1. A SMALL TWO COUNTRY, TWO SECTOR DSGE MODEL

1.1. *Aggregate investment and consumption*

There are two tradable intermediate goods ($n = A, B$) that are produced in two countries ($i = 1, 2$) which can be used for consumption and investment

(1a) $$I_t^i = [\gamma^{\frac{1}{\theta}}(J_{A,t}^i)^{1-\frac{1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(J_{B,t}^i)^{1-\frac{1}{\theta}}]^{\frac{\theta}{1-\theta}}$$

(1b) $$P^i = \left[\gamma(P_A^i)^{1-\theta} + (1-\gamma)(P_B^i)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

(1c) $$C_t^i = [\gamma^{\frac{1}{\theta}}(C_{A,t}^i)^{1-\frac{1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_{B,t}^i)^{1-\frac{1}{\theta}}]^{\frac{\theta}{1-\theta}}$$

The elasticity of substitution between good A and good B is θ, the elasticity of substitution between Home and Foreign produced goods is η.

The demand equations are:

(2a) $$C_A^i = (\gamma)\left[\frac{P_A^i}{P^i}\right]^{-\theta} C^i$$

(2b) $$C_B^i = (1-\gamma)\left[\frac{P_B^i}{P^i}\right]^{-\theta} C^i$$

(3a) $$J_A^i = (\gamma)\left[\frac{P_A^i}{P^i}\right]^{-\theta} I^i$$

(3b) $$J_B^i = (1-\gamma)\left[\frac{P_B^i}{P}\right]^{-\theta} I^i$$

1.2. *Maximization problem*

There are two countries $i = 1, 2$ that produce two tradable goods, A and B. The consumer of country 1 maximizes:

$$\begin{aligned}
\mathcal{L} = {} & E_t \sum_{j=0}^{\infty} \beta^j \left[\frac{C_{t+j}^{1-\phi}}{1-\phi}\right] \\
+ {} & \beta^j \lambda_{t+j}[P_{A,t+j} w_{A,t+j} L_{A,t+j} + P_{B,t+j} w_{B,t+j} L_{B,t+j} + P_{A,t+j} r_{A,t+j} K_{A,t+j} + P_{B,t+j} r_{B,t+j} K_{B,t+j} \\
- {} & P_{t+j} C_{t+j} - P_{A,t+j} I_{A,t+j} - P_{B,t+j} I_{B,t+j} \\
- {} & e_{t+j} \sum_{s(t+j+1)} \xi_{s(t+j+1)} B_{s(t+j+1)} + e_{t+j} B_{t+j} \\
+ {} & \beta^j q_{A,t+j}[(1-\delta)K_{A,t+j} + I_{A,t+j}] \\
+ {} & \beta^j q_{B,t+j}[(1-\delta)K_{B,t+j} + I_{B,t+j}]
\end{aligned}$$

(4)

where e denotes the nominal exchange rate.

The firm that produces A maximizes:

(5) $$\Pi_A^i = \underbrace{(K_{A,t}^i)^{\alpha_A}(A_t^i L_{A,t}^i)^{1-\alpha_A}}_{Y_A^i} - r_{A,t}^i K_{A,t}^i - w_{A,t}^i L_{A,t}^i$$

The firm that produces B maximizes:

(6) $$\Pi_B^i = \underbrace{K_{B,t}^{\alpha_B}(A_t^i L_{B,t}^i)^{1-\alpha_B}}_{Y_B^i} - r_{B,t}^i K_{B,t} - w_{B,t}^i L_{B,t}^i$$

Technology follows an autoregressive process:

(7) $$ln(A^i) = \rho ln(A_{t-1}^i) + \epsilon_t^i$$

where $\rho < 1$ and ϵ, ϵ^* are white noise.

