Appendix C

Equilibrium Conditions

The appendix contains a detailed description of the estimated DSGE model. The exposition is based on the technical notes of Ireland (2003).[1]

C.1 The Economic Environment

- Households:
  The representative household chooses $\{c_t, l_t, M_t, B_t, k_{t+1}, i_t\}_{t=0}^{\infty}$ to maximize utility

$$E\sum_{t=0}^{\infty}\beta^t\{a_t[\gamma/(\gamma-1)]\ln[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}]+\chi\ln(1-l_t)\},$$

  subject to the budget constraint

$$\frac{M_{t-1}+T_t+B_{t-1}+W_tl_t+Q_tk_t+D_t}{P_t}\geq c_t+i_t+\frac{\phi_k}{2}\left(\frac{k_{t+1}}{k_t}-1\right)^2k_t+\frac{B_t/r_t+M_t}{P_t},$$

  and the law of motion for capital

$$k_{t+1}=(1-\delta)k_t+x_ti_t. \tag{3.1}$$

[1]The technical notes are available at http://www.irelandp.com/progs/endogenous.zip.

Further, following Buiter and Sibert (2007), we prevent the household from excessive debts by imposing the no-Ponzi-game condition:

$$\lim_{t\to\infty} B_t \prod_{s=0}^{t} \frac{1}{r_s} \geq 0.$$

Accordingly the Lagrangian can be written as follows:

$$\begin{aligned}\Lambda \;=\; & E\sum_{t=0}^{\infty}\Bigg(\beta^t\Bigg\{a_t\left[\gamma/(\gamma-1)\right]\ln\left[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}\left(\frac{M_t}{P_t}\right)^{(\gamma-1)/\gamma}\right]+\chi\ln(1-l_t)\Bigg\}\\ & -\beta^t\lambda_t\Bigg\{c_t+\left[\frac{k_{t+1}-(1-\delta)k_t}{x_t}\right]+\frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t}-1\right)^2 k_t+\frac{B_t/r_t+M_t}{P_t}\\ & -\left(\frac{M_{t-1}+T_t+B_{t-1}+W_tl_t+Q_tk_t+D_t}{P_t}\right)\Bigg\}\Bigg).\end{aligned}$$

The first-order conditions are obtained by setting the partial derivatives of Λ with respect to $c_t, l_t, M_t, B_t, k_{t+1}$, and λ_t equal to zero, yielding

$$\Lambda_{c_t} = a_t - \lambda_t c_t^{1/\gamma}\left[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}\left(\frac{M_t}{P_t}\right)^{(\gamma-1)/\gamma}\right] = 0, \qquad (3.7)$$

$$\Lambda_{l_t} = \chi - \lambda_t\left(\frac{W_t}{P_t}\right)(1-l_t) = 0, \qquad (3.8)$$

$$\begin{aligned}\Lambda_{M_t} = & \left(\frac{M_t}{P_t}\right)^{1/\gamma}\left[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}\left(\frac{M_t}{P_t}\right)^{(\gamma-1)\gamma}\right]\left[\lambda_t-\beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right)\right]\\ & - a_te_t^{1/\gamma} = 0,\end{aligned} \qquad (3.9)$$

$$\Lambda_{B_t} = \lambda_t - \beta r_t E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right) = 0, \qquad (3.10)$$

$$\begin{aligned}\Lambda_{k_{t+1}} = \lambda_t &\left[\frac{1}{x_t} + \phi_K\left(\frac{k_{t+1}}{k_t} - 1\right)\right] \\ &- \left\{\beta E_t\left[\lambda_{t+1}\left(\frac{Q_{t+1}}{P_{t+1}} + \frac{1-\delta}{x_{t+1}}\right)\right]\right. \\ &- \left(\frac{\beta\phi_K}{2}\right) E_t\left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)^2\right] \\ &\left. + \beta\phi_K E_t\left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right]\right\} = 0,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\Lambda_{\lambda_t} = c_t + \left[\frac{k_{t+1} - (1-\delta)k_t}{x_t}\right] + \frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t + \frac{B_t/r_t + M_t}{P_t} \\ - \left(\frac{M_{t-1} + T_t + B_{t-1} + W_t l_t + Q_t k_t + D_t}{P_t}\right) = 0.\end{aligned} \tag{3.12}$$

