

Long-Run Risk through Consumption Smoothing

Georg Kaltenbrunner and Lars Lochstoer^{*,†}

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Abstract

We examine how long-run consumption risk arises endogenously in a standard production economy model where the representative agent has Epstein-Zin preferences. We show that even when technology growth is i.i.d., optimal consumption smoothing induces long run risk - highly persistent variation in expected consumption growth. As a consequence, the model can account for a high price of risk although both consumption growth volatility and the coefficient of relative risk aversion are low. The asset pricing implications of endogenous long-run risk depend crucially on the persistence of technology shocks and investors' preference for the timing of resolution of uncertainty.

*Lochstoer (Corresponding author): Columbia Graduate School of Business; Uris Hall 405B, 3022 Broadway, New York, NY 10027. Email: ll2609@columbia.edu. Kaltenbrunner: McKinsey & Co.

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Long-run consumption risk has recently been proposed as a mechanism for explaining important asset pricing moments such as the unconditional Sharpe ratio of equity market returns, the equity premium, the level and volatility of the risk free rate and the cross-section of stock returns (see Bansal and Yaron, 2004; Hansen, Heaton and Li, 2005; and Parker and Julliard, 2005). The existence of such risks, however, is still under debate (e.g., Cochrane, 2007). Here, we investigate how long-run consumption risk arises *endogenously* in a standard, one-sector production economy framework and how this additional risk factor can help production economy models to jointly explain the dynamic behavior of consumption, investment and asset prices.¹

Households are assumed to have identical Epstein-Zin preferences. Unlike in the power utility case, where risk is only associated with the shock to *realized* consumption growth, investors with Epstein-Zin preferences also demand a premium for holding assets correlated with shocks to *expected* consumption growth. Epstein-Zin preferences allow the elasticity of intertemporal substitution to be decoupled from the coefficient of relative risk aversion, which leads to a preference for the timing of resolution of uncertainty. If investors prefer *late* resolution of uncertainty, shocks to expected consumption growth carry a negative price of risk, while if investors prefer *early* resolution of uncertainty, these shocks carry a positive price of risk. This source of risk is labelled "long-run risk" in Bansal and Yaron (2004). We show in this paper that such long-run risks arise endogenously in production economies, even when technology growth is i.i.d., because consumption smoothing induces highly persistent time-variation in expected consumption growth rates. This feature can help the model generate a high Sharpe ratio of equity returns, even when the volatility of consumption growth and the coefficient of relative risk aversion are low.

To evaluate the magnitude and nature of the endogenous long-run risk, we calibrate models with either transitory or permanent technology shocks to match both consumption and output growth volatility. The two technology specifications we consider are commonly employed in the macro literature, but their long-run risk implications are different. In each case, the endogenous time-variation in expected consumption growth is a small, but highly persistent, fraction of realized consumption growth, similar to the exogenous process specified in Bansal and Yaron (2004). However, the endogenous correlation between shocks to realized consumption growth and long-run expected consumption growth is perfectly negative in the

¹For extensive discussions of the poor performance of standard production economy models in terms of jointly explaining asset prices and macroeconomic moments, refer to Rouwenhorst (1995), Lettau and Uhlig (2000), Uhlig (2004), and Cochrane (2005), amongst others.

transitory shock case and perfectly positive in the permanent shock case. This correlation and whether the representative agent prefers early or late resolution of uncertainty are crucial for the asset pricing implications of long-run risk in the model.

To understand how long-run consumption risk arises endogenously, consider first the case of a positive, *permanent* shock to productivity. A permanently higher level of productivity implies that the optimal level of capital is also permanently higher. Therefore, investors temporarily increase investment relative to output. Since current consumption then is low relative to future expected consumption, expected consumption growth is high. In a model calibrated to match the relative volatility of consumption and output growth, both realized and expected consumption growth respond positively to the technology shock. With power utility preferences, the shock to expected consumption growth is not priced, but with a preference for early resolution of uncertainty the price of shocks to expected consumption growth is positive. The overall price of risk in the latter economy is therefore higher than in the standard power utility economy.

Next, consider a positive, *transitory* shock to technology. In this case, technology is expected to revert down to its long-run trend. Thus, while the shock to realized consumption growth is *positive*, the shock to expected future long-run consumption growth is *negative* as consumption reverts to the long-run trend.² If agents have a preference for early resolution of uncertainty, and thus dislike shocks to both realized and expected consumption growth, the long-run risk component now acts as a hedge for shocks to realized consumption growth and the overall price of risk is then *lower* than in the power utility counterpart. With a preference for late resolution of uncertainty, the implications for the price of risk are reversed in both cases as the agent now enjoys variation in expected consumption growth.

We find that both the transitory and the permanent shock models can be calibrated to match the high price of risk, the low level of the risk free rate, and the low volatility of consumption found in the data, with a low coefficient of relative risk aversion. Due to the above described differences in the endogenous consumption dynamics, the transitory and the permanent shock models do so with low and high elasticity of intertemporal substitution, respectively. In particular, a preference for late (early) resolution of uncertainty is needed in

²This description is intentionally loose to emphasize the intuition. The consumption response to transitory technology shocks is often hump-shaped. A positive shock to realized consumption growth is then followed by high expected consumption growth in the near term, but lower expected consumption growth in the long term - the negative correlation arises at lower frequencies. The low frequency effect dominates for standard values of the discount factor and leads to a lower price of risk unless the transitory shocks are extremely persistent.

the transitory (permanent) shock model.

The dynamic behavior of firm payouts, however, does not correspond to that observed for aggregate public equity dividends in either model. Therefore, the average excess returns to the capital claims do not match the historical level of the stock market equity premium. To evaluate the models' predictions for a claim that is comparable to the stock market, we compute the average return and return volatility of a claim to a dividend stream that is calibrated to historical moments of the aggregate stock market dividends. In the permanent shock model, the average excess return and return volatility of this dividend claim matches the empirical values for the equity market and for the transitory shock model neither moments match the empirical values.

Related Literature. It is well-known that agents optimally smooth consumption over time (see, e.g., Friedman, 1957, and Hall, 1978). More recently, Den Haan (1995) demonstrates that the risk free rate in production economy models is highly persistent (close to a random walk) even when the level of technology is i.i.d., while Campbell (1994) solves a log-linear approximation to the standard real business cycle model with power utility preferences, and presents analytical expressions for the optimal consumption choice. In the asset pricing literature, Bansal and Yaron (2004) show that a small, persistent component of consumption growth can have quantitatively important implications for asset prices if the representative agent has Epstein-Zin preferences. We show that a consumption process similar to what Bansal and Yaron assume can be generated *endogenously* in a standard production economy model with Epstein-Zin preferences, i.i.d. technology shocks and the same elasticity of substitution as in Bansal and Yaron (2004). The model thus provides a theoretical justification for the existence of long-run consumption risk, which it is difficult to establish empirically as pointed out by Harvey and Shepard (1990) and Hansen, Heaton and Li (2005). Earlier papers that emphasize a small, highly persistent component in the pricing kernel include Backus and Zin (1994) and Cochrane and Hansen (1992). Naik (1994) investigates the impact of time-varying fundamental risk in a general equilibrium production economy with Epstein-Zin preferences. He also highlights the importance of the elasticity of intertemporal substitution for asset pricing implications.

The paper also contributes to the literature Cochrane (2007) terms 'production-based asset pricing'. The starting point of this literature is a representative agent, one-sector production economy model (e.g., Long and Plosser, 1983) and the observation that this model, while being able to generate realistic processes for consumption and investment, fails markedly at explaining asset prices. Both Jermann (1998) and Boldrin, Christiano, Fisher

(2001) augment the basic production economy framework with habit preferences in order to remedy its shortcomings and succeed to a considerable extent to jointly explain macroeconomic time series and asset prices. However, both models display excessive volatility of the risk free rate and very high levels of risk aversion.

Tallarini (2000) also considers a one-sector stochastic growth model with Epstein-Zin preferences, but he restricts himself to the special case of unit elasticity of intertemporal substitution and no capital adjustment costs. By increasing the coefficient of relative risk aversion to very high levels, Tallarini matches some asset pricing moments such as the market price of risk (Sharpe ratio) as well as the level of the risk free rate. We take an opposite strategy and restrict the coefficient of relative risk aversion to be low, while we let the elasticity of intertemporal substitution vary. This allows us to study how endogenous long-run consumption risk arises and how it interacts with capital adjustment costs and the price of risk. Also, Tallarini does not consider the case of transitory technology shocks in the production economy model in his paper.

In recent research, Croce (2007) investigates the welfare implications of long-run risk in a general equilibrium production economy similar to the one we analyze. Panageas and Yu (2006) focus on the impact of major technological innovations and real options on endogenous consumption and the cross-section of asset prices. Finally, Campanale, Castro, and Clementi (2007) look at asset prices in general equilibrium production economies where the representative agent's preferences are in the Chew-Dekel class. Unlike us, they do not consider the role of long-run risk.

1 The Model

In this paper, we analyze how long-run consumption risk arises endogenously in a representative agent dynamic stochastic general equilibrium model, given standard assumptions on the production technology. The aim of the paper is two-fold: first, to understand why and how long-run consumption risk arises in equilibrium; second, to evaluate the magnitude of the long-run consumption risk in a model calibrated to match the standard moments of macro-economic variables, using reasonable preference parameter values.

1.1 Households

We assume there exists a representative household with Epstein-Zin preferences over a non-durable consumption good C_t with the utility function V_t satisfying:

$$V_t = \left\{ (1 - \beta) C_t^{1-1/\psi} + \beta E_t [V_{t+1}^{1-\gamma}]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \quad (1)$$

where ψ is the Elasticity of Intertemporal Substitution (EIS), γ is the coefficient of relative risk aversion for atemporal wealth gambles, and $\psi \neq 1$.³ The representative household has a preference for early resolution of uncertainty if $\gamma > 1/\psi$, a preference for late resolution of uncertainty if $\gamma < 1/\psi$, or is indifferent to the resolution of uncertainty if $\gamma = 1/\psi$. In the latter case, the preferences given in equation (1) collapse to the familiar power utility case.

The stochastic discount factor in this economy is given by (see, e.g., Epstein and Zin, 1989; Hansen et al., 2007; the Technical Appendix to this paper):

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1/\psi} \left(\frac{V_{t+1}}{E_t(V_{t+1}^{1-\gamma})^{1/(1-\gamma)}} \right)^{1/\psi-\gamma}. \quad (2)$$

1.2 Firms

The representative firm has the standard Cobb-Douglas production function:

$$Y_t = (Z_t H_t)^{1-\alpha} K_t^\alpha, \quad (3)$$

where Y_t is aggregate output, Z_t is an exogenous, labor-enhancing technology level, H_t is hours worked, and K_t denotes the capital stock. Since the representative household experiences no disutility of labor, hours worked will always be the maximum possible, which we normalize to be 1.

