**Estimating nonzero off-diagonal covariances for exogenous shocks in `Dynare`.**

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Say that the shocks you would like to incorporate into the model are

$$\underbrace{\begin{bmatrix} \varepsilon_{zt} \\ \varepsilon_{gt} \\ \varepsilon_{rt} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t} \sim IID \left( \mathbf{0}_{3\times 1}, \underbrace{\begin{bmatrix} \sigma_z^2 & \cdot & \cdot \\ \sigma_{gz} & \sigma_g^2 & \cdot \\ \sigma_{rz} & \sigma_{rg} & \sigma_g^2 \end{bmatrix}}_{\boldsymbol{\Sigma}_\varepsilon} \right)$$

So, in the `shocks` block you enter

```
shocks;
var ez; stderr sz;
var eg; stderr sg;
var er; stderr sr;
var ez, eg = sgz;
var ez, er = srz;
var er, eg = srg;
end;
```

Trying to estimate these off-diagonal covariances in `Dynare` produces the following perplexing error message:

```
ERROR: some estimated parameters (sz, sg, sr, sgz, srz, srg) also appear
in the expressions defining the variance/covariance matrix of shocks;
this is not allowed.
```

In the case of zero off-diognal entries in the variance-covariance matrix $\boldsymbol{\Sigma}_\varepsilon$, i.e. $\sigma_{gz} = \sigma_{rz} = \sigma_{rg} = 0$, if we wanted to estimate the variance of an innovation like $\varepsilon_{zt}$, we could simply define another variable $\varepsilon_{zt} = \sigma_z \eta_{zt}$ where $\eta_{zt}$ is iid mean zero with variance 1. This would be invoked in the `model` and `shocks` blocks as

```
model;
⋮
... sz*etaz ...
⋮
end;
```

```
⋮
shocks;
var etaz; stderr 1;
var etag; stderr 1;
var etar; stderr 1;
end;
```

It is not as obvious what to do in the more general case of $\sigma_{gz} = \sigma_{rz} = \sigma_{rg} \neq 0$. However, it is in fact possible to incoporate nonzero off-diagonal entries by simple reparameterization. Hypothesize that there exists a relationship:

$$\begin{bmatrix} \varepsilon_{zt} \\ \varepsilon_{rt} \\ \varepsilon_{gt} \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}}_{\boldsymbol{\Phi}} \underbrace{\begin{bmatrix} \eta_{zt} \\ \eta_{gt} \\ \eta_{rt} \end{bmatrix}}_{\boldsymbol{\eta}_t}$$

where $\boldsymbol{\eta}_t \sim IID(\mathbf{0}_{3\times 1}, \boldsymbol{I}_3)$. For this relationship to be valid implies the equality $\boldsymbol{\Sigma}_\varepsilon = \boldsymbol{\Phi}\boldsymbol{\Phi}'$, i.e.,

$$\begin{bmatrix} 0 & \cdot & \cdot \\ 0 & 0 & \cdot \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_z^2 - a^2 & \cdot & \cdot \\ \sigma_{gz} - ab & \sigma_g^2 - b^2 - c^2 & \cdot \\ \sigma_{rz} - ad & \sigma_{rg} - bd - ce & \sigma_r^2 - d^2 - e^2 - f^2 \end{bmatrix}$$

solving this system of 6 equations and 6 unknowns one-by-one in the vech direction of the previous matrix $\boldsymbol{\Sigma}_\varepsilon - \boldsymbol{\Phi}\boldsymbol{\Phi}'$ results in

1. $a = \sigma_z$
2. $b = \frac{\sigma_{gz}}{\sigma_z}$
3. $d = \frac{\sigma_{rz}}{\sigma_z}$
4. $c = \sqrt{\sigma_g^2 - \frac{\sigma_{gz}^2}{\sigma_z^2}}$
5. $e = \frac{1}{\sqrt{\sigma_g^2 - \frac{\sigma_{gz}^2}{\sigma_z^2}}} \left(\sigma_{rg} - \frac{\sigma_{gz}\sigma_{rz}}{\sigma_z^2}\right)$
6. $f = \sqrt{\sigma_r^2 - \frac{\sigma_{rz}^2}{\sigma_z^2} - \frac{1}{\sigma_g^2 - \frac{\sigma_{gz}^2}{\sigma_z^2}} \left(\sigma_{rg} - \frac{\sigma_{gz}\sigma_{rz}}{\sigma_z^2}\right)^2}$

Thus, definining $\boldsymbol{\eta}_t$ as iid mean zero covariance $\boldsymbol{I}_3$ in the `shocks` block of the code, and $\boldsymbol{\varepsilon}_t$ related to $\boldsymbol{\eta}_t$ by $\boldsymbol{\Phi}$ in the `model` block of the code using the above definitions for $a$-$f$, achieves the desired result. Reparameterization for higher-dimension $\varepsilon_t$ can be carried out using the same iterative procedure outlined above.