Note that we can rewrite (3.9) by using (3.7) and (3.10) to obtain

$$c_t e_t - \left(\frac{M_t}{P_t}\right)\left(1 - \frac{1}{r_t}\right)^\gamma = 0. \tag{3.9$'$}$$

Finally, we impose the standard transversality conditions to guarantee that money, bonds and capital do not grow too quickly:

$$\begin{aligned}\lim_{t\to\infty} \beta^t \lambda_t \frac{M_t}{P_t} &= 0, \\ \lim_{t\to\infty} \beta^t \lambda_t \frac{B_t}{P_t} &= 0, \\ \lim_{t\to\infty} \beta^t \lambda_t k_{t+1} &= 0.\end{aligned}$$

- Finished goods-producing firms:
  The representative finished goods-producing firm seeks to maximize its profits

$$P_t y_t - \int_0^1 P_t(i) y_t(i) di$$

subject to the constant returns to scale technology

$$y_t \leq \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)}.$$

Therefore, the firm's optimization problem can be written as

$$\max_{y_t(i)} \Pi_t = P_t \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} - \int_0^1 P_t(i) y_t(i) di,$$

which leads to the following first-order condition characterizing the demand for intermediate goods:

$$\frac{\partial \Pi_t}{\partial y_t(i)} = y_t(i) - \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t = 0.$$

By plugging this expression into the constant elasticity of substitution aggregator of intermediate goods we obtain the price aggregator

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{1/(1-\theta)}.$$

- Intermediate goods-producing firms:
  Each intermediate goods-producing firm seeks to maximize its present discounted value of profits

$$E \sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t],$$

by choosing $\{l_t(i), k_t(i), y_t(i), P_t(i)\}_{t=0}^{\infty}$ subject to the Cobb-Douglas technology constraint

$$y_t(i) \leq k_t(i)^{\alpha} [z_t l_t(i)]^{1-\alpha}$$

and the above demand for intermediate goods

$$y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t.$$

We can use the latter expression to rewrite the real value of dividends

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t} \right] y_t(i) - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t} \right] - \frac{\phi_P}{2} \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 y_t$$

as

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t}\right] - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t. \tag{3.13}$$

Therefore, the Lagrangian for the firms' intertemporal optimization problem can be written as:

$$\begin{aligned}\Lambda \;=\; & E\sum_{t=0}^{\infty}\Bigg(\beta^t\lambda_t\Bigg\{\left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t}\right] - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t\Bigg\} \\ & -\beta^t\xi_t\Bigg\{\left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha}\Bigg\}\Bigg).\end{aligned}$$

Setting the partial derivatives of Λ with respect to $l_t(i), k_t(i), P_t(i)$, and ξ_t equal to zero leads to the first order conditions:

$$\Lambda_{l_t(i)} = \frac{\lambda_t W_t l_t(i)}{P_t} - (1-\alpha)\xi_t k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha} = 0, \tag{3.14}$$

$$\Lambda_{k_t(i)} = \frac{\lambda_t Q_t k_t(i)}{P_t} - \alpha\xi_t k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha} = 0, \tag{3.15}$$

$$\begin{aligned}\Lambda_{P_t(i)} = & \;\phi_P\lambda_t\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]\left[\frac{P_t}{\pi P_{t-1}(i)}\right] \\ & - (1-\theta)\lambda_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta} - \theta\xi_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta-1} \\ & - \beta\phi_P E_t\left\{\lambda_{t+1}\left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)P_t}{\pi P_t(i)^2}\right]\left(\frac{y_{t+1}}{y_t}\right)\right\} = 0,\end{aligned} \tag{3.16}$$

and

$$\Lambda_{\xi_t} = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha} = 0. \tag{3.17}$$

- The monetary authority sets the gross nominal interest rate according to the generalized Taylor rule:

$$\ln\left(\frac{r_t}{r}\right) = \omega_\tau \ln\left(\frac{\tau_t}{\tau}\right) + \omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right) + \ln(\upsilon_t). \tag{3.18}$$