The representative firm's capital accumulation equation is given by:

$$K_{t+1} = (1 - \delta) K_t + \phi(I_t/K_t) K_t, \quad (4)$$

where δ is the capital depreciation rate, and $\phi(\cdot)$ is a weakly concave function that allows

³If $\psi = 1$, the preferences are given by $V_t = C_t^{1-\beta} E_t[V_{t+1}^{1-\gamma}]^{\beta/(1-\gamma)}$. We do not explicitly consider this case, which is studied extensively in Tallarini (1999), although we allow for values of the EIS arbitrarily close to unity.

for convex capital adjustment costs. In particular:

$$\phi(X) = a_1 + \frac{a_2}{1 - 1/\xi} X^{1-1/\xi}, \quad (5)$$

where ξ is the elasticity of the investment rate to Tobin's q . If ξ is low, capital adjustment costs are high; if $\xi = \infty$, capital adjustment costs are zero. The constants a_1 and a_2 are set such that there are no adjustment costs in the non-stochastic steady state following Boldrin, Christiano, and Fisher (2001).⁴ Let I_t denote gross investment (i.e., $I_t = Y_t - C_t$). The production economy in this paper is standard relative to the real business cycle literature in that (a) it takes one period for investment to be reflected in capital, and (b) there is a representative firm with a constant returns to scale production technology.

The manager of the representative firm acts competitively and maximizes firm value through optimal investment given the current capital stock, the current level of technology, and the stochastic discount factor. We solve the firm's problem in the Appendix, but note here that the firm's optimality condition for investment is summarized by requiring that:

$$E_t [M_{t+1} R_{i,t+1}] = 1, \quad (6)$$

where $R_{i,t+1}$ is the return on investment defined as:

$$R_{i,t+1} \equiv \phi'(I_t/K_t) \left\{ \alpha \left(\frac{Z_{t+1}}{K_{t+1}} \right)^{1-\alpha} + \frac{1 - \delta + \phi(I_{t+1}/K_{t+1})}{\phi'(I_{t+1}/K_{t+1})} - \frac{I_{t+1}}{K_{t+1}} \right\}. \quad (7)$$

The return on investment is the same as the return to holding a claim to the firm's payouts (an unlevered equity claim) given the assumed production technology (see, e.g., Restoy and Rockinger, 1994).⁵ Both capital and labor are paid their marginal product, and it is straightforward to show that wages, ω , and firm payouts (dividends), D , are given by $\omega_t = (1 - \alpha) Y_t$ and $D_t = \alpha Y_t - I_t$, respectively.

⁴In particular, we set $a_2 = (\exp(\mu) - 1 + \delta)^{1/\xi}$ and $a_1 = \frac{1}{\xi-1} (1 - \delta - \exp(\mu))$. It is straightforward to verify that $\phi'(\frac{I_t}{K_t}) > 0$ and $\phi''(\frac{I_t}{K_t}) < 0$ for $\xi > 0$ and $\frac{I_t}{K_t} > 0$. Furthermore, $\phi(\frac{I}{K}) = \frac{I}{K}$ and $\phi'(\frac{I}{K}) = 1$, where $\frac{I}{K} = (\exp(\mu) - 1 + \delta)$ is the steady state investment-capital ratio. Investment is always positive since the marginal cost of investing goes to infinity as investment goes to zero.

⁵In particular, the production function and the implied adjustment cost function satisfy the conditions of Proposition 1 in Restoy and Rockinger (1994).

1.3 Technology

Following, e.g., Campbell (1994), we define technology as:

$$Z_t = \exp(\mu t + z_t) \tag{8}$$

$$z_t = \varphi z_{t-1} + \varepsilon_t, \tag{9}$$

where $\varepsilon_t \sim N(0, \sigma^2)$ for all t , and $|\varphi| \leq 1$. Thus, technology shocks can be either permanent ($|\varphi| = 1$) or transitory ($|\varphi| < 1$). We consider two parameters for φ : 0.95 and 1. These are benchmark values in the real business cycle literature. However, as is well known, the choice of transitory versus permanent technology shocks can have a substantial impact on the optimal consumption-savings decision.⁶

2 Two Benchmark Models

In this section, we present two calibrated benchmark models with transitory and permanent technology shocks, respectively. In both models, long-run consumption risk arises in equilibrium. In the following sections, we inspect the mechanisms in the model that gives rise to such endogenous long-run risk, we evaluate the empirical plausibility of the endogenous consumption dynamics, as well as the asset pricing implications of the model. Our discussion is centered around different values of the elasticity of intertemporal substitution and the two specifications of technology (permanent vs. transitory). Unless otherwise stated, the model is solved numerically by means of the value function iteration algorithm. As is standard in the real business cycle literature, one unit of time in the model corresponds to a quarter of a year. The model is real and in per capita form, so all calibration is done with respect to real, per capita empirical counterparts. To facilitate easy comparison with Bansal and Yaron (2004), the sample moments are calculated using U.S. data from 1929 to 1998.

2.1 Calibration

We use values for the production technology that are standard in the macro literature, taken from Boldrin, Christiano, and Fisher (2001). In particular, the capital share (α) is 0.36, the

⁶See, for example, Campbell (1994), who considers permanent and transitory shocks, Cooley and Prescott (1995), transitory shocks, Jermann (1998), permanent and transitory shocks, Prescott (1986), permanent shocks, Rouwenhorst (1995), permanent and transitory shocks. See Prescott (1986) for an empirical discussion of the persistence of Solow residuals.

Table 1
Calibration

Table 1: Calibrated values of parameters that are constant across models.

Quarterly Model Calibration		
Parameter	Description	Value
α	Elasticity of capital	0.36
δ	Depreciation rate of capital	0.021
γ	Coefficient of relative risk aversion	5
μ	Mean technology growth rate	0.4%
σ	Volatility of technology shock	4.1%
φ	Persistence of the technology shock	{0.95, 1}

quarterly depreciation rate (δ) is 0.021, and the quarterly log technology growth rate (μ) is 0.4%. The model is an exogenous growth model with a balanced growth path. Thus, all endogenous variables in the long run grow at the same rate as technology.

While we allow the elasticity of intertemporal substitution to vary, we fix the coefficient of relative risk aversion (γ) to 5 across all models. This value is in the middle of the range of reasonable values for the coefficient of relative risk aversion, as suggested by Mehra and Prescott (1985). As pointed out by Tallarini (2000), the consumption dynamics are mainly a function of the elasticity of intertemporal substitution, and only to a second order affected by the risk aversion coefficient. Different from us, Tallarini fixes the elasticity of intertemporal substitution to unity and instead allows for high levels risk aversion. Our calibration strategy is thus complementary to his and similar to the approach in Bansal and Yaron (2004).

There are four remaining parameters: the elasticity of intertemporal substitution (ψ), capital adjustment costs (ξ), the time discount parameter (β), and the volatility of technology shocks (σ). For each model, we determine these four parameters by matching four moments: the volatility of consumption growth, the relative volatilities of consumption and output growth, the equity Sharpe ratio, and the level of the risk-free rate. These moments are directly related to the dynamic behavior of consumption and, in particular, the amount of consumption risk. While the volatility of consumption growth is of obvious importance, the relative volatility of output and consumption growth is a key quantity in a production economy. This ratio is a measure of how much consumption is smoothed, by optimally adjusting capital investments, relative to the exogenous technology shocks. The level of the risk-free rate is pinned down by β , which determines how much weight future utility,

and therefore also any long-run risk, is given. Finally, the maximum Sharpe ratio in the economy is a function of the *amount* of short-run and long-run consumption risk, as well as the *prices* of these risks. Table 1 reports calibrated values of model parameters that are, unless otherwise stated, constant across models.

2.2 Two Models with Long-Run Risk

Table 2 gives two benchmark calibrations, LRR I and LRR II, corresponding to a model with transitory or permanent technology shocks, respectively. Panel A shows that both models are able to match the volatility of consumption, the relative volatilities of consumption and output growth, the level of the risk free rate, and the Sharpe ratio.⁷ The latter fact is noteworthy given a coefficient of relative risk aversion of only 5. With a consumption volatility of 2.72%, a power utility model would give a Sharpe ratio of only 0.14, whereas both calibrations of the Epstein-Zin model slightly overshoot the sample annual Sharpe ratio of 0.33. Panel B of Table 2 reports that about 60% of the magnitude of the Sharpe ratio in both models is due to long-run risk. In this calculation, the amount of short-run risk is defined as $\gamma \times Std(\Delta c_t)$, whereas the amount of long-run risk is the residual.

While both the calibrated models generate a substantial amount of endogenous long-run risk in the consumption process, the way in which they do so is quite different. In particular, the calibrated transitory shock model has low *EIS* (0.05) and high capital adjustment costs, while the calibrated permanent shock model has high *EIS* (1.5) and low capital adjustment costs. In fact, since in the transitory shock model $\gamma < \frac{1}{\psi}$, while in the permanent shock model $\gamma > \frac{1}{\psi}$, the *price* of long-run consumption risk has the opposite sign in the two models: negative and positive, respectively. Therefore, it must be that the nature of the long-run risk that arise in a transitory and a permanent shock model are different.

Panel C shows important statistics the models were not calibrated to fit: the relative volatility of investment growth and the volatility of the risk-free rate, as well as the mean and standard deviation of the returns to a real default-free 10-year zero-coupon bond and the claim to firm payouts (the capital claim). Both models display investment volatility significantly higher than output volatility, although in neither model it is as high as in the

⁷It is important to note that the transitory shock model matches the risk free rate only by allowing a discount rate (β) greater than one. Prices in this economy are still well-defined, however, since the economy is growing (see Kocherlakota, 1990). One may principally object to a value of β greater than one. If we were to restrict $\beta < 1$, the annualized risk free rate in the transitory shock model would increase to over 25% since the *EIS* is low. The permanent shock model, however, has $\beta = 0.998$ and is not subject to this issue.

Table 2
Two Models with Endogenous Long-Run Consumption Risk

Table 2: This table reports key annualized moments for two calibrations of the stochastic growth model. The models have permanent and transitory technology shocks, respectively. The level of risk aversion (γ) is 5 in both models. Both models are calibrated to match the relative volatility of consumption to output, the volatility of output, the level of the risk free rate, and the Sharpe ratio of equity returns. The returns to capital are in both models the same as the return to investment, i.e. the claim to total firm payouts. This is compared to U.S. aggregate equity returns since there is no data that correspond directly to such a claim. The part of the Sharpe ratio due to "short-run" risk is defined as $\gamma \times Std(\Delta c_t)$. The empirical moments are taken from the annual U.S. sample from 1929-1998, corresponding to the sample in Bansal and Yaron (2004).