1.3. *First order conditions*

The first order conditions in country 1 are:

(8a) $$\frac{\partial \mathcal{L}}{\partial C_{t+j}} = 0 \Leftrightarrow \frac{C_{t+j}^{-\phi}}{P_{t+j}} = \lambda_{t+j}$$

(8b) $$\frac{\partial \mathcal{L}}{\partial I_{A,t+j}} = 0 \Leftrightarrow q_{A,t+j} = \lambda_{t+j} P_{A,t+j}$$

(8c) $$\frac{\partial \mathcal{L}}{\partial I_{B,t+j}} = 0 \Leftrightarrow q_{B,t+j} = \lambda_{t+j} P_{B,t+j}$$

(8d) $$\frac{\partial \mathcal{L}}{\partial K_{A,t+j+1}} = 0 \Leftrightarrow \beta \lambda_{t+j+1} P_{A,t+j+1} r_{A,t+j+1} = 0$$

(8e) $$\frac{\partial \mathcal{L}}{\partial K_{B,t+j+1}} = 0 \Leftrightarrow \beta \lambda_{t+j+1} P_{B,t+j+1} r_{B,t+j+1} = 0$$

(8f) $$\frac{\partial \mathcal{L}}{\partial B_{s(t+j+1)}} = 0 \Leftrightarrow \lambda_{s(t+j)} e_{s(t+j)} \xi_{s(t+j+1)} = \beta Prob[s(t+j+1)] \lambda_{s(t+j+1)} e_{s(t+j+1)}$$

the first order conditions in country 2 (here denoted by * to avoid (further) confusion) are:

(9a) $$\frac{\partial \mathcal{L}^*}{\partial C^*_{t+j}} = 0 \Leftrightarrow \frac{(C^*_{t+j})^{-\phi}}{P^*_{t+j}} = \lambda^*_{t+j}$$

(9b) $$\frac{\partial \mathcal{L}^*}{\partial I^*_{A,t+j}} = 0 \Leftrightarrow q^*_{A,t+j} = \lambda^*_{t+j} P^*_{A,t+j}$$

(9c) $$\frac{\partial \mathcal{L}^*}{\partial I^*_{B,t+j}} = 0 \Leftrightarrow q^*_{B,t+j} = \lambda^*_{t+j} P^*_{B,t+j}$$

(9d) $$\frac{\partial \mathcal{L}^*}{\partial K^*_{A,t+j+1}} = 0 \Leftrightarrow \beta \lambda^*_{t+j+1} P^*_{A,t+j} r^*_{A,t+j+1} = 0$$

(9e) $$\frac{\partial \mathcal{L}^*}{\partial K^*_{B,t+j+1}} = 0 \Leftrightarrow \beta \lambda^*_{t+j+1} P^*_{B,t+j} r^*_{B,t+j+1} = 0$$

(9f) $$\frac{\partial \mathcal{L}^*}{\partial B_{s(t+j+1)}} = 0 \Leftrightarrow \lambda^*_{s(t+j)} \xi_{s(t+j+1)} = \beta Prob[s(t+j+1)] \lambda^*_{s(t+j+1)}$$

The first order conditions of the firms are:

(10a) $$\frac{\partial \Pi_{A,t+j}}{\partial K_{A,t+j}} = 0 \Leftrightarrow r_{A,t+j} = \alpha_A \frac{Y_{A,t+j}}{K_{A,t+j}}$$

(10b) $$\frac{\partial \Pi_{A,t+j}}{\partial L_{A,t+j}} = 0 \Leftrightarrow w_{A,t+j} = (1-\alpha_A) \frac{Y_{A,t+j}}{L_{A,t+j}}$$

(10c) $$\frac{\partial \Pi_{B,t+j}}{\partial K_{B,t+j}} = 0 \Leftrightarrow r_{B,t+j} = \alpha_B \frac{Y_{B,t+j}}{K_{B,t+j}}$$

(10d) $$\frac{\partial \Pi_{B,t+j}}{\partial L_{B,t+j}} = 0 \Leftrightarrow w_{B,t+j} = (1-\alpha_B) \frac{Y_{B,t+j}}{L_{B,t+j}}$$

(10e) $$\frac{\partial \Pi^*_{A,t+j}}{\partial K^*_{A,t+j}} = 0 \Leftrightarrow r^*_{A,t+j} = \alpha_A \frac{Y^*_{A,t+j}}{K^*_{A,t+j}}$$