C.2 The Nonlinear System

C.2.1 Symmetric Equilibrium

The dynamic system is described by the nonlinear difference equations (3.1) – (3.8), (3.9′), (3.10) – (3.18). To close the model, we complete the following two steps. First, we consider a symmetric equilibrium where all intermediate goods-producing firms make identical decisions. This assumption implies $P_t(i) = P_t$, $y_t(i) = y_t$, $l_t(i) = l_t$, $k_t(i) = k_t$, and $D_t(i) = D_t$ for $t = 0, 1, 2...$ and all $i \in [0, 1]$. Second, the market clearing condition for both the bond market, $B_t = B_{t-1} = 0$, and the money market, $M_t = M_{t-1} + T_t$, must hold for all $t = 0, 1, 2....$ By substituting these conditions into (3.1) – (3.18) and defining the average product of labor as $n_t = y_t/l_t$ and the money growth rate as $\tau_t = \frac{M_t}{M_{t-1}}$ we get:

$$k_{t+1} = (1 - \delta)k_t + x_t i_t, \tag{3.1}$$

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{3.2}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3.3}$$

$$\ln(e_t) = (1 - \rho_e) \ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3.4}$$

$$\ln(z_t) = (1 - \rho_z) \ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{3.5}$$

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{3.6}$$

$$a_t = \lambda_t c_t^{1/\gamma} \left[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma} \left(\frac{M_t}{P_t} \right)^{(\gamma-1)/\gamma} \right], \tag{3.7}$$

$$\chi = \lambda_t \left(\frac{W_t}{P_t} \right) (1 - l_t), \tag{3.8}$$

$$c_t e_t = \left(\frac{M_t}{P_t} \right) \left(1 - \frac{1}{r_t} \right)^\gamma, \tag{3.9′}$$

$$\lambda_t = \beta r_t E_t \left(\lambda_{t+1} \frac{P_t}{P_{t+1}} \right), \tag{3.10}$$

$$\lambda_t\left[\frac{1}{x_t}+\phi_K\left(\frac{k_{t+1}}{k_t}-1\right)\right]=\beta E_t\left[\lambda_{t+1}\left(\frac{Q_{t+1}}{P_{t+1}}+\frac{1-\delta}{x_{t+1}}\right)\right]$$
$$-\left(\frac{\beta\phi_K}{2}\right)E_t\left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}}-1\right)^2\right] \quad (3.11)$$
$$+\beta\phi_K E_t\left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}}-1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right],$$

$$c_t+\left[\frac{k_{t+1}-(1-\delta)k_t}{x_t}\right]+\frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t}-1\right)^2 k_t=\left(\frac{W_t l_t+Q_t k_t+D_t}{P_t}\right), \quad (3.12)$$

$$\frac{D_t}{P_t}=y_t-\frac{W_t l_t+Q_t k_t}{P_t}-\frac{\phi_P}{2}\left(\frac{P_t}{\pi P_{t-1}}-1\right)^2 y_t, \quad (3.13)$$

$$\lambda_t\left(\frac{W_t}{P_t}\right)l_t=(1-\alpha)\xi_t k_t^{\alpha}[z_t l_t]^{1-\alpha}, \quad (3.14)$$

$$\lambda_t\left(\frac{Q_t}{P_t}\right)k_t=\alpha\xi_t k_t^{\alpha}[z_t l_t]^{1-\alpha}, \quad (3.15)$$

$$\phi_P\lambda_t\left[\frac{P_t}{\pi P_{t-1}}-1\right]\left[\frac{P_t}{\pi P_{t-1}}\right]=(1-\theta)\lambda_t+\theta\xi_t$$
$$+\beta\phi_P E_t\left[\lambda_{t+1}\left(\frac{P_{t+1}}{\pi P_t}-1\right)\left(\frac{P_{t+1}}{\pi P_t}\right)\left(\frac{y_{t+1}}{y_t}\right)\right], \quad (3.16)$$