Statistic	U.S. Data 1929 – 1998	Transitory shocks: $\varphi = 0.95$	Permanent shocks: $\varphi = 1$
		Long-Run Risk I $\psi = 0.05, \gamma = 5$ $\beta = 1.064, \xi = 0.70$	Long-Run Risk II $\psi = 1.5, \gamma = 5$ $\beta = 0.998, \xi = 18$
Panel A - Calibrated Moments			
<i>Volatility of Consumption Growth</i>			
$\sigma [\Delta c]$ (%)	2.72	2.72	2.72
<i>Relative Volatility of Consumption and Output (GDP)</i>			
$\sigma [\Delta c] / \sigma [\Delta y]$	0.52	0.52	0.52
<i>Level of Risk Free Rate</i>			
$E [R_f]$ (%)	0.86	0.85	0.82
<i>Sharpe ratio</i>			
$E [R_i - R_f] / \sigma [R_i - R_f]$	0.33	0.34	0.36
Panel B - Decomposing the Sharpe ratio			
	<i>Short-Run Risk</i>	0.14 (40%)	0.14 (38%)
	<i>Long-Run Risk</i>	0.20 (60%)	0.22 (62%)
Panel C - Other Moments			
<i>Relative volatility of Investment and Output</i>			
$\sigma [\Delta i] / \sigma [\Delta y]$	3.32	2.36	1.83
<i>Volatility of the Risk Free Rate</i>			
$\sigma [R_f]$ (%)	0.97	4.60	0.45
<i>Mean and volatility of 10-year default free bond</i>			
$E [R_{10yr} - R_f]$ (%)	n/a	9.19	-0.87
$\sigma [R_{10yr} - R_f]$ (%)	n/a	27.37	2.41
<i>Returns to Aggregate Capital</i>			
$E [R_i - R_f]$ (%)	6.33	8.06	0.24
$\sigma [R_i - R_f]$ (%)	19.42	24.06	0.66

historical data. This is, however, where the similarities between the model output ends. Broadly speaking, the transitory shock model displays too volatile asset returns, while the permanent shock model displays reasonable long-term bond return volatility, but low return volatility of the claim to firm payouts. In particular, the low EIS in the transitory shock model makes the risk-free rate very sensitive to time-variation in expected consumption growth, which in turn makes it too volatile (4.6%). In the permanent shock model with its high EIS, the volatility of the risk-free rate is only 0.5%. The risk-free rate is very persistent, and so this difference is amplified for longer maturity bonds. The return volatility of the 10-year bond is 27%(!) in LRR I relative to the more reasonable 2.4% in LRR II. These discount rate dynamics carry over to the capital claim, which is too volatile in LRR I (24%) and not volatile enough in LRR II (0.7%). An alternative way of explaining this difference in return volatility is to refer to the level of capital adjustment costs, which is much higher in the transitory shock model. Importantly, note that the capital claim is *less* volatile than the 10-year bond in *both* models. This is due to counter-cyclical firm payouts. Empirically, aggregate stock market dividends are pro-cyclical, which calls into question the direct comparison of the capital claim to aggregate public equities.⁸ In the next section, we analyze the mechanisms within the model that give rise to endogenous long-run risk and the differences between the transitory and permanent shock models.

3 Inspecting the Mechanism

To establish intuition for the different mechanisms that determine the amount of short-run versus long-run consumption risk in the model, it is instructive to consider a log-linear approximation to the model around the non-stochastic steady state. This allows for an analytical solution to the equilibrium consumption choice in terms of fundamental parameters. Further, it allows for an analytical description of how shocks to the stochastic discount factor is related to shocks to technology, as well as an analytical definition and decomposition of short-run versus long-run consumption risk.

⁸We verify these statements regarding firm payouts and discuss their implications in more detail in Section 3.6. In particular, we there price a claim with dividends calibrated to the historical behavior of aggregate stock market dividends.

3.1 A Log-Linear Solution

To obtain the log-linear approximate solution, we follow Campbell (1994) who develops a log-linear solution to a special case of the model presented here with power utility preferences and no capital adjustment costs. Since the solution technique essentially is the same as in Campbell (1994), we relegate its description to the Appendix. As the concern is the *dynamic behavior* of key quantities in the model, we consider only deviations from the deterministic trend. Also, intercepts are not reported for clarity of exposition. In this section, lower case letters denote the natural log of their upper case counterparts normalized by the deterministic trend (e.g., $z_t \equiv \ln(Z_t/e^{ut})$).

Remark 1 *In equilibrium, the log of the value function, consumption, and capital are all affine in the model's two state variables, log capital and log technology:*

$$v_t = A_1 k_t + A_2 z_t, \quad (10)$$

$$c_t = B_1 k_t + B_2 z_t, \quad (11)$$

$$k_{t+1} = D_1 k_t + D_2 z_t, \quad (12)$$

where A_1 , A_2 , B_1 , B_2 , D_1 , and D_2 are analytical functions given in the Appendix of the model's preference and production technology parameters.

Since the approximation is log-linear and around the *non-stochastic* steady state, the model solution given in Remark 1 is not a function of the risk aversion parameter γ . Therefore, in the case of zero capital adjustment costs, the model solution is the same as that of Campbell (1994), despite the more general Epstein-Zin preferences. The equilibrium consumption dynamics are thus in this case driven by the EIS only. This highlights the importance of solving the exact model for quantitative analysis. However, the general intuition gained in this section carries over to the exact model.

The model solution from Remark 1 can be used to obtain standard time-series representations for the key variables in the model.

Corollary 2 *The log of output and consumption follow ARMA(2,1) processes in equilibrium:*

$$y_t = \frac{(1 - \alpha) + (\alpha D_2 - (1 - \alpha) D_1) L}{(1 - D_1 L)(1 - \varphi L)} \varepsilon_t, \quad (13)$$

and

$$c_t = \frac{B_2 + (B_1 D_2 - B_2 D_1) L}{(1 - D_1 L)(1 - \varphi L)} \varepsilon_t, \quad (14)$$

where L denotes the lag operator. Log capital follows an $AR(2)$ process:

$$k_{t+1} = \frac{D_2}{(1 - D_1 L)(1 - \varphi L)} \varepsilon_t. \quad (15)$$

Proof. See Appendix. ■

There are three important conclusions to be drawn from Corollary 2. First, all variables are homoskedastic. There is no internal time-variation in the propagation of the technology shocks, which are homoskedastic by assumption. While this is the outcome of the log-linear approximation, we show in a Technical Appendix that the exact model only generates economically small time-variation in the conditional volatility of the above quantities, and thus also in the price of risk. Second, notice that capital, output, and consumption all have autoregressive roots D_1 and φ . That is, the equilibrium dynamics of capital accumulation, as given by D_1 , carries over to the dynamic behavior of consumption and output. Capital accumulation is endogenous and its impact is *in addition* to the impact of technology, as given by φ . Third, if the technology shocks are transitory ($\varphi = 0.95$), shocks to the macro variables are also transitory, since $|D_1| < 1$. If, however, technology shocks are permanent ($\varphi = 1$), the above variables have a unit root and the economy exhibits a stochastic trend. Since the Bansal and Yaron (2004) model, and in fact most consumption-based asset pricing models, feature a unit root in log consumption, permanent shocks are needed to exhibit dynamic long-run behavior close to what is assumed in the typical long-run risk setup.

Corollary 3 *If log technology follows a random walk ($\varphi = 1$), log consumption growth follows an $ARMA(1,1)$ and expected log consumption growth follows an $AR(1)$:*

$$\Delta c_t = D_1 \Delta c_{t-1} + (B_1 D_2 - B_2 D_1) \varepsilon_{t-1} + B_2 \varepsilon_t \quad (16)$$

$$= x_{t-1} + B_2 \varepsilon_t, \quad (17)$$

where $x_{t-1} \equiv E_{t-1}[\Delta c_t]$, and

$$x_t = D_1 x_{t-1} + B_1 D_2 \varepsilon_t. \quad (18)$$

Proof. See Appendix. ■

Thus, the endogenous consumption dynamics arising in a benchmark production economy with i.i.d. shocks to technology growth are of the same form as those assumed in *Case*

I (homoskedastic consumption growth) in Bansal and Yaron (2004), except that the correlation between shocks to realized and expected consumption growth is perfectly negative or positive, while in Bansal and Yaron these shocks are assumed to be uncorrelated. Optimal capital accumulation (that is, D_1 and D_2) leads to a predictable component in consumption growth even though technology growth is unpredictable. With Epstein-Zin preferences, this predictable component can be either positively or negatively priced.⁹

Note that optimal consumption smoothing makes the predictability of consumption growth not only different from that of technology growth, but also different from that of output growth. For instance, in the permanent shock case it can be shown that expected output growth follows an AR(1) with the same persistence as expected consumption growth (D_1). In a well-calibrated model, the volatility of realized consumption growth is about half of that of output growth. Thus, such a model will have $B_2 \approx \frac{1}{2}(1 - \alpha)$, where B_2 and $(1 - \alpha)$ are the loadings of consumption and output growth on the technology shock, respectively. In this case, since $B_1 = 1 - B_2$ with permanent shocks (see Appendix), and since the calibrated value of α implies that $\frac{1}{2}(1 - \alpha) \approx \alpha$, we have that $B_1 D_2 \approx 2(\alpha D_2)$. That is, by Corollary 3, the volatility of expected consumption growth will, in a well-calibrated model, be about twice the volatility of expected output growth. Since *realized* output growth is about twice as volatile as realized consumption growth, the predictability of consumption growth is about 16 times larger than the predictability of output growth, in an R^2 sense.

Next, before we analyze how the relevant loadings in the above processes are related to the model parameters, we define short-run versus long-run consumption risk and their relative impact on the pricing kernel.

