_{ss}^{\eta-2}\alpha\eta(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_{ss}^{\eta-1}\right]^{\frac{\eta}{\eta-1}}}{(1-\alpha\pi_{ss}^{\eta})^2}\left[\frac{\pi_{ss}-1}{1-\alpha\pi_{ss}^{\eta-1}}\right]\geq 0$$

for $\pi_{ss}\geq 1$

16.18 Understanding term premium in the model

We start by computing an expression for term premium as in Rudebusch & Swanson (2012), after which we look at the excess holding period return

16.18.1 Term premium

We define term premia as $TP_{t,k}=r_{t,k}-\tilde{r}_{t,k}$, where $\tilde{r}_{t,k}$ is the yield-to-maturity on a zero-coupon bond $\tilde{P}_{t,k}$ under risk-neutral valuation by the financial intermediary, i.e. $\tilde{P}_{t,k}=e^{-r_t}\mathbb{E}_t\left[\tilde{P}_{t+1,k-1}\right]$. Hence,

$TP_{t,k}=r_{t,k}-\tilde{r}_{t,k}$

$$= \frac{-1}{k}\log P_{t,k}-\left(\frac{-1}{k}\log\hat{P}_{t,k}\right)$$

$$= \frac{1}{k}\left(\log\hat{P}_{t,k}-\log P_{t,k}\right)$$

$$\approx \frac{1}{k}\left(\log\hat{P}_{ss,k}+\frac{1}{\hat{P}_{ss,k}}\left(\hat{P}_{t,k}-\hat{P}_{ss,k}\right)-\left(\log P_{ss,k}+\frac{1}{P_{ss,k}}\left(P_{t,k}-P_{ss,k}\right)\right)\right)$$

$\log x_t\approx\log x_{ss}+\frac{1}{x_{ss}}\left(x_t-x_{ss}\right)$

$$= \frac{1}{k}\left(\log P_{ss,k}+\frac{1}{P_{ss,k}}\left(\hat{P}_{t,k}-P_{ss,k}\right)-\left(\log P_{ss,k}+\frac{1}{P_{ss,k}}\left(P_{t,k}-P_{ss,k}\right)\right)\right)$$

because $P_{ss,k}=\hat{P}_{ss,k}$

$$= \frac{1}{k}\left(\frac{1}{P_{ss,k}}\hat{P}_{t,k}-\frac{1}{P_{ss,k}}P_{t,k}\right)$$

$$= \frac{-1}{kP_{ss,k}}\left(P_{t,k}-\hat{P}_{t,k}\right)$$

Next note as in

$P_{t,k}-\hat{P}_{t,k}=E_t\left[M_{t,t+1}P_{t+1,k-1}\right]-e^{-r_t}E_t\left[\tilde{P}_{t+1,k-1}\right]$

$$= Cov_t\left(M_{t,t+1},P_{t+1,k-1}\right)+E_t\left[M_{t,t+1}\right]E_t\left[P_{t+1,k-1}\right]-e^{-r_t}E_t\left[\tilde{P}_{t+1,k-1}\right]$$

$$= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right) + e^{-r_t-\omega\times xhr_{t,L}} E_t\left[P_{t+1,k-1}\right] - e^{-r_t} E_t\left[\tilde{P}_{t+1,k-1}\right]$$

because $E_t\left[M_{t,t+1}\right] = e^{-r_t-\omega\times xhr_{t,L}}$

$$= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right) + e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} E_t\left[P_{t+1,k-1}\right] - E_t\left[\tilde{P}_{t+1,k-1}\right]\right\}$$

$$= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right) + E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} P_{t+1,k-1} - \tilde{P}_{t+1,k-1}\right\}$$

$$= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right) + E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} E_{t+1}\left[M_{t+1,t+2} P_{t+2,k-2}\right] - e^{-r_{t+1}} E_{t+1}\left[\tilde{P}_{t+2,k-2}\right]\right\}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right) + e^{-\omega\times xhr_{t,L}} E_{t+1}\left[M_{t+1,t+2}\right] E_{t+1}\left[P_{t+2,k-2}\right] - e^{-r_{t+1}} E_{t+1}\left[\tilde{P}_{t+2,k-2}\right]\right\}\end{aligned}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right) + e^{-\omega\times xhr_{t,L}} e^{-r_{t+1}-\omega\times xhr_{t+1,L}} E_{t+1}\left[P_{t+2,k-2}\right] - e^{-r_{t+1}} E_{t+1}\left[\tilde{P}_{t+2,k-2}\right]\right\}\end{aligned}$$