$$y_t=k_t^{\alpha}[z_t l_t]^{1-\alpha}, \quad (3.17)$$

$$\ln\left(\frac{r_t}{r}\right)=\omega_\tau\ln\left(\frac{\tau_t}{\tau}\right)+\omega_\pi\ln\left(\frac{\pi_t}{\pi}\right)+\omega_y\ln\left(\frac{y_t}{y}\right)+\ln(\upsilon_t), \quad (3.18)$$

$$n_t=\frac{y_t}{l_t}, \quad (3.19)$$

and

$$\tau_t=\frac{M_t}{M_{t-1}}. \quad (3.20)$$

Note that we can rewrite (3.12) by using (3.13) to obtain:

$$y_t=c_t+i_t+\frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t}-1\right)^2 k_t+\frac{\phi_P}{2}\left[\frac{P_t}{\pi P_{t-1}}-1\right]^2 y_t. \quad (3.12')$$

Further, (3.17) can be used to rewrite (3.14) and (3.15) as

$$\lambda_t\left(\frac{W_t}{P_t}\right)l_t=(1-\alpha)\xi_t y_t \quad (3.14')$$

and

$$\lambda_t \left(\frac{Q_t}{P_t}\right) k_t = \alpha \xi_t y_t. \tag{3.15$'$}$$

C.2.2 Change of Variables

We can rewrite the nonlinear system by defining $\pi_t = \frac{P_t}{P_{t-1}}$, $m_t = \frac{M_t}{P_t}$, $w_t = \frac{W_t}{P_t}$, $q_t = \frac{Q_t}{P_t}$, and $d_t = \frac{D_t}{P_t}$. With these re-defined variables, (3.1)−(3.8), (3.9′), (3.10), (3.11), (3.12′), (3.13), (3.14′), (3.15′), (3.16) – (3.20) become:

$$k_{t+1} = (1-\delta)k_t + x_t i_t, \tag{3.1}$$

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{3.2}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3.3}$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3.4}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{3.5}$$

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{3.6}$$

$$a_t = \lambda_t c_t^{1/\gamma} \left[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma} m_t^{(\gamma-1)/\gamma}\right], \tag{3.7}$$

$$\chi = \lambda_t w_t (1-l_t), \tag{3.8}$$

$$c_t e_t = m_t \left(1 - \frac{1}{r_t}\right)^\gamma, \tag{3.9$'$}$$

$$\lambda_t = \beta r_t E_t \left(\frac{\lambda_{t+1}}{\pi_{t+1}}\right), \tag{3.10}$$

$$\begin{aligned} \lambda_t \left[\frac{1}{x_t} + \phi_K \left(\frac{k_{t+1}}{k_t} - 1\right)\right] &= \beta E_t \left[\lambda_{t+1}\left(q_{t+1} + \frac{1-\delta}{x_{t+1}}\right)\right] \\ &- \left(\frac{\beta\phi_K}{2}\right) E_t \left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)^2\right] \\ &+ \beta\phi_K E_t \left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right], \end{aligned} \tag{3.11}$$

$$y_t = c_t + i_t + \frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t + \frac{\phi_P}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{3.12$'$}$$

$$d_t = y_t - w_t l_t - q_t k_t - \left(\frac{\phi_P}{2}\right)\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{3.13}$$

$$\lambda_t w_t l_t = (1-\alpha)\xi_t y_t, \tag{3.14$'$}$$

$$\lambda_t q_t k_t = \alpha \xi_t y_t, \tag{3.15$'$}$$

$$\begin{aligned}\phi_P \lambda_t \left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) &= (1-\theta)\lambda_t + \theta\xi_t \\ &+ \beta\phi_P E_t\left[\lambda_{t+1}\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{\pi_{t+1}}{\pi}\right)\left(\frac{y_{t+1}}{y_t}\right)\right],\end{aligned} \tag{3.16}$$

$$y_t = k_t^{\alpha}[z_t l_t]^{1-\alpha}, \tag{3.17}$$

$$\ln\left(\frac{r_t}{r}\right) = \omega_\tau \ln\left(\frac{\tau_t}{\tau}\right) + \omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right) + \ln(\upsilon_t), \tag{3.18}$$