3.2 Short-run versus long-run consumption risk

Consider the following form of the stochastic discount factor:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{V_{t+1}/C_{t+1}}{E_t((V_{t+1}/C_{t+1})^{1-\gamma} (C_{t+1}/C_t)^{1-\gamma})^{1/(1-\gamma)}} \right)^{\frac{1}{\psi} - \gamma}. \quad (19)$$

The first term corresponds to the power utility case, where the price of consumption risk is equal to the relative risk aversion, γ . We refer to shocks to this factor (realized consumption

⁹With transitory technology shocks, the expression for expected consumption growth does not simplify as in the permanent shock case, and it is easiest to just work directly with the result given in Corollary 2. In a model with imperfectly correlated transitory *and* permanent technology shocks (not reported), the correlation between shocks to expected and realized consumption need not be perfectly positive or negative.

growth) as *short-run consumption risk*. Shocks to the second term, i.e., shocks to continuation utility normalized by current consumption, becomes a risk factor if $\gamma \neq 1/\psi$. We refer to the latter as *long-run consumption risk*. The purpose of this decomposition, as opposed to the decomposition implicit in equation (2), is to relate long-run consumption risk to shocks to *expected consumption growth*, consistent with the Bansal and Yaron (2004) definition of long-run risk. The two ways of writing the pricing kernel are of course equivalent. The log stochastic discount factor can now be written:

$$m_t = E_{t-1}[m_t] - \gamma \varepsilon_t^c - \left(\gamma - \frac{1}{\psi} \right) \varepsilon_t^{vc}, \quad (20)$$

where $\varepsilon_t^c \equiv \Delta c_t - E_{t-1}[\Delta c_t]$ and $\varepsilon_t^{vc} \equiv \Delta v c_t - E_{t-1}[\Delta v c_t]$, and where $v c_t \equiv \ln \frac{V_t}{C_t}$. Notice that the volatility of shocks to consumption growth, ε^c , and the normalized value function, ε^{vc} , as well as their correlation are important determinants of the conditional volatility of the log stochastic discount factor. We refer to the latter as the maximum Sharpe ratio in the economy or the price of risk.¹⁰ Using the log-linear approximate expressions for the value function and consumption from Remark 1, we have that:

$$\varepsilon_t^c \approx B_2 \varepsilon_t, \quad (21)$$

and

$$\varepsilon_t^{vc} \approx (A_2 - B_2) \varepsilon_t. \quad (22)$$

The two shocks are perfectly correlated with the exogenous technology shock, which has a standard deviation of σ . Thus, the *amount* of short-run consumption risk is given by $B_2 \sigma$, while the *amount* of long-run consumption risk is given by $(A_2 - B_2) \sigma$. The *price* of short-run consumption risk is given by γ , as usual, while the *price* of long-run consumption risk is given by $\gamma - 1/\psi$.

From Epstein and Zin (1989), the log wealth-consumption ratio is given by $w c_t \equiv \ln \frac{W_t}{C_t} = \ln \frac{1}{1-\beta} + (1 - 1/\psi) v c_t$. Thus, we have that:

$$(A_2 - B_2) \varepsilon_t = \frac{w c_t - E_{t-1}[w c_t]}{1 - 1/\psi}, \quad (23)$$

which shows that long-run consumption risk is reflected in shocks to the log wealth-consumption

¹⁰Strictly speaking, the maximum Sharpe ratio is given by $\sigma(M)/E(M)$. With a log approximation (as in, e.g., Campbell, 1999), the maximum Sharpe ratio is approximately $\sigma(m)$.

ratio. The following proposition relates these shocks to revisions in investors' expectations of future consumption growth.

Proposition 4 *In the log-linear model, since all shocks are homoskedastic and using a log-linear approximation similar to that in Campbell (1999), we can write:¹¹*

$$wc_t - E_{t-1}[wc_t] \approx (E_t - E_{t-1}) \sum_{j=1}^{\infty} \kappa_1^j (1 - 1/\psi) \Delta c_{t+j}, \quad (24)$$

where $\kappa_1 \equiv \frac{W/C-1}{W/C} < 1$, and W/C is the non-stochastic steady state wealth to consumption ratio. Thus, using equation (23), we have that:

$$(A_2 - B_2) \varepsilon_t \approx (E_t - E_{t-1}) \sum_{j=1}^{\infty} \kappa_1^j \Delta c_{t+j}. \quad (25)$$

Proof. See Appendix. ■

Thus, long-run consumption risk can be defined using shocks to the continuation utility normalized by consumption, shocks to the wealth-consumption ratio, or shocks to expected future consumption growth. Proposition 4 implies that understanding how endogenous long-run risk arises in this model is equivalent understanding how the dynamic behavior of *expected* consumption growth is determined.

3.3 The Elasticity of Intertemporal Substitution.

In this section, we investigate the endogenous consumption dynamics in the log-linear model for a wide range of values of the elasticity of intertemporal substitution. Following Campbell (1994), we parameterize the time-discounting parameter (β) in each log-linear model by requiring that the quarterly log interest rate in the model equals 0.015 (see Appendix for details). To as clearly as possible convey the intuition for how long-run consumption risk arises endogenously, we here normalize the volatility of technology shocks to unity and set the capital adjustment costs to zero.

Transitory Technology Shocks. Panel A of Figure 1 shows the impulse-response functions of technology and consumption to a positive, transitory technology shock ($\varphi = 0.95$). The consumption response is given for both a low and a high value of the EIS (0.1 and 1.5).

¹¹The notation $(E_t - E_{t-1})(x)$ is short-hand for $E[x|\Omega_t] - E[x|\Omega_{t-1}]$, where Ω_j denotes investors' information set at time j .

Figure 1 - Transitory and Permanent Shocks

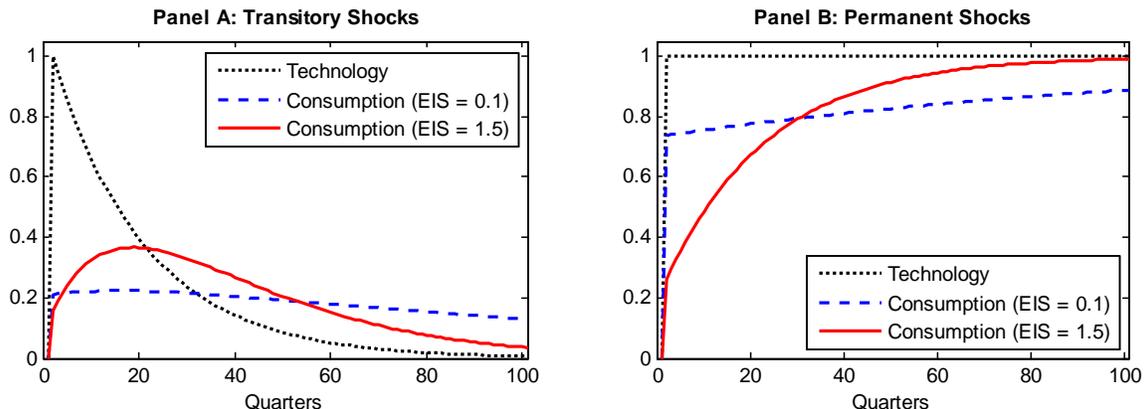


Figure 1: Panel A shows the impulse-response of technology (dotted line) and consumption to a transitory technology shock. Panel B shows the impulse-response of technology (dotted line) and consumption to a permanent technology shock. The consumption response is given for the case of EIS = 0.1 (dashed line) and EIS = 1.5 (solid line).

Agents in this economy want to take advantage of the temporary increase in the productivity of capital. To do so, they invest immediately in capital at the expense of current consumption. As a result, the consumption response is hump-shaped, and it is more so if the EIS is high. An agent with low EIS is more concerned with achieving a smooth consumption path.

The impulse-responses illustrate how time-varying expected consumption growth arises endogenously in the model: a *positive* shock to realized consumption growth (the initial consumption response) is associated with positive short-run expected consumption growth, but *negative* long-run expected consumption growth as consumption reverts back to the steady state. Thus, the shock to long-run expected consumption growth is negatively correlated with the shock to realized consumption growth. The volatility and persistence of *expected* consumption growth are increasing and decreasing in the EIS, respectively.

Panel A of Table 3 confirms this intuition and reports, for different values of the EIS, the amount of short- and long-run consumption risk, consumption growth volatility, expected consumption growth volatility, and the first-order autocorrelation of expected consumption growth. In particular, the volatility of expected consumption growth, $\sigma(x)$, is indeed increasing in the EIS. For most values of the EIS, the amount of long-run consumption risk, given by $A_2 - B_2$, is as expected negatively related to the amount of short-run risk, given by B_2 , and shocks to short- and long-run consumption risk are perfectly negatively correlated.

Permanent Technology Shocks. Panel B of Figure 1 shows that the long-run consumption dynamics are very different when technology shocks are permanent. In this case, technology adjusts immediately to the new steady state, and the permanently higher productivity of capital implies that the optimal long-run levels of both capital and consumption are also higher. Agents invest immediately in order to build up capital at the expense of current consumption, and consumption gradually increases towards the new steady state after the initial shock. Thus, a positive shock to realized consumption growth (the initial consumption response) is associated with a *positive* shock to long-run expected consumption growth. In this case, the two shocks are therefore perfectly positively correlated.

Panel B of Table 3 shows that the volatility of consumption growth, and thus the amount of *short*-run risk, is monotonically decreasing in the EIS as in the transitory shock case, while the volatility of *expected* consumption growth, $\sigma(x)$, and the amount of *long*-run risk, are increasing in the EIS. With a higher EIS, the agent is increasingly willing to substitute consumption today for consumption in the future.

Additional observations. From Table 3, it is clear that long-run consumption risk is a robust feature of the endogenous consumption choice. In particular, the economy exhibits no long-run risk only in very special cases. Most prominently, only if the agent is infinitely averse to substituting between consumption today and in the future (i.e., the EIS equals zero), consumption growth is i.i.d. and there is no long-run risk.¹² This is the production economy version of the Permanent Income Hypothesis (see, e.g., Friedman, 1957; Hall, 1978) as also pointed out by Campbell (1994). However, the case of an EIS of zero can be ruled out as neither the transitory nor the permanent shock model in this case match the relative volatility of consumption to output growth. For any other value of the EIS, expected consumption growth is time-varying ($\sigma(x) > 0$).

It is a robust stylized fact that consumption growth is less volatile than output growth. In the sample used for the calibration in this paper, the ratio of consumption growth volatility to output growth volatility is 0.52.¹³ This is an important magnitude to consider, because it pins down the initial consumption response given a technology shock. From Figure 1

¹²In the case of $EIS = 0$, the log-linear solution is actually not valid in the case of transitory technology shocks, as pointed out by Campbell (1994), and also it requires $\beta \rightarrow \infty$ to maintain finite interest rates. However, the log-linearization is valid for values of the EIS close to zero, and the discussion above strictly speaking only applies to such cases. The reported numbers are for $EIS = 0.0001$.

¹³Jermann (1998) reports a relative volatility of 0.49, while King and Rebelo (2000) reports a relative volatility of 0.74. The empirical estimate of the relative volatility depends on the sample period and on the filtering method used (for instance, the Hodrick-Prescott filter is often used in the macro literature).