because $E_{t+1}\left[M_{t+1,t+2}\right] = e^{-r_{t+1}-\omega\times xhr_{t+1,L}}$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{\begin{array}{c} e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right) \\ +e^{-r_{t+1}}\left\{e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} E_{t+1}\left[P_{t+2,k-2}\right] - E_{t+1}\left[\tilde{P}_{t+2,k-2}\right]\right\}\end{array}\right\}\end{aligned}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right)\right\}\\ &+E_t e^{-r_t-r_{t+1}}\left\{e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} P_{t+2,k-2} - \tilde{P}_{t+2,k-2}\right\}\end{aligned}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right)\right\}\\ &+E_t e^{-r_t-r_{t+1}}\left\{e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} E_{t+2}\left[M_{t+2,t+3} P_{t+3,k-3}\right] - e^{-r_{t+2}} E_{t+2}\left[\tilde{P}_{t+3,k-3}\right]\right\}\end{aligned}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right)\right\}\\ &+E_t e^{-r_t-r_{t+1}}\left\{\begin{array}{c} e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} Cov_{t+2}\left(M_{t+2,t+3}, P_{t+3,k-3}\right) \\ +e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} E_{t+2}\left[M_{t+2,t+3}\right] E_{t+2}\left[P_{t+3,k-3}\right] \\ -e^{-r_{t+2}} E_{t+2}\left[\tilde{P}_{t+3,k-3}\right]\end{array}\right\}\end{aligned}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right)\right\}\\ &+E_t e^{-r_t-r_{t+1}}\left\{\begin{array}{c} e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} Cov_{t+2}\left(M_{t+2,t+3}, P_{t+3,k-3}\right) \\ +e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} e^{-r_{t+2}-\omega\times xhr_{t+2,L}} E_{t+2}\left[P_{t+3,k-3}\right] \\ -e^{-r_{t+2}} E_{t+2}\left[\tilde{P}_{t+3,k-3}\right]\end{array}\right\}\end{aligned}$$

$$\begin{aligned}&= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)\\ &+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}} Cov_{t+1}\left(M_{t+1,t+2}, P_{t+2,k-2}\right)\right\}\\ &+E_t e^{-r_t-r_{t+1}}\left\{\begin{array}{c} e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}\right)} Cov_{t+2}\left(M_{t+2,t+3}, P_{t+3,k-3}\right) \\ +e^{-r_{t+2}}\left\{e^{-\omega\times\left(xhr_{t,L}+xhr_{t+1,L}+xhr_{t+2,L}\right)} E_{t+2}\left[P_{t+3,k-3}\right] - E_{t+2}\left[\tilde{P}_{t+3,k-3}\right]\right\}\end{array}\right\}\end{aligned}$$

$$= Cov_t\left(M_{t,t+1}, P_{t+1,k-1}\right)$$

$$+E_t e^{-r_t}\left\{e^{-\omega\times xhr_{t,L}}Cov_{t+1}\left(M_{t+1,t+2},P_{t+2,k-2}\right)\right\}$$
$$+E_t e^{-r_t-r_{t+1}}\left\{e^{-\omega\times(xhr_{t,L}+xhr_{t+1,L})}Cov_{t+2}\left(M_{t+2,t+3},P_{t+3,k-3}\right)\right\}$$
$$+E_t e^{-r_t-r_{t+1}-r_{t+2}}\left\{e^{-\omega\times(xhr_{t,L}+xhr_{t+1,L}+xhr_{t+2,L})}P_{t+3,k-3}-\tilde{P}_{t+3,k-3}\right\}$$

and hence in general

$$\begin{aligned} P_{t,k}-\hat{P}_{t,k} &= E_t\left[\sum_{j=0}^{k-1}e^{-\sum_{m=0}^{j-1}r_{t+m}+\omega\times xhr_{t+m,L}}Cov_{t+j}\left(M_{t+j,t+j+1},P_{t+j+1,k-j-1}\right)\right] \\ &\quad +E_t\left[e^{-\sum_{m=0}^{k-1}\omega\times xhr_{t+m,L}}-1\right] \end{aligned}$$

Thus, we get
$TP_{t,k}=\frac{-1}{kP_{ss,k}}\left(P_{t,k}-\hat{P}_{t,k}\right)$
$\Updownarrow$

$$TP_{t,k}=\frac{-1}{kP_{ss,k}}\left\{\begin{array}{c} E_t\left[\sum_{j=0}^{k-1}e^{-\sum_{m=0}^{j-1}(r_{t+m}+\omega\times xhr_{t+m,L})}Cov_{t+j}\left(M_{t+j,t+j+1},P_{t+j+1,k-j-1}\right)\right] \\ +E_t\left[e^{-\sum_{m=0}^{k-1}\omega\times xhr_{t+m,L}}-1\right]\end{array}\right\}$$