$$n_t = \frac{y_t}{l_t}, \tag{3.19}$$

and

$$\tau_t = \left(\frac{m_t}{m_{t-1}}\right)\pi_t. \tag{3.20}$$

C.3 Steady States

In absence of the five shocks, i.e., $\varepsilon_{xt} = \varepsilon_{at} = \varepsilon_{et} = \varepsilon_{zt} = \varepsilon_{vt} = 0$ for all $t = 0, 1, 2, ...$, the economy converges to a steady state, where each of the 20 variables is constant. We use (3.2), (3.3), (3.4), (3.5), and (3.6) to solve for

$$\begin{aligned} x &= 1, \\ a &= 1, \\ e &= e, \\ z &= z, \\ \upsilon &= 1. \end{aligned}$$

Assuming that the steady state money growth rate τ is determined by policy, (3.10) and (3.20) can be used to solve for

$$\pi = \tau$$

and

$$r = \frac{\pi}{\beta}.$$

Next, (3.11) and (3.16) can be used to solve for

$$q = \frac{1}{\beta} - 1 + \delta$$

and

$$\xi = \left[\frac{(\theta - 1)}{\theta}\right] \lambda.$$

Equations (3.7) and (3.9′) can be used to solve for

$$c = \left[1 + e\left(\frac{r}{r-1}\right)^{\gamma-1}\right]^{-1} \left(\frac{1}{\lambda}\right)$$

and

$$m = e\left(\frac{r}{r-1}\right)^{\gamma} c.$$

Use (3.1), (3.12′), (3.15′), and (3.16) to solve for

$$y = \left[1 - \delta\left(\frac{\alpha}{q}\right)\left(\frac{\theta - 1}{\theta}\right)\right]^{-1}.$$

Use (3.15′) and (3.16) to solve for

$$k = \left(\frac{\alpha}{q}\right)\left(\frac{\theta - 1}{\theta}\right) y.$$

Equations (3.1), (3.13), (3.14′),(3.17), and (3.19) can be used to solve for

$$i = \delta k,$$

$$l = \frac{1}{z}\left(\frac{y}{k^{\alpha}}\right)^{1/(1-\alpha)},$$

$$w = (1 - \alpha)\left(\frac{\theta - 1}{\theta}\right)\left(\frac{y}{l}\right),$$

$$d = y - wl - qk,$$

and

$$n = \frac{y}{l}.$$

Finally, (3.8), (3.14′), and (3.16) can be used to solve for

$$\lambda = \frac{\chi + (1-\alpha)\left[1 + e\left(\frac{r}{r-1}\right)^{\gamma-1}\right]^{-1}\left[\left(\frac{\theta}{\theta-1}\right) - \delta\left(\frac{\alpha}{q}\right)\right]^{-1}}{(1-\alpha)z\left(\frac{\theta-1}{\theta}\right)^{1/(1-\alpha)}\left(\frac{\alpha}{q}\right)^{\alpha/(1-\alpha)}}.$$

C.4 The Linearized System

To linearize the nonlinear system (3.1) − (3.20), we perform a log-linear approximation of the model at steady state values.[2] Let $\widehat{var}_t \equiv log\left(\frac{var_t}{var}\right)$ denote the log-deviation of some variable var_t from its steady state var, where $log\left(\frac{var_t}{var}\right) \approx \frac{var_t - var}{var}$. A first-order Taylor approximation of equation (3.1) − (3.8), (3.9′), (3.10), (3.11), (3.12′), (3.13), (3.14′), (3.15′), (3.16) − (3.20) at the steady state gives:

$$k\hat{k}_{t+1} = (1-\delta)k\hat{k}_t + i\hat{x}_t + i\hat{i}_t, \tag{3.1}$$

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_{xt}, \tag{3.2}$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{3.3}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{3.4}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{3.5}$$