Table 3
The EIS and endogenous short- and long-run consumption risk

Table 3: This table reports relevant statistics for the endogenous consumption dynamics in the log-linear approximation for different values of the EIS (ψ). There are zero capital adjustment costs. The variance of technology shocks is in the log-linear model normalized to one. Panel A shows the statistics for the case of transitory technology shocks ($\varphi = 0.95$), while Panel B shows the case of permanent technology shocks ($\varphi = 1$). The EIS = 1 row is actually the average statistics from models with EIS = 0.99 and EIS = 1.01.

<i>EIS</i>	<i>Consumption Risk:</i>		<i>Consumption dynamics:</i>			
	<i>Short-run</i>	<i>Long-run</i>	$\frac{\sigma(\Delta c)}{\sigma(\Delta y)}$	$\sigma(\Delta c)$	$\sigma(x)$	ρ_x
ψ	<i>risk, B_2</i>	<i>risk, A_2-B_2</i>				
Panel A: $\varphi = 0.95$						
0	0.15	0.00	0.23	0.15	0.00	<i>n/a</i>
0.1	0.21	-0.06	0.33	0.21	0.01	0.99
0.2	0.23	-0.07	0.36	0.23	0.02	0.98
0.5	0.23	-0.07	0.36	0.23	0.03	0.95
1.0	0.20	-0.04	0.33	0.21	0.06	0.93
1.5	0.16	-0.00	0.27	0.17	0.08	0.91
2.5	0.07	0.09	0.20	0.13	0.11	0.89
Panel B: $\varphi = 1$						
0	0.85	0.00	1.33	0.85	0.00	<i>n/a</i>
0.1	0.74	0.11	1.16	0.74	0.02	0.99
0.2	0.67	0.18	1.05	0.67	0.03	0.99
0.5	0.53	0.32	0.84	0.54	0.05	0.98
1.0	0.38	0.47	0.61	0.39	0.08	0.96
1.5	0.26	0.59	0.45	0.29	0.11	0.96
2.5	0.08	0.77	0.28	0.18	0.16	0.94

and Table 3, there is an inverse relation between the amount of long-run risk and the initial consumption response (short-run risk). Thus, matching the relative volatility of consumption and output effectively pins down the amount of short-run versus long-run risk in the economy.

We have established that changing the EIS changes the immediate consumption response, and therefore also the relative volatility of consumption and output growth. From Table 3, the permanent shock model can only match this moment with a high EIS, while the transitory shock model always generates too low a consumption response. As we show next, capital adjustment costs (CAC) help match this quantity for a broad range of values of the EIS.

3.4 Capital adjustment costs.

Panel A of Figure 2 shows the impulse-response from the log-linear model of consumption to a positive, one standard deviation transitory shock to technology when the EIS is low (0.1). Different from Figure 1, the consumption response is plotted for both zero and high capital adjustment costs (CAC). Here "high" adjustment costs correspond to adjustment costs that on average constitute approximately 1% of output. Panel B shows the corresponding graphs for the case of a permanent technology shock and a high EIS (1.5). In both cases, increasing CAC make firms invest less aggressively in response to technology shocks, which in turn increases the initial consumption response. Thus, increasing CAC increases short-run risk. For the permanent shock case, it also decreases the amount of long-run risk, as consumption now moves immediately closer to its new expected steady state. In the transitory shock case, increasing CAC makes the amount of long-run risk more negative, since the immediate consumption response in this case takes consumption farther away from its expected steady state value. In sum, capital adjustment costs allow us to vary the amount of short-run versus long-run consumption risk in the model, while holding the *prices* of short- and long-run consumption risk fixed.

Figure 2 - The Effect of Capital Adjustment Costs

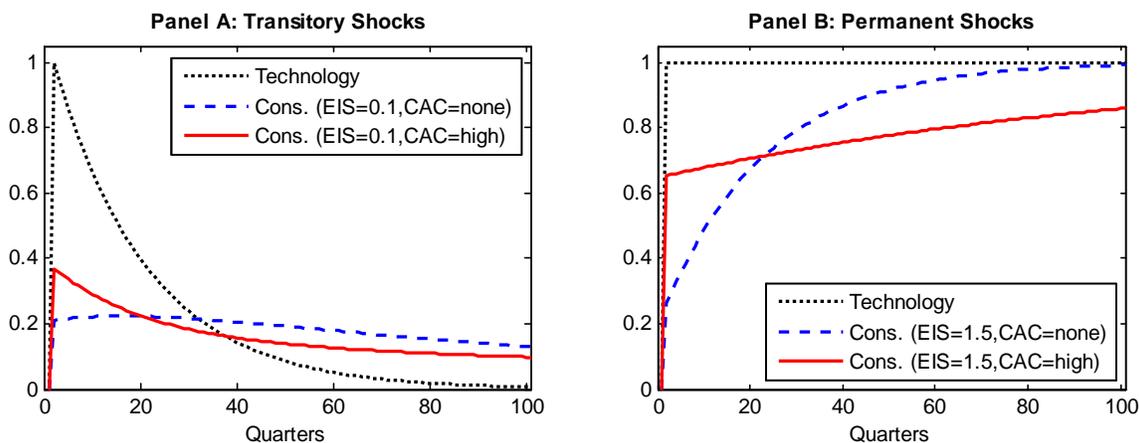


Figure 2: Panel A shows the impulse-response of technology (dotted line) and consumption to a transitory technology shock when the EIS = 0.1. Panel B shows the impulse-response of technology (dotted line) and consumption to a permanent technology shock when the EIS = 1.5. The dashed line shows the consumption response when there are no capital adjustment costs (CAC), while the solid line shows the consumption response when there are high adjustment costs.

3.5 Exactly Solved Models: Asset Pricing Moments

In this section, we evaluate the asset pricing properties of models with different EIS calibrated to macroeconomic data. Since the exercise here is quantitative, we consider exactly (i.e., numerically) solved models. Table 4 shows macroeconomic and asset pricing moments for models with either transitory or permanent technology shocks and two different values of the EIS, $\psi \in \{0.05, 1.5\}$. Per the intuition given for the log-linear model, capital adjustment costs varies across models so that each model, if possible, matches the historical volatilities of consumption and output growth. The resulting capital adjustment costs are the highest in the model with transitory technology shocks and a low EIS; on average 0.88% of output. The permanent shock model, on the other hand, cannot, even with no capital adjustment costs, match both the volatility of output and consumption growth unless the EIS is high (1.5). In this case, adjustment costs are only on average 0.04% of output. It should be noted that the calibrated capital adjustment costs used in this paper are within the wide range of such costs reported by the empirical literature. For instance, Hall (2004) argues that aggregate capital adjustment costs are close to zero, while Eberly, Rebelo, and Vincent (2009) estimates a model with quadratic, homogeneous adjustment costs on Compustat firms and find that adjustment costs represent on average 4.6% of firm revenue net of variable costs.

Table 4 further reports the price of risk (the maximal Sharpe ratio) and the prices of both short-run and long-run risk. Even though the coefficient of relative risk aversion and the volatility of consumption growth are the same across all models, the price of risk varies from close to zero to 0.36! With power utility and the calibrated consumption growth volatility, the price of risk would be $0.14 \approx 5 \times 2.72\%$. Deviations from this value are due to the effect of long-run risk in the model. In the case of transitory technology shocks, the price of risk is decreasing in the EIS. When the EIS is 1.5 (Model 2) the agent prefers early resolution of uncertainty. The price of risk is then low as the two risk factors, shocks to realized and expected future consumption growth, are negatively correlated and therefore hedge each other per the intuition given in the log-linear model. When the EIS is 0.05 (LRR I), the agents instead prefer late resolution of uncertainty and therefore *like* shocks to expected consumption growth. For these agents, a world where shocks to realized consumption (which they dislike) and expected consumption (which they like) are negatively correlated, is a more risky world. The same logic applies for the case of permanent shocks, where the two shocks are positively correlated. In this case, it is the *high EIS* model (LRR II) that has a high price of risk. The pattern in the price of risk carries over to the Sharpe ratio of returns.

Table 4
Calibrated Models - The EIS and The Persistence of Shocks

Table 4: This table reports relevant macroeconomic and asset pricing moments for models with either transitory ($\varphi = 0.95$) or permanent ($\varphi = 1.00$) technology shocks and different levels of the elasticity of intertemporal substitution (ψ). The coefficient of relative risk aversion (γ) is 5 across all models. We calibrate the discount factor (β) for each model to match the level of the risk free rate. The volatility of the technology shock (σ) is set so that the volatility of consumption growth is the same across models. Capital adjustment costs (ξ) are set so that (if possible) the relative volatility of consumption to output growth is matched. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis for the empirical moment values. The sample is the same as in Bansal and Yaron (2004).

		LRR I	Model 2	Model 3	LRR II
		Transitory Shocks ($\varphi = 0.95$)		Permanent Shocks ($\varphi = 1.00$)	
		$\psi = 0.05$	$\psi = 1.5$	$\psi = 0.05$	$\psi = 1.5$
		$\beta = 1.064$	$\beta = 0.998$	$\beta = 1.067$	$\beta = 0.998$
	U.S. Data	$\xi = 0.70$	$\xi = 6.75$	$\xi = \infty$	$\xi = 18.0$
Statistic	1929-1998	$\sigma = 4.05\%$	$\sigma = 4.05\%$	$\sigma = 1.61\%$	$\sigma = 4.11\%$
Macro Moments					
$\sigma[\Delta c]$ (%)	2.72	2.72	2.72	2.72	2.72
$\sigma[\Delta c]/\sigma[\Delta y]$	0.52	0.52	0.52	1.32	0.52
$\sigma[\Delta i]/\sigma[\Delta y]$	3.32	2.36	1.90	0.34	1.83
<i>AdjCosts/Output</i> (%)	<i>n/a</i>	0.88	0.04	0.00	0.04
The Price of Risk					
$\sigma(M)/E(M)$	<i>n/a</i>	0.34	0.02	0.08	0.36
<i>Price short-run risk</i> (γ)		5	5	5	5
<i>Price long-run risk</i> ($\gamma - 1/\psi$)		-15	4.3	-15	4.3
Short- and Long-Term Real Risk Free Bonds					
$E[R_f]$ (%)	0.86	0.85	1.84	2.93	0.82
$\sigma[R_f]$ (%)	0.97	4.60	0.17	0.56	0.45
$E[R_{10yr} - R_f]$ (%)	<i>n/a</i>	9.19	0.02	-0.16	-0.87
$\sigma[R_{10yr} - R_f]$ (%)	<i>n/a</i>	27.37	0.69	2.05	2.41
The Consumption Claim					
$E[R_a - R_f]$ (%)	<i>n/a</i>	9.68	0.05	-0.38	1.59
$\sigma[R_a - R_f]$ (%)	<i>n/a</i>	29.15	1.85	4.74	4.41
$SR[R_a]$	<i>n/a</i>	0.34	0.03	-0.08	0.36
The Capital Claim					
$E[R_i - R_f]$ (%)	6.33	8.06	0.04	0.00	0.24
$\sigma[R_i - R_f]$ (%)	19.42	24.06	1.59	0.06	0.66
$SR[R_i]$	0.33	0.34	0.03	0.08	0.36