(10f) $$\frac{\partial \Pi^*_{A,t+j}}{\partial L^*_{A,t+j}} = 0 \Leftrightarrow w^*_{A,t+j} = (1-\alpha_A) \frac{Y^*_{A,t+j}}{L^*_{A,t+j}}$$

(10g) $$\frac{\partial \Pi^*_{B,t+j}}{\partial K^*_{B,t+j}} = 0 \Leftrightarrow r^*_{B,t+j} = \alpha_B \frac{Y^*_{B,t+j}}{K^*_{B,t+j}}$$

(10h) $$\frac{\partial \Pi^*_{B,t+j}}{\partial L^*_{B,t+j}} = 0 \Leftrightarrow w^*_{B,t+j} = (1-\alpha_B) \frac{Y^*_{B,t+j}}{L^*_{B,t+j}}$$

1.4. *Complete Markets condition*

Combining the 2 bond and 2 consumption equations from both countries we obtain (j=0):

(11) $$\frac{\lambda^*_{s(t+1)}}{\lambda^*_{s(t)}} = \frac{\lambda_{s(t+1)}}{\lambda_{s(t)}} \frac{e_{s(t+1)}}{e_{s(t)}} \Leftrightarrow \frac{\frac{C_{t+1}^{-\phi}}{C_t^{-\phi}}}{\frac{(C^*_{t+1})^{-\phi}}{(C^*_t)^{-\phi}}} = \frac{e_t \frac{P^*_t}{P_t}}{e_{t+1} \frac{P^*_{t+1}}{P_{t+1}}} \Leftrightarrow \frac{\frac{U'_{t+1}}{U'_t}}{\frac{U'^*_{t+1}}{U'^*_t}} = \frac{\frac{e_t P^*_t}{P_t}}{\frac{e_{t+1} P^*_{t+1}}{P_{t+1}}}$$

Because the two countries are ex-ante symmetrical, the price levels equalize between countries because of PPP, the above equation reduces to:

(12) $$\lambda = \lambda^*$$

1.5. *Market clearing and resource constraints*

Aggregate output:

(13a) $$Y = P_A Y_A + P_B Y_B$$

(13b) $$Y^* = P_A^* Y_A^* + P_B^* Y_B^*$$

Resource contraints:

(14a) $$Y_A + Y_A^* = C_A + C_A^* + J_A + J_A^*$$

(14b) $$Y_B + Y_B^* = C_B + C_B^* + J_B + J_B^*$$

Aggregate investment:

(15a) $$I = I_A + I_B$$

(15b) $$I^* = I_A^* + I_B^*$$

1.6. *Steady State values*

The law of motion of capital implies: $K_{n,t+1} = (1-\delta)K_{n,t} + I_{n,t} \Leftrightarrow K_n^{ss} = (1-\delta)K_n^{ss} + I_n^{ss} \Leftrightarrow K_n^{ss} = \frac{I_n^{ss}}{\delta} \Leftrightarrow \delta = \frac{I_n^{ss}}{K_n^{ss}} \Leftrightarrow I_n^{ss} = \delta K_n^{ss}$. Therefore, the consumer first order conditions reduce to

(16a) $$q_n^{ss} = \lambda^{ss} P_n^{ss}$$

(16b) $$P_n^{ss} \beta \lambda^{ss} r_n^{ss} + \beta P_n^{ss} \lambda^{ss} (1-\delta) - P_n^{ss} \lambda^{ss} = 0$$

Combining these conditions with the first order conditions from the firm problem and assuming that $A_{ss} = 1$ it follows that in steady state

(17a) $$r_n^{ss} = \frac{1}{\beta} - 1 + \delta = \alpha_n \left[\frac{K_n^{ss}}{A^{ss} L_n^{ss}}\right]^{\alpha_n - 1} \Leftrightarrow \frac{K_n^{ss}}{L_n^{ss}} = \left[\frac{r_n^{ss}}{\alpha_n}\right]^{\frac{1}{\alpha_n - 1}}$$