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \tag{3.6}$$

$$\gamma r\hat{a}_t = \gamma r\hat{\lambda}_t + r\left[1 + (\gamma-1)\lambda c\right]\hat{c}_t + (r-1)\lambda m\hat{e}_t + (\gamma-1)(r-1)m\hat{m}_t, \tag{3.7}$$

$$\lambda wl\hat{l}_t = \chi\hat{\lambda}_t + \chi\hat{w}_t, \tag{3.8}$$

$$(r-1)\hat{c}_t + (r-1)\hat{e}_t = (r-1)\hat{m}_t + \gamma\hat{r}_t, \tag{3.9′}$$

$$\hat{\lambda}_t = \hat{r}_t + E_t\hat{\lambda}_{t+1} - E_t\pi_{t+1}, \tag{3.10}$$

$$\hat{\lambda}_t - \hat{x}_t - \phi_k\hat{k}_t = E_t\hat{\lambda}_{t+1} + \beta q E_t\hat{q}_{t+1} - \beta(1-\delta)E_t\hat{x}_{t+1} + \beta\phi_K E_t\hat{k}_{t+2} - (1+\beta)\phi_K\hat{k}_{t+1}, \tag{3.11}$$

$$y\hat{y}_t = c\hat{c}_t + i\hat{i}_t, \tag{3.12′}$$

$$d\hat{d}_t = y\hat{y}_t - wl\hat{w}_t - wl\hat{l}_t - qk\hat{q}_t - qk\hat{k}_t, \tag{3.13}$$

[2] Canova (2007), DeJong and Dave (2007), and Zietz (2008) provide a detailed description of logarithmic approximations.

$$\hat{\lambda}_t + \hat{w}_t + \hat{l}_t = \hat{\xi}_t + \hat{y}_t, \tag{3.14$'$}$$

$$\hat{\lambda}_t + \hat{q}_t + \hat{k}_t = \hat{\xi}_t + \hat{y}_t, \tag{3.15$'$}$$

$$\phi_P \hat{\pi}_t = (1-\theta)\hat{\lambda}_t + (\theta - 1)\hat{\xi}_t + \beta\phi_P E_t \hat{\pi}_{t+1}, \tag{3.16}$$

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{z}_t + (1-\alpha)\hat{l}_t, \tag{3.17}$$

$$\hat{r}_t = \omega_\tau \hat{\tau}_t + \omega_\pi \hat{\pi}_t + \omega_y \hat{y}_t + \hat{v}_t, \tag{3.18}$$

$$\hat{n}_t = \hat{y}_t - \hat{l}_t, \tag{3.19}$$

and

$$\hat{\tau}_t = \hat{m}_t - \hat{m}_{t-1} + \hat{\pi}_t. \tag{3.20}$$

To facilitate the model's solution we follow Ireland (2003) and use (3.20) to rewrite (3.7) and (3.9$'$) as

$$\begin{aligned} \gamma r \hat{a}_t = {} & \gamma r \hat{\lambda}_t + r[1 + (\gamma - 1)\lambda c]\hat{c}_t + (r-1)\lambda m \hat{e}_t \\ & + (\gamma-1)(r-1)\lambda m \hat{\tau}_t + (\gamma-1)(r-1)\lambda m \hat{m}_{t-1} - (\gamma-1)(r-1)\lambda m \hat{\pi}_t \end{aligned} \tag{3.7$'$}$$

and

$$(r-1)\hat{e}_t + (r-1)\hat{c}_t = (r-1)\hat{\tau}_t + (r-1)\hat{m}_{t-1} - (r-1)\hat{\pi}_t + \gamma \hat{r}_t. \tag{3.9$''$}$$

Further, we make use of (3.1) and (3.2) to rewrite (3.11) as

$$\begin{aligned} & \hat{\lambda}_t - \{1 + \beta[\delta\phi_K - (1-\delta)]\rho_x\}\hat{x}_t - \phi_K \hat{k}_t \\ = {} & E_t \hat{\lambda}_{t+1} + \beta q E_t \hat{q}_{t+1} + \phi_K[\beta(1-\delta) - (1+\beta)]\hat{k}_{t+1} \\ & + \beta\delta\phi_K E_t \hat{i}_{t+1}. \end{aligned} \tag{3.11$'$}$$