Table 4 also gives the first two moments of the real risk-free rate, the return to a 10-year zero coupon default-free bond, the return to the consumption claim, and the return to the capital claim. For all claims, the return volatility is strongly increasing in capital adjustment costs. This is a well known effect as non-zero CAC means that marginal q deviates from 1. For model "LRR I" (the preferred transitory shock calibration), the volatility of the consumption and capital claims are high at 29.15% and 24.06% per year. Model "LRR II" (the preferred permanent shock calibration), which has much lower capital adjustment costs, displays a return volatility of the consumption claim of 4.41%, while the capital claim has a return volatility of only 0.66%. Correspondingly, the annualized excess return to the capital claim is 8.06% in LRR I and 0.24% in LRR II. While all the models are able to match the level of the risk free rate relatively well, the volatility of the risk-free rate in the transitory shock model with low EIS is too high. The high volatility and persistence of the risk free rate provides a discount rate explanation for why this model generates very high volatility of the return to the consumption and capital claims, while the models with low risk-free rate volatility generates low corresponding return volatilities. This is corroborated by the return volatility of the 10-year bond. This claim has no cash flow risk, but its return volatility is very high (27%) in the model with high adjustment costs and low EIS (LRR I), but only 2.4% in the model with low adjustment costs and high EIS (LRR II). There is no economically significant predictability in excess returns to the capital claim (reported in a technical appendix), and so discount rate variation is mainly due to variation in the risk-free rate. Finally, we note that bonds are hedges in the permanent shock model as the risk-free rate decreases in response to negative technology shocks, while the opposite is the case in the transitory shock model. This follows directly from the different behavior of expected consumption growth in these models as discussed earlier.

3.6 Pricing a Levered Equity Claim

Many papers define equity market dividends as a levered claim to the consumption stream, in order to fit the observed volatility of dividend growth, the high equity return volatility and the equity risk premium. With a leverage factor of about 3 on the consumption claim, the resulting "equity" return premium for model LRR II would be around 4.5% with a return volatility of about 13%. It is surprising then that the capital claim displays lower return volatility than the consumption claim in both models. In fact, in both LRR I and LRR II the returns to the claim to firm payouts are less volatile than the returns to real long-term

default-free bonds, which have no cash flow risk. Intuitively, one might expect the residual claim to be more volatile.

The endogenous payout process differs from the endogenous consumption process along important dimensions. While firm payouts are given by $D_t = \alpha Y_t - I_t$, dividends to the total wealth portfolio (i.e., aggregate consumption) are given by $C_t = Y_t - I_t$. In the calibrated models, output, investment, and consumption all respond positively to a positive technology shock. However, since $\alpha = 0.36 < 1$, dividends are less pro-cyclical than consumption and may in fact decrease in response to a shock. Figure 3 shows the impulse responses of firm payouts. In both models, payouts respond negatively to a positive technology shock. In fact, the annual correlation between time-averaged consumption and payout growth is -1 in both models. Bansal and Yaron (2004) report that the corresponding correlation between consumption and *public equity market dividend* growth is 0.55 in the data. Payout growth in the permanent shock model is also too volatile, with an annual volatility around 40%.¹⁴ A high payout volatility arises as the average *level* of firm payout is quite small. While the initial payout response is negative, the shock to payout *growth* is positive, which explains why in both models the return to capital responds positively to technology shocks. However, the initial negative cash flow response is the reason why the payout claim is less volatile than both bonds and the consumption claim.

Figure 3 - Endogenous Firm Payouts

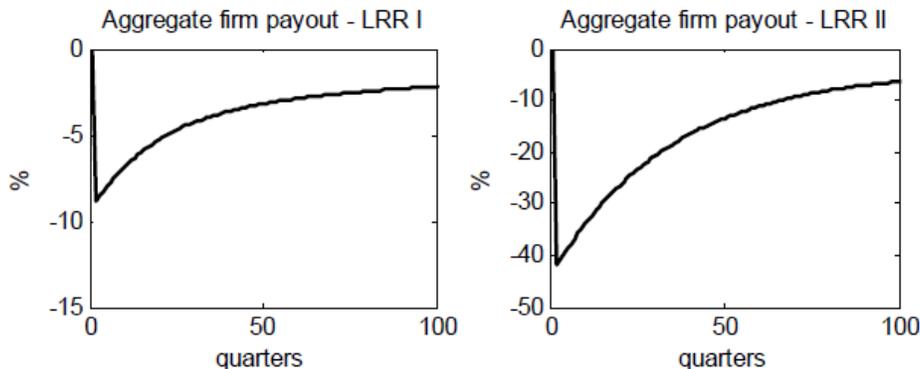


Figure 3: The plots show the impulse-responses of firm payouts for the LLR I (transitory technology shocks) and the LLR II (permanent technology shocks) models.

¹⁴Since in both models dividends are sometimes negative, it is not clear how to compare the endogenous dividends to the volatility in dividend growth rates observed in the data. As a partial resolution, the dividend *growth* in the simulated data is constructed as: $\frac{D_{t+1}-D_t}{D_t} = \frac{D_{t+1}-D_t}{Y_t} E[Y_t/D_t]$.

Pricing a Claim to Stock Market Dividends. There are good reasons to not regard the payouts from the one-sector representative firm in this model as equivalent to the dividends from the aggregate stock market. In reality, public equity is only a small fraction of all productive capital. In particular, private equity, small businesses, real estate and the government contribute substantial fractions of total output. For instance, proprietors income, which can be considered dividends from small businesses, is historically on average a larger fraction of output than dividend payments from the total corporate sector, which in turn is a larger fraction than the dividends that are paid out by the public equity sector. To give a sense of the differences, the annual real earnings volatility of the S&P500 firms from 1948 to 2005 was 18% (using the Shiller data), while the volatility of all corporate profits from the Flow of Funds data was only 10%. The correlation between the two is only 0.6 over this period. The volatility of the growth in the sum of proprietors income, rental income and all corporate sector profits have a volatility of only 4%. This is much closer to the volatility of output growth, which is the same as the earnings volatility of the firm within the model.

While it is beyond the scope of this paper to introduce additional productive sectors, Table 5 documents the properties of a claim to a cash flow stream that is calibrated to the historical behavior of the U.S. equity market dividend growth. In particular, we calibrate dividend growth to be consistent with the historical mean, volatility, and annual autocorrelation of dividend growth, as well as the contemporaneous correlation with consumption growth. To do so, we define market log dividend growth as follows:

$$\Delta d_{m,t+1} = d_0 + d_1 (zc_t - E[zc]) + d_2 \varepsilon_{t+1} + \sigma_d \eta_{t+1}, \quad (26)$$

where $\eta \sim N(0, 1)$ is i.i.d. and uncorrelated with the technology shock. Thus, σ_d is the idiosyncratic dividend volatility, while d_2 captures the systematic shocks to dividends, and d_1 captures common predictability of consumption and dividends. Table 5 shows that with $d_0 = 0.004$, $d_1 = 0.055$, $d_3 = 0.6$, and $\sigma_d = 0.058$, the model implied moments for dividend growth are consistent with both the historical data and the moments of the exogenous dividend process in Bansal and Yaron (2004).

In the transitory shock model (LRR I), the returns to this dividend claim are too volatile and therefore the average excess returns are too high. This is due to the very high volatility of the risk-free rate, which induces too high variability in discount rates. In the permanent shock model (LRR II), the average excess returns and return volatility of this claim are very close to those observed for the aggregate stock market. In sum, the endogenous consumption

dynamics from the permanent shock model (LRR II), where technology growth is i.i.d., delivers a stochastic discount factor that, when faced with a claim to a dividend process that is consistent with the historical properties of U.S. equity market dividends, gives a risk premium and return volatility that correspond to those of the U.S. equity market.

Table 5
Calibrated Stock Market Dividend Claim

Table 5: This table reports annual time-averaged moments for a dividend claim calibrated to match the observed properties of aggregate stock market dividends, as well as the dividend claim in Bansal and Yaron (2004). The annual risk premium, return volatility and Sharpe ratio is given for each model relative to that found in historical data.

	<i>Data</i>		<i>LRR I</i>	<i>LRR II</i>	<i>B&Y</i>
	1929-1998	(<i>s.e.</i>)	$\varphi = 0.95$ $\gamma=5, \psi=0.05$	$\varphi = 1.00$ $\gamma=5, \psi=1.5$	<i>Case I</i> $\gamma=10, \psi=1.5$
$E(\Delta d)$ (%)	1.80	(0.30)	1.81	1.75	1.80
$\sigma(\Delta d)$ (%)	11.49	(1.98)	11.33	11.59	12.11
$corr(\Delta d_t, \Delta d_{t-1})$	0.21	(0.13)	0.33	0.34	0.33
$corr(\Delta d_t, \Delta c_t)$	0.55	(0.34)	0.54	0.55	0.34
$E(R_m - R_f)$ (%)	6.33	(2.15)	10.98	6.11	4.20
$\sigma(R_m - R_f)$ (%)	19.42	(3.07)	33.04	19.35	16.21
$SR(R_m - R_f)$	0.33		0.33	0.32	0.26

4 Endogenous Consumption Predictability

In this section, we take a closer look at the equilibrium consumption dynamics in the exactly solved benchmark models LRR I and LRR II. Since long-run risk can be related to consumption predictability (see Proposition 4), we compare the consumption predictability implied by the benchmark models to what we see in the data and propose an instrument, suggested by the production economy model, for predicting consumption growth.

4.1 Endogenous Consumption versus Bansal and Yaron (2004)

Table 6 shows the quarterly volatility of realized and expected consumption growth, the first order autocorrelation of expected consumption growth, and the conditional correlation

between realized and expected consumption growth.¹⁵ The moments are taken from the two exactly solved calibrated models LRR I and LRR II. In the left column we report, for comparison, the preference parameters used in Bansal and Yaron (2004) and the consumption statistics implied by their calibration of an *exogenous* consumption process.

The models are calibrated to have similar quarterly consumption volatility.¹⁶ More interestingly, note that the volatility and autocorrelation of expected consumption growth in both the production economy models are of a similar magnitude as that in Bansal and Yaron's calibration: expected consumption growth is a relatively small, but highly persistent component of realized consumption growth. In both cases the persistence is slightly higher than in Bansal and Yaron (2004), while the volatility is smaller.