(17b) $$P_A w_A = P_B w_B = P_n (1-\alpha_n) \left[\frac{K_n^{ss}}{L_n^{ss}}\right]^{\alpha_n}$$

From the two preceding equations and assuming that $P_A^{ss} = 1$ it follows that

(18a) $$P_B^{ss} = \frac{(1-\alpha_A)\left[\frac{r^{ss}}{\alpha_A}\right]^{\frac{\alpha_A}{\alpha_A - 1}}}{(1-\alpha_B)\left[\frac{r^{ss}}{\alpha_B}\right]^{\frac{\alpha_B}{\alpha_B - 1}}}$$

(18b) $$P^{ss} = \left[\gamma 1^{1-\theta} + (1-\gamma)(P_B^{ss})^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

Assume that labor supply equals one ($L_A = L_B = L_A^* = L_B^* = 1$). Using this equation we can back out the wages:

(19a) $$w_A^{ss} = (1-\alpha_A)\left[\frac{r^{ss}}{\alpha_A}\right]^{\frac{\alpha_A}{\alpha_A - 1}}$$

(19b) $$w_B^{ss} = \frac{w_A^{ss}}{P_B^{ss}} = (1-\alpha_B)\left[\frac{r^{ss}}{\alpha_B}\right]^{\frac{\alpha_B}{\alpha_B - 1}}$$

From the preceding equation we can back out the steady state consumption demands

(20a) $$C^{ss} = Y_{ss} - I_{ss}$$

(20b) $$C_A^{ss} = \gamma \left[\frac{P_A^{ss}}{P^{ss}}\right]^{-\theta} C^{ss}$$

(20c) $$C_B^{ss} = (1-\gamma)\left[\frac{P_B^{ss}}{P^{ss}}\right]^{-\theta} C^{ss}$$

If the two countries are symmetric, $P = P^*$ and therefore $e = e^* = 1$.

(21a) $$K_n^{ss} = \left[\frac{r_n^{ss}}{\alpha_n}\right]^{\frac{1}{\alpha_n - 1}} A^{ss} L_n^{ss} = \frac{I_n^{ss}}{\delta}$$

(21b) $$I_n^{ss} = \delta \left[\frac{r_n^{ss}}{\alpha_n}\right]^{\frac{1}{\alpha_n - 1}} A^{ss} L_n^{ss}$$

(21c) $$Y_n^{ss} = \left[\frac{r_n^{ss}}{\alpha_n}\right]^{\frac{\alpha_n}{\alpha_n - 1}} A^{ss} L_n^{ss}$$

To backing out the sectoral labor supplies is more difficult. Aggregate domestic output is given by $Y = P_A Y_A + P_B Y_B$. Because supply equals demand in equilibrium, $Y_B = C_{BH} + J_{BH} + C^*_{BF} + J^*_{BF}$ and $Y_A = C_{AH} + J_{AH} + C^*_{AF} + J^*_{AF}$. We can substitute our demand equations (6) and (7) into the market clearing conditions (49) and (50) to obtain

(22a) $$Y_A = \gamma \left[\frac{P_A}{P}\right]^{-\theta} [\omega C + \omega I + (1-\omega) I^* + (1-\omega) C^*] \Leftrightarrow Y_A P_A = \gamma P^\theta P_A^{1-\theta} [...]$$

(22b) $$Y_B = (1-\gamma) \left[\frac{P_B}{P}\right]^{-\theta} [\omega C + \omega I + (1-\omega) I^* + (1-\omega) C^*] \Leftrightarrow Y_B P_B = (1-\gamma) P^\theta P_B^{1-\theta} [...]$$

(22c) $$Y = \gamma P^\theta P_A^{1-\theta} [...] + (1-\gamma) P^\theta P_B^{1-\theta} [...] = [\gamma P_A^{1-\theta} + (1-\gamma) P_B^{1-\theta}] P^\theta [...] = P^{1-\theta} P^\theta [...]$$