Heterogeneous Agents Real Business Cycle

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1 Introduction

This mini-paper describes the Heterogeneous Agents Real Business Cycle within the lecture notes of Eric Sims. In this case we add second household to our mode, one lends to another. Below we consider maximization problems step-by-step.

2 Maximization problem of the first household

$$\max_{C_t, B_{t+1}} = E_0 \sum_{t=0}^{\infty} \beta^t (u(C_t)) \tag{1}$$

s.t.

$$C_t + B_{t+1} \leq \Pi_t + (1 + r_{t-1}) B_t \tag{2}$$

$$\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t (u(C_t) + \lambda_t (\Pi_t + (1 + r_{t-1}) B_t - C_t - B_{t+1}))(3)$$

$$\frac{\partial L}{\partial C_t} = 0 \Leftrightarrow u'(C_t) = \lambda_t \tag{4}$$

$$\frac{\partial L}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} (1 + r_t) \tag{5}$$

$$u'(C_t) = \beta E_t (u'(C_{t+1})(1 + r_t)) \tag{6}$$

Transversality conditions

$$\lim_{t=\infty} \beta^t B_{t+1} u'(C_t) = 0 \tag{7}$$

3 Second household maximization problem

$$\max_{C_t, N_t, D_{t+1}} = E_0 \sum_{t=0}^{\infty} \beta^t (u(C_t) + v(1 - N_t)) \tag{8}$$

s.t.

$$C_t + (1 + r_{t-1}) D_t \leq w_t N_t + D_{t+1} \tag{9}$$

$$\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t (u(C_t) + v(1 - N_t) + \lambda_t (w_t N_t + D_{t+1} - (1 + r_{t-1}) D_t - C_t)) \tag{10}$$

$$\frac{\partial L}{\partial C_t} = 0 \Leftrightarrow u'(C_t) = \lambda_t \tag{11}$$

$$\frac{\partial L}{\partial N_t} = 0 \Leftrightarrow v'(1 - N_t) = \lambda_t w_t \tag{12}$$

$$\frac{\partial L}{\partial D_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1 + r_t) \tag{13}$$

Then we get the expression for:

$$u'(C_t) = \beta E_t(u'(C_{t+1})(1 + r_t)) \tag{14}$$

$$\mu = 0 \tag{15}$$

$$v'(1 - N_t) = \lambda_t w_t \tag{16}$$

Transversality conditions

$$\lim_{t=\infty} \beta^t D_{t+1} u'(C_t) = 0 \tag{17}$$

4 The Firm Problem

$$M_t = \beta^t \frac{E_0 u'(C_t)}{u'(C_0)} \tag{18}$$

$$\max_{N_t, I_t, D_{t+1}, K_{t+1}} V_0 = E_0 \sum_{t=0}^{\infty} M_t(A_t F(K_t, N_t) - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1})D_t) \tag{19}$$

s.t.

$$K_{t+1} = I_t + (1 - \delta)K_t \tag{20}$$

We can re-write the problem by imposing that the constraint hold each period:

$$\max_{N_t, K_{t+1}, D_{t+1}} V_0 = E_0 \sum_{t=0}^{\infty} M_t(A_t F(K_t, N_t) - w_t N_t - K_{t+1} + (1 - \delta)K_t + D_{t+1} - (1 + r_{t-1})D_t) \tag{21}$$

$$\frac{\partial V_0}{\partial N_t} = 0 \Leftrightarrow A_t F_N(K_t, N_t) = w_t \tag{22}$$

$$\frac{\partial V_0}{\partial K_{t+1}} = 0 \Leftrightarrow u'(C_t) = \beta E_t(u'(C_{t+1})((A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta)) \tag{23}$$

$$\frac{\partial V_0}{\partial D_{t+1}} = 0 \Leftrightarrow u'(C_t) = \beta E_t(u'(C_{t+1})(1 + r_t)) \tag{24}$$

5 Equilibrium Analysis of the Decentralized Model with Heterogeneous Agents

$$u'(C_t) = \beta E_t(u'(C_{t+1})((A_{t+1} F_K(k_{t+1}, N_{t+1}) + (1 - \delta)) \tag{25}$$

$$v'(1 - N_t) = u'(C_t) A_t F_N(K_t, N_t) \tag{26}$$

$$K_{t+1} = A_t f(K_t, N_t) - C_t + (1 - \delta)K_t \tag{27}$$

$$ln A_t = \rho ln A_{t-1} + \epsilon_t \tag{28}$$

$$Y_t = A_t f(K_t, N_t) \tag{29}$$

$$I_t = Y_t - C_t \tag{30}$$

$$u'(C_t) = \beta E_t u'(C_{t+1})(1 + r_t) \tag{31}$$

$$w_t = A_t F_N(K_t, N_t) \tag{32}$$

$$R_t = A_t F_N(K_t, N_t) \tag{33}$$

These first order conditions can then be re-written imposing these function forms to get:

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left(\alpha A_{t+1} K_{t+1}^{\alpha} - 1 N_{t+1}^{1} - \alpha + (1 - \delta) \right) \right) \tag{34}$$

$$\frac{\theta}{1 - N_t} = \frac{1}{C_t}(1 - \alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{35}$$

$$lnA_t = \rho lnA_{t-1} + \epsilon_t \tag{36}$$

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{37}$$

$$I_t = Y_t - C_t \tag{38}$$

$$I_t = K_{t+1} - (1 - \delta) K_t \tag{39}$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} (1 + r_t) \tag{40}$$

$$w_t = (1 - \alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{41}$$

$$R_t = \alpha A_t K_t^{\alpha-1} N_t^{-\alpha} \tag{42}$$

6 Steady State Analysis

Next step is to express the steady states for all above mentioned variables in the model. Steady state output is:

$$Y = \left(\frac{K}{N}\right)^{\alpha} N \tag{43}$$

Investment-output ratio in case of Russian economy I used data from Federal State Statics Service. Then the steady state investment-output ratio is:

$$\frac{I}{Y} = \delta \left(\frac{K}{N}^{\alpha} \right) \tag{44}$$

The steady state capital-labor ratio is:

$$\frac{K}{N} = \left(\frac{\alpha}{\frac{1}{\beta} - (1 - \delta)} \right)^{\frac{1}{1-\alpha}} \tag{45}$$

$$N = \frac{\frac{1-\alpha}{\theta} \left(\frac{K}{N}\right)^{\alpha}}{\frac{\theta+1-\alpha}{\theta} \left(\frac{K}{N}\right)^{\alpha} - \delta \left(\frac{K}{N}\right)} \tag{46}$$

Hence, we can write the steady state investment-output ratio as:

$$\frac{I}{Y} = \frac{\alpha\delta}{\frac{1}{\beta} - (1 - \delta)} \tag{47}$$

Now solve the accumulation equation for steady state consumption per worker:

$$\frac{C}{N} = \left(\frac{K}{N}^{\alpha} - \delta\frac{K}{N}\right) \tag{48}$$

Hence solve the equation for steady state investment:

$$I = \delta\frac{K}{N}N \tag{49}$$

Now we can write steady state for the wage:

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha} \tag{50}$$

For solving real-interest rate we need to write for steady state R:

$$R = \alpha A\left(\frac{K}{N}\right)^{\alpha-1} \tag{51}$$