A key difference between the models is in the conditional correlation of shocks to realized and expected consumption growth. Bansal and Yaron assume this correlation is zero, while in the transitory shock model it is -1 and in the permanent shock model it is 1. The volatility of the stochastic discount factor is more sensitive to the amount of long-run risk in the production economy, because the long-run risk shocks (ε^{vc}) are perfectly correlated with the shock to realized consumption growth (ε^c) as both shocks are driven by the same technology shock (see Equations (20) to (22)). This is reflected in the coefficient of relative risk aversion, which is 10 in Bansal and Yaron, but only 5 in the production economy model.

4.2 The Autocorrelation of Consumption Growth

Figure 4 compares the model implied autocorrelations of annual, time-averaged log consumption growth from the model versus the empirical counter-parts based on BEA data of real, per capita aggregate consumption data from 1929 to 2005 (76 observations). This is the longest single source of consumption data available. Consumption growth in the data is positively autocorrelated at the 1 and 2 year lags with autocorrelation coefficients 0.44 and 0.16, respectively. The 3 and 4 year autocorrelation coefficients, however, are negative (-0.11 and -0.24), while the remaining 5 to 10 year autocorrelation coefficients hover around zero. Comparing this to the autocorrelation pattern generated by the transitory shock model (LRR I), the 1 and 2 year autocorrelations are not only on average too low relative the data, but the

¹⁵In a technical appendix, we show that there is some heteroskedasticity in both shocks to expected and realized consumption growth, but these effects are second order. We find expected consumption growth by numerically integrating realized consumption growth over the technology shock, given the state of the economy.

¹⁶Bansal and Yaron (2004) calibrate their model such that annual *time-averaged* consumption volatility is 2.72%. The models in this paper are calibrated to have an actual consumption volatility of 2.72% p.a.

Table 6
Expected Quarterly Consumption Growth Dynamics

Table 6: This table reports the volatility of quarterly realized consumption growth, expected consumption growth, the first order autocorrelation of expected consumption growth and the contemporaneous correlation between shocks consumption growth and expected consumption growth. These statistics are given for the Bansal and Yaron (2004) model (their "Case I"; quarterly and not time-averaged), the preferred transitory shock model (LRR I) and the preferred permanent shock model (LRR II). The statistics are computed numerically for the production economy models. These statistics give a reasonable summary of the endogenous consumption dynamics. The table also gives the price of long- and short-run consumption risk in each model.

	Bansal & Yaron	LRR I	LRR II
	$(\Delta c_t = \mu + x_{t-1} + \nu_t,$ $x_t = \rho x_{t-1} + \eta_t)$	Transitory Shocks $\varphi = 0.95$	Permanent Shocks $\varphi = 1.00$
	$\psi = 1.5$ $\beta = 0.998$ $\gamma = 10$	$\psi = 0.05$ $\beta = 1.064$ $\gamma = 5, \xi = 0.70$	$\psi = 1.5$ $\beta = 0.998$ $\gamma = 5, \xi = 18.0$
Statistic (quarterly)			
$\sigma[\Delta c_t]$ (%)	1.440	1.360	1.362
$\sigma[x_t]$ (%)	0.495	0.126	0.336
$\sigma[x_t]/\sigma[\Delta c_t]$	0.344	0.093	0.247
<i>Autocorr</i> (x_t)	0.938	0.959	0.970
<i>Corr</i> $_{t-1}(\Delta c_t, x_t)$	0	-1	1
<i>Price of short-run risk</i> (γ)	10	5	5
<i>Price of long-run risk</i> ($\gamma - 1/\psi$)	4.3	-15	4.3
<i>SR</i> [R_i]	0.33	0.34	0.36

empirical autocorrelations are outside the model's two standard error bounds. The transitory shock model predicts small, negative autocorrelation in consumption growth. However, the 1 year autocorrelation estimate is on average positive due to the time-averaging of the consumption data. The empirical autocorrelations for years 3 to 10 are within the transitory shock model's two standard error bounds. The permanent shock model, on the other hand, predicts positive autocorrelation of consumption growth, and for the 1 and 2 year lags the average autocorrelations in the model are very close to the empirical point estimates. The model is not consistent with the negative autocorrelations in the data at the 3 and 4 year lags, but the remaining lags are within the model's two standard error bounds.

A joint Simulated Method of Moments goodness of fit test of whether the average autocorrelation coefficients generated by each model are *jointly* significantly different from those

in the data gave a p -value of 7% for the permanent shock model and 30% for the transitory shock model. Thus, while neither model can be rejected at the 5%-level based on the autocorrelation properties of the consumption process given the samples we have available, we note that the permanent shock model implies too high autocorrelations in aggregate consumption growth on average relative to the point estimates from the data. We also tested whether the autocorrelations of each model are jointly significantly different from those generated by *i.i.d.* quarterly consumption growth that is time-averaged to get to annual growth rates. They are not. The p -value for both models is in this case greater than 0.9.¹⁷

Figure 4 - Autocorrelation of Annual Consumption growth

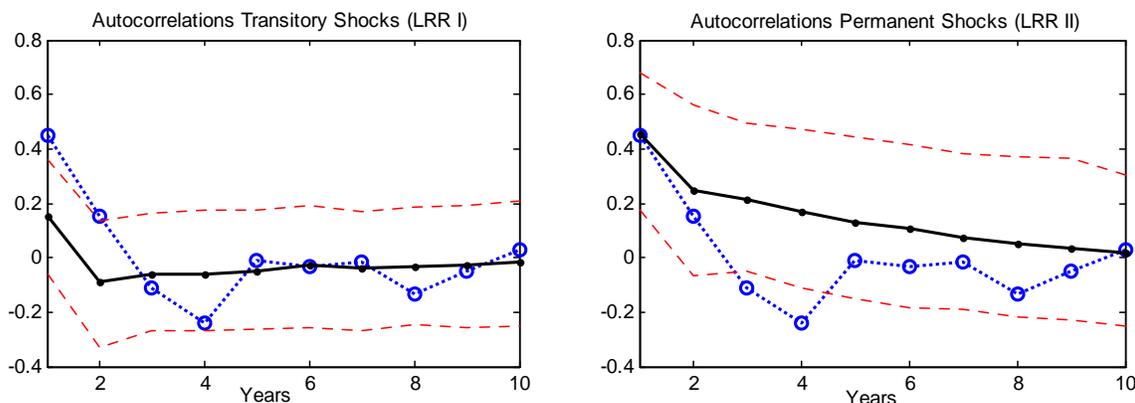


Figure 4: The dots along the solid line in the left graph shows the average autocorrelations of annual log consumption growth estimated from 76 year simulated samples from the transitory shock model (LRR I). The dashed lines are two standard deviation confidence bounds. The dotted line with circles give the empirical sample autocorrelations for annual consumption growth from 1929 to 2005 (76 observations). The right graph is similar, but for the model with permanent shocks (LRR II).

5 Conclusion

We analyze a one-sector DSGE model with capital adjustment costs and Epstein-Zin preferences and show how long-run risk arises endogenously as a consequence of the optimal consumption-savings decision. Consumption smoothing induces time-variation in expected consumption growth, even in the case when log technology follows a random walk. The quantitative implications of endogenous long-run risk can be large when the model is calibrated

¹⁷In a Technical Appendix to this paper, posted on the corresponding author's website, we provide evidence of long-horizon consumption growth predictability consistent with the permanent shock model, but not consistent with the transitory shock model.

to match key macro economic moments and help the model jointly explain a high price of risk and a low volatility of consumption growth with a low coefficient of risk aversion.

We find that the elasticity of intertemporal substitution, which strongly affects the dynamics of the macroeconomic variables, also strongly affects the price of risk and the Sharpe ratio of returns in all our calibrations of the model. Thus, the elasticity of substitution provides a tight link between quantity dynamics and asset prices in our implementation of the standard stochastic growth model, different from Tallarini (2000) who emphasizes the role of high risk aversion and a separation between quantity and asset price dynamics.

It would be useful, in terms of relating the asset pricing implications of the model directly to the historical returns on public equity, to consider a multi-sector model that accounts for the fact that the public equity market is only a relatively small fraction of all productive capital in the economy. Also, the interaction between labor markets and asset markets in the presence of endogenous long-run risk warrants further attention. Much of the real business cycle literature is concerned with also explaining the pro-cyclicality of hours worked. We abstract from this aspect of the model, following Jermann (1998), by assuming a fixed labor supply. Introducing preferences over leisure leads to non-trivial interaction between the consumption and leisure choice, even when consumption and leisure are separable in the intratemporal utility function. With Epstein-Zin preferences *intra*temporal separability does not carry over to *inter*temporal separability, except in the power utility special case. Long-run risks will in such a model not only be related to predictability in consumption growth, but also to predictability in leisure. We leave this topic for future research.

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A Appendix

A.1 The Return to Investment and the Firm's Problem

The firm maximizes firm value. Let $M_{t,t+1}$ denote the stochastic discount factor. The firm's problem is then:

$$\max_{\{I_t, K_{t+1}, H_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} M_{0,t} \left\{ (Y_t - W_t H_t - I_t) - q_t \left(K_{t+1} - (1 - \delta) K_t - \phi \left(\frac{I_t}{K_t} \right) K_t \right) \right\} \right], \quad (1)$$

where q_t denotes the shadow price of the capital accumulation constraint, equivalent to marginal q . Each period in time the firm decides how much labor to employ and how much to invest, taking marginal q as given. The first order conditions with respect to H_t , I_t and K_{t+1} are, respectively:

$$0 = (1 - \alpha) (Y_t / H_t) - W_t, \quad (2)$$

$$0 = -1 + q_t \phi' (I_t / K_t), \quad (3)$$

$$0 = -q_t + E_t \left[M_{t+1} \left\{ \alpha \frac{Y_{t+1}}{K_{t+1}} + q_{t+1} \left((1 - \delta) - \phi' \left(\frac{I_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} + \phi \left(\frac{I_{t+1}}{K_{t+1}} \right) \right) \right\} \right]. \quad (4)$$

Substituting out q_t and q_{t+1} in (4) and imposing equilibrium $H_t = 1$ yields:

$$1 = E_t [M_{t+1} R_{i,t+1}], \quad (5)$$

where the firm's return to investment, R_i , is as given in Equation (7) in the main text. The labor market clearing wage is $W_t = (1 - \alpha) Y_t$.

