1. Investment with a linear-quadratic objective.

(a) We have

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} (aY_t - K_t)^2 + \frac{b}{2} I_t^2 - q_t (K_t - (1-\delta) K_{t-1} - I_t) \right)$$

$$\begin{aligned} K_t &: aY_t - K_t - q_t + \beta q_{t+1} (1-\delta) = 0 \\ I_t &: bI_t - q_t = 0 \end{aligned}$$

(b) Substitute for q_t and I_t to get a 2nd-order difference equation in K_t of the form

$$-E_t \{K_{t+1}\} + (\phi + 1 - \delta) K_t - \beta^{-1} K_{t-1} = \mu Y_t$$

where ϕ and μ are functions of b, β, δ, and a.

$$aY_t - K_t \left(1 + b + \beta b (1-\delta)^2\right) + b (1-\delta) K_{t-1} + \beta b (1-\delta) E_t \{K_{t+1}\} = 0$$

So

$$K_t \left(\frac{1+b}{\beta b (1-\delta)} + 1 - \delta \right) - \beta^{-1} K_{t-1} - E_t \{K_{t+1}\} = \frac{a}{\beta b (1-\delta)} Y_t$$

(c) Let $\mu \equiv \frac{a}{\beta b (1-\delta)}$ and $\psi = \frac{1+b}{\beta b (1-\delta)} + 1 - \delta$. We have

$$K_t \psi - \beta^{-1} K_{t-1} - E_t \{K_{t+1}\} = \mu Y_t$$

The problem tells you the solution will be of the form

$$K_t = \lambda K_{t-1} + \lambda \mu \beta \sum_{i=0}^{\infty} (\beta \lambda)^i E_t \{Y_{t+i}\},$$

which we can express as

$$(1 - \lambda L) \left(1 - \beta \lambda L^{-1}\right) K_t (\beta \lambda)^{-1} = \mu Y_t.$$

We have

$$(\beta \lambda)^{-1} (1 - \lambda L) \left(1 - \beta \lambda L^{-1}\right) = \left(\lambda + (\beta \lambda)^{-1} - \beta^{-1} L - L^{-1}\right)$$

so we need

$$\lambda + (\beta \lambda)^{-1} = \psi$$

or $\lambda^2 - \psi \lambda + \beta^{-1} = 0$, i.e. λ is the solution to

$$\lambda^2 - \psi \lambda + \beta^{-1} = 0$$

such that $|\lambda| < 1$. Therefore (assuming $\psi^2 \geq 4\beta^{-1}$)

$$\lambda = \left(\psi - \sqrt{\psi^2 - 4\beta^{-1}} \right) / 2.$$