Note that if $\omega=0$, then we get

$$\begin{aligned} TP_{t,k} &= \frac{-1}{kP_{ss,k}}\left\{E_t\left[\sum_{j=0}^{k-1}e^{-\sum_{m=0}^{j-1}r_{t+m}}Cov_{t+j}\left(M_{t+j,t+j+1},P_{t+j+1,k-j-1}\right)\right]+E_t\left[e^0-1\right]\right\} \\ &= \frac{-1}{kP_{ss,k}}E_t\left[\sum_{j=0}^{k-1}e^{-\sum_{m=0}^{j-1}r_{t+m}}Cov_{t+j}\left(M_{t+j,t+j+1},P_{t+j+1,k-j-1}\right)\right] \end{aligned}$$

as in Rudebusch & Swanson (2012), their Eq 35. Given that $\omega\times xhr_{t+m,L}$ generally is positive, we see that $\omega>0$ adds more discounting to $Cov_{t+j}\left(M_{t+j,t+j+1},P_{t+j+1,k-j-1}\right)$ but it also makes $e^{-\sum_{m=0}^{k-1}\omega\times xhr_{t+m,L}}$ substantially less than 1 (i.e. increases the final term in absolute value) and this serves to increase term premia. Note that this final term arises because we price the risk-neutral bond $\tilde{P}_{t,k}$ by r_t and not by the deposite rate $r_t+\omega\times xhr_{t,L}$ which is only provided to households and not to the bond trading financial intermediary.

16.18.2 The excess holding period return

To understand the dynamic properties of term premium in the model consider the ex ante excess holding period return:
$xhr_{t,L}\equiv\mathbb{E}_t\left[\log\left(P_{t+1,L-1}/P_{t,L}\right)\right]-r_t$

$=\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)-\log P_{t,L}\right]-r_t$

$=\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)-\log\left\{\mathbb{E}_t\left[M_{t,t+1}P_{t+1,L-1}\right]\right\}\right]-r_t$

$=\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)-\log\left\{\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right]+Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)\right\}\right]-r_t$

Now recall that a first-order approximation implies
$\log\left(x_t+y_t\right)=\log\left(x_{ss}+y_{ss}\right)+\frac{1}{x_{ss}+y_{ss}}\left(x_t-x_{ss}\right)+\frac{1}{x_{ss}+y_{ss}}\left(y_t-y_{ss}\right)$
Applied in our case with $x_t=\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right]$ and $y_t=Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)$, implying that $x_t+y_t=P_{t,L}$, we have
$xhr_{t,L}\approx\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)-\left\{\log P_{ss,L}+\frac{1}{P_{ss,L}}\left(\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right]-M_{ss,ss+1}P_{ss,L-1}\right)+\frac{1}{P_{ss,L}}Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)\right\}\right]$

$-r_t$

$= \mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right) - \log P_{ss,L} - \frac{1}{P_{ss,L}}\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right] + \frac{M_{ss,ss+1}P_{ss,L-1}}{P_{ss,L}} - \frac{1}{P_{ss,L}}Cov_t\left(M_{t,t+1}, P_{t+1,L-1}\right)\right]$
$-r_t$

$= \mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)\right] - \log P_{ss,L} - \frac{1}{P_{ss,L}}\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right] + \frac{M_{ss,ss+1}P_{ss,L-1}}{P_{ss,L}} - \frac{1}{P_{ss,L}}Cov_t\left(M_{t,t+1}, P_{t+1,L-1}\right)$
$-r_t$

$= \mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)\right] - \log P_{ss,L} - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L}}e^{-r_t-\omega\times xhr_{t,L}} + 1 - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}} - r_t$
because $\mathbb{E}_t\left[M_{t,t+1}e^{r_t+\omega\times xhr_{t,L}}\right] = 1 \iff e^{-r_t-\omega\times xhr_{t,L}} = \mathbb{E}_t\left[M_{t,t+1}\right]$
and $P_{ss,L} = M_{ss,ss+1}P_{ss,L-1}$

$= \mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L}}e^{-r_t-\omega\times xhr_{t,L}} - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}} + 1 - r_t$

Now consider the first-order approximation round x_{ss}:
$e^{x_t} \approx e^{x_{ss}} + e^{x_{ss}}\left(x_t - x_{ss}\right) = e^{x_{ss}}\left(1 + x_t - x_{ss}\right)$
Applied to $x_t = -r_t - \omega \times xhr_{t,L}$ and $x_{ss} = -r_{ss}$, we get
$e^{-r_t-\omega\times xhr_{t,L}} \approx e^{-r_{ss}}\left(1 - r_t - \omega \times xhr_{t,L} + r_{ss}\right)$
Thus we get
$xhr_{t,L} \approx \mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L}}\left(e^{-r_{ss}}\left(1 - r_t - \omega \times xhr_{t,L} + r_{ss}\right)\right) - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}}$
$+1 - r_t$

$= \mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L}}M_{ss,ss+1}\left(1 - r_t - \omega \times xhr_{t,L} + r_{ss}\right) - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}}$
$+1 - r_t$

