Digression: Difference equations

- Before solving the whole matrix equation (112), we first explore **univariate difference equations**
- Key distinction
  - Pre-determined variables (such as state variables)
  - Control variables (or forward-looking variables)
- Introduce **generic notation** for the univariate case:
  - y_t now denotes the endogenous variable
  - ϵ_t denotes an exogenous variable
- Both are scalars
- A **solution** relates y_t to past, present, and future values of ϵ_t (and a boundary condition)

Pre-determined Variables

- Assume y_t is a state variable determined by

$$y_t = ay_{t-1} + \epsilon_t \tag{114}$$

- Solve this equation by backward iteration

$$y_t = a(ay_{t-2} + \epsilon_{t-1}) + \epsilon_t \tag{115}$$

$$= a^t y_0 + a^{t-1}\epsilon_1 + a^{t-2}\epsilon_2 + ... + a\epsilon_{t-1} + \epsilon_t. \tag{116}$$

- y_t is a function of the realizations of ϵ_s for $s = 1, \ldots, t$ and initial condition y_0

- With predetermined variables, we have a stable solution only if (we have an initial condition and) $|a| < 1$. If $|a| > 1$ there is no stable solution

Control/Forward-Looking Variables I

- Assume y_t is a control variable determined by (e.g. an Euler equation)

$$a y_t = E_t y_{t+1} + \epsilon_t \tag{117}$$

- Again solution depends on a
- Case $|a| > 1$: Rewrite and iterate forward

$$y_t = \frac{1}{a} E_t y_{t+1} + \frac{1}{a}\epsilon_t$$

$$y_t = \underbrace{\lim_{k\to\infty}\left(\frac{1}{a}\right)^{k+1} E_t(y_{t+k+1})}_{=0 \text{ by (113) and } a>1} + \sum_{i=0}^{\infty}\left(\frac{1}{a}\right)^{i+1} E_t \epsilon_{t+i} \tag{118}$$

- *Exercise 14:* Show that for ϵ_t following an AR(1) process

$$\epsilon_{t+1} = \rho\epsilon_t + \nu_{t+1}\ ,\ \nu_t \overset{iid}{\sim} \left(0, \sigma^2\right) \tag{119}$$

one obtains a unique and stable solution:

$$y_t = \frac{1}{a-\rho}\epsilon_t = \pi\epsilon_t \tag{120}$$

Control/Forward-Looking Variables II

- Case $|a| < 1$: forward-iteration of (117) does not yield stable solution
- Compute backward-solution; define expectational error: $\xi_{t+1} = y_{t+1} - E_t y_{t+1}$, and rewrite (117):

$$a y_t = y_{t+1} - \xi_{t+1} + \epsilon_t \tag{121}$$

$$\text{or: } y_t = a y_{t-1} + \xi_t - \epsilon_{t-1} \tag{122}$$

- Solving backward until

$$y_t = a^t y_0 + \sum_{s=1}^{t} a^{t-s} \xi_s + \sum_{s=1}^{t} a^{t-s} \epsilon_{s-1}, \tag{123}$$

  which is a **stable solution** for $|a| < 1$.
- Note that it **holds for any realization of the sequence of expectational errors** $\{\xi_s\}_{s=1\ldots t}$, i.e. there is no unique solution (indeterminacy)

Univariate Case: Summary

- We need an explosive autoregressive coefficient $(|a| > 1)$ when dealing with a forward looking variable to obtain a unique and stable solution
- We need a stable autoregressive coefficient $(|a| < 1)$ when dealing with a backward looking variable to obtain a stable solution (initial condition makes it unique)