$= \mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1 - r_t - \omega \times xhr_{t,L} + r_{ss}\right) - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}} + 1 - r_t$
Remember that $M_{ss,ss+1}P_{ss,L-1} = P_{ss,L} \iff \frac{M_{ss,ss+1}}{P_{ss,L}} = \frac{1}{P_{ss,L-1}}$
$\Updownarrow$

$xhr_{t,L}\left(1 - \omega \times \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\right) = \mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1 - (r_t - r_{ss})\right) - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}} + 1 - r_t$
$\Updownarrow$

$$xhr_{t,L} = \frac{1}{1 - \omega \times \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}}\left[\mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}\left(1 - (r_t - r_{ss})\right) - \frac{Cov_t\left(M_{t,t+1}, P_{t+1,L-1}\right)}{P_{ss,L}} + 1 - r_t\right]$$

Note that in the steady state, we have
$xhr_{ss,L} = \frac{1}{1-\omega}\left[\mathbb{E}_t\left[\log\left(\frac{P_{ss,L-1}}{P_{ss,L}}\right)\right] - \left(1 - (r_{ss} - r_{ss})\right) + 1 - r_t\right]$

$= \frac{1}{1-\omega}\left[\log\left(\frac{P_{ss,L-1}}{P_{ss,L}}\right) - r_t\right]$

$= \frac{1}{1-\omega}\left[\log\left(\frac{P_{ss,L-1}}{M_{ss,ss+1}P_{ss,L-1}}\right) - r_{ss}\right]$
because $M_{ss,ss+1}P_{ss,L-1} = P_{ss,L}$

$= \frac{1}{1-\omega}\left[\log\left(\frac{1}{M_{ss,ss+1}}\right) - r_{ss}\right]$

$= \frac{1}{1-\omega}\left[-\log\left(M_{ss,ss+1}\right) - r_{ss}\right]$

$= \frac{1}{1-\omega}\left[-\log\left(e^{-r_{ss}}\right) - r_{ss}\right]$
using $M_{ss,ss+1} = e^{-r_{ss}}$

$= \frac{1}{1-\omega}\left[r_{ss} - r_{ss}\right]$

$= 0$
as desired.

Let us re-write the above expression a bit:
$xhr_{t,L} = xhr_{t,L} = \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{P_{ss,L}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1-(r_t - r_{ss})\right) - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{P_{ss,L}} + 1 - r_t\right]$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t\left[\log\left(\frac{P_{t+1,L-1}}{M_{ss,ss+1}P_{ss,L-1}}\right)\right] - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1-(r_t - r_{ss})\right) - \frac{Cov_t(M_{t,t+1},P_{t+1,L-1})}{M_{ss,ss+1}P_{ss,L-1}} + 1 - r_t\right]$
using $M_{ss,ss+1}P_{ss,L-1} = P_{ss,L}$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t \log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) - \log M_{ss,ss+1} - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1-(r_t - r_{ss})\right) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) + 1 - r_t\right]$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t \log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) - \log e^{-r_{ss}} - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1-(r_t - r_{ss})\right) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) + 1 - r_t\right]$
using $M_{ss,ss+1} = e^{-r_{ss}}$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t \log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) + r_{ss} - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1-(r_t - r_{ss})\right) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) + 1 - r_t\right]$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t \log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\left(1-(r_t - r_{ss})\right) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) + 1 - (r_t - r_{ss})\right]$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t \log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right) + \left(1 - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\right)\left(1-(r_t - r_{ss})\right) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)\right]$

Now a first order approximation of $\log(x_t)$ around $x_{ss} = 1$ gives
$\log(x_t) \approx \log(1) + \frac{1}{x_{ss}}(x_t - x_{ss}) = x_t - 1$
Applied in our case to $\log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)$ we get
$xhr_{t,L} \approx \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\mathbb{E}_t\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}} - 1\right) + \left(1 - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\right)\left(1-(r_t - r_{ss})\right) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)\right]$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\left(\frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}} - 1\right) + \left(1 - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\right) - \left(1 - \frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}\right)(r_t - r_{ss}) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)\right]$

$= \frac{1}{1-\omega\times\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}}\left[\left(\frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}} - 1\right)(r_t - r_{ss}) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)\right]$
That is, up to a first order approximation, we have

$$xhr_{t,L} \approx \frac{1}{1-\omega\times\frac{\mathbb{E}_t[P_{t+1,L-1}]}{P_{ss,L-1}}}\left[\left(\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}} - 1\right)(r_t - r_{ss}) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)\right]$$

Importantly, we have a multiplication effect in our mode if $\omega > 0$, meaning that the effect of $Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)$ is amplified. This explains why we can settle with lower variation in $Cov_t(\cdot)$ and hence $Var_t(M_{t+1})$ to explain term premia in our model with feedback effects.

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