A.2 Log-Linear Approximate Solution

Solution for Remark 1: In the following we suppress all constant terms for ease of exposition. Lower case variables are logs of their upper case counterparts normalized by the time-trend. The solution method is parallel to that in Campbell (1994), except that we (a) introduce capital adjustment costs, and (b) solve for the general case of Epstein-Zin preferences. The derivations presented here are minimal, and a Technical Appendix is available from the authors for more details of the solution. We log-linearize around the steady-state consumption to output ratio and assume when convenient that the interest rate $R \approx 1 + r$, where $r = \ln R$. Further, we parameterize the time-preference parameter (β) implicitly in each model with different EIS (ψ) by requiring that $r = 0.015$ (quarterly), as in Campbell (1994), which is why β does not show up in the solution. For the purposes of the log-linear solution, we assume quadratic capital adjustment costs, $\phi(X) = X - \frac{\xi}{2} (X)^2$.¹ The steady state investment rate satisfies $I/K = \frac{1 - \sqrt{1 - 2\xi(\delta + \mu)}}{\xi}$, where we assume

¹This is different from the capital adjustment costs in the model in the main body of the paper, which has zero capital adjustment costs in the non-stochastic steady state. However, this case would lead to no capital adjustment costs at all in a

that $1 - 2\xi(\delta + \mu) > 0$. We choose the solution that yields the lower investment rate as this is more efficient for the consumer. After much algebra, the log-linear capital accumulation equation obtains:

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 z_t + (1 - \lambda_1 - \lambda_2) c_t, \text{ where} \quad (6)$$

$$\lambda_2 = \lambda_6 \frac{\mu + \delta}{1 + \mu} \left[\frac{1 - \alpha}{1 - \exp(c - y)} \right], \quad \lambda_1 = \lambda_6 \frac{\mu + \delta}{1 + \mu} \left[\frac{\exp(c - y) - (1 - \alpha)}{1 - \exp(c - y)} \right] + 1, \quad (7)$$

$$\text{and } \lambda_6 = 2 - \frac{1 - \sqrt{1 - 2\xi(\delta + \mu)}}{\xi(\delta + \mu)}. \quad (8)$$

The log-linear return on investments is:

$$r_{i,t+1} \approx \tilde{\lambda}_3 (z_{t+1} - k_{t+1}) + \lambda_4 (i_t - k_t) + \lambda_5 (i_{t+1} - k_{t+1}), \text{ where} \quad (9)$$

$$\tilde{\lambda}_3 = (1 - \alpha) \frac{(r + \delta) + \left((\delta + \mu) - \frac{1 - \sqrt{1 - 2\xi(\delta + \mu)}}{\xi} \right)}{1 + r}, \quad \lambda_4 = -\frac{1 - \sqrt{1 - 2\xi(\delta + \mu)}}{\sqrt{1 - 2\xi(\delta + \mu)}}, \quad (10)$$

$$\text{and } \lambda_5 = \left(\frac{1 + \mu}{1 + r} \right) \frac{1 - \sqrt{1 - 2\xi(\delta + \mu)}}{\sqrt{1 - 2\xi(\delta + \mu)}}. \quad (11)$$

The log-linearization of the return on wealth is well-known (e.g., Bansal and Yaron, 2004) and yields:

$$r_{a,t+1} \approx \kappa_0 + \kappa_1 h_{t+1} - h_t + \Delta c_{t+1}, \quad (12)$$

where $h_t \equiv w_t^{ex} - c_t$, is the log wealth (W_t^{ex}) to consumption (C_t) ratio. This is not the same wealth-consumption ratio as in Equation (23) in the main text, as W^{ex} is wealth net of current consumption. We log-linearize around the non-stochastic steady state, so $\kappa_1 = \frac{\overline{W/C} - 1}{\overline{W/C}} > 0$ and close to 1, where $\overline{W/C}$ is the non-stochastic steady-state wealth to consumption ratio where wealth is measured *cum-dividend* as in Equation (23) in the main text.

Solving the Model. There are three state variables in this economy: the time trend and the current levels of capital and technology. Since we are ignoring intercepts throughout the analysis and since lower case variables are logs of their upper case counterparts normalized by the time-trend, we conjecture that:

$$h_t = \tilde{A}_1 k_t + \tilde{A}_2 z_t, \quad (13)$$

$$c_t = B_1 k_t + B_2 z_t. \quad (14)$$

Using Equations (6) and (14), we have that:

$$k_{t+1} = D_1 k_t + D_2 z_t, \quad (15)$$

log-linear approximation around the non-stochastic steady state.

where $D_1 = \lambda_1 + (1 - \lambda_1 - \lambda_2) B_1$ and $D_2 = \lambda_2 + (1 - \lambda_1 - \lambda_2) B_2$. Given B_1 and B_2 , D_1 and D_2 can be obtained.

From our conjecture and the log-linear relations, we can write (first substitute for Δk_{t+1} , then for c_t):

$$\Delta c_{t+1} = [(1 - \lambda_1 - \lambda_2) B_1^2 + (\lambda_1 - 1) B_1] k_t + [B_1 \lambda_2 + (1 - \lambda_1 - \lambda_2) B_1 B_2 + B_2 (\varphi - 1)] z_t + B_2 \varepsilon_{t+1}. \quad (16)$$

Define the constants d_{ck} , and d_{cz} implicitly by writing

$$\Delta c_{t+1} = d_{ck} k_t + d_{cz} z_t + B_2 \varepsilon_{t+1}. \quad (17)$$

Since the shocks are i.i.d. normal, log consumption growth is normally distributed with a time-varying mean and constant variance. The log return on wealth can then be written (after much algebra) as:

$$\begin{aligned} r_{a,t+1} = & \left((\kappa_1 \lambda_1 - 1) \tilde{A}_1 + d_{ck} + \kappa_1 (1 - \lambda_1 - \lambda_2) \tilde{A}_1 B_1 \right) k_t + \dots \\ & \left(\kappa_1 \lambda_2 \tilde{A}_1 + d_{cz} + (\kappa_1 \varphi - 1) \tilde{A}_2 + \kappa_1 (1 - \lambda_1 - \lambda_2) \tilde{A}_1 B_2 \right) z_t + \left(\kappa_1 \tilde{A}_2 + B_2 \right) \varepsilon_{t+1}. \end{aligned} \quad (18)$$

Implicitly define the constants d_{rk} , and d_{rz} by writing

$$r_{a,t+1} = d_{rk} k_t + d_{rz} z_t + \left(\kappa_1 \tilde{A}_2 + B_2 \right) \varepsilon_{t+1}. \quad (19)$$

Using the equilibrium conjectured capital and consumption processes, the technology process $z_{t+1} = \varphi z_t + \varepsilon_{t+1}$, and the expression for the investment return we have (after much algebra) that:

$$r_{i,t+1} = \left\{ (\lambda_4 + \lambda_5 D_1) \left[\frac{\alpha}{1 - \exp(c - y)} - 1 - \frac{\exp(c - y)}{1 - \exp(c - y)} B_1 \right] - \tilde{\lambda}_3 D_1 \right\} k_t \dots \quad (20)$$

$$+ \left\{ (\varphi - D_2) \tilde{\lambda}_3 + (\lambda_4 + \lambda_5 \varphi) \left[\frac{(1 - \alpha)}{1 - \exp(c - y)} - \frac{\exp(c - y)}{1 - \exp(c - y)} B_2 \right] \dots \right\} z_t \dots \quad (21)$$

$$+ \left\{ \tilde{\lambda}_3 + \lambda_5 \left[\frac{(1 - \alpha)}{1 - \exp(c - y)} - \frac{\exp(c - y)}{1 - \exp(c - y)} B_2 \right] \right\} \varepsilon_{t+1}. \quad (22)$$

From the above, implicitly define d_{ik} , d_{iz} and λ_3 by writing

$$r_{i,t+1} = d_{ik} k_t + d_{iz} z_t + \lambda_3 \varepsilon_{t+1}. \quad (23)$$

Moment restrictions. The log stochastic discount factor is:

$$m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{a,t+1}. \quad (24)$$

From moment restrictions of the return on wealth and the return on investment ($1 = E_t [\exp(m_{t+1} + r_{a,t+1})]$ and $1 = E_t [\exp(m_{t+1} + r_{i,t+1})]$), the fact the the log SDF and returns are normally distributed, and by the method of undetermined coefficients, we obtain a system of four equations and four unknowns and can solve for the variables $\tilde{A}_1, \tilde{A}_2, B_1, B_2$:

$$\text{Equation 1 } (k_t): -\frac{1}{\psi} d_{ck} + d_{rk} = 0.$$

$$\text{Equation 2 } (z_t): -\frac{1}{\psi} d_{cz} + d_{rz} = 0.$$

$$\text{Equation 3 } (k_t): -\frac{\theta}{\psi} d_{ck} + (\theta - 1) d_{rk} + d_{ik} = 0.$$

$$\text{Equation 4 } (z_t): -\frac{\theta}{\psi} d_{cz} + (\theta - 1) d_{rz} + d_{iz} = 0.$$

The solution to the system of equations is:

$$\tilde{A}_1 = \left(\frac{1}{\psi} - 1 \right) \frac{[(1 - \lambda_1 - \lambda_2) B_1^2 + (\lambda_1 - 1) B_1]}{[(\kappa_1 \lambda_1 - 1) + \kappa_1 (1 - \lambda_1 - \lambda_2) B_1]}, \quad (25)$$

$$\tilde{A}_2 = \frac{\left(1 - \frac{1}{\psi}\right) [B_1 \lambda_2 + (1 - \lambda_1 - \lambda_2) B_1 B_2 + B_2 (\varphi - 1)] + \kappa_1 \lambda_2 \tilde{A}_1 + \kappa_1 (1 - \lambda_1 - \lambda_2) \tilde{A}_1 B_2}{(1 - \kappa_1 \varphi)}, \quad (26)$$

$$B_1 = \frac{1}{2Q_2} \left[-Q_1 - \sqrt{Q_1^2 - 4Q_0 Q_2} \right], \quad (27)$$

$$B_2 = \dots$$

$$\frac{-\frac{1}{\psi} B_1 \lambda_2 - \left[\tilde{\lambda}_3 (\lambda_2 - \varphi) + \lambda_5 \lambda_2 \left(1 - \frac{\alpha}{1 - \exp(c-y)} \right) + \lambda_5 \lambda_2 \frac{\exp(c-y)}{1 - \exp(c-y)} B_1 - (\lambda_4 + \lambda_5 \varphi) \frac{(1-\alpha)}{1 - \exp(c-y)} \right]}{\left[\frac{1}{\psi} [(1 - \lambda_1 - \lambda_2) B_1 + (\varphi - 1)] + \frac{\exp(c-y)}{1 - \exp(c-y)} [\lambda_4 + \lambda_5 B_1 (1 - \lambda_1 - \lambda_2) + \lambda_5 \varphi] + (1 - \lambda_1 - \lambda_2) \left[\tilde{\lambda}_3 + \lambda_5 \left(1 - \frac{\alpha}{1 - \exp(c-y)} \right) \right] \right]} \quad (28)$$

where

$$Q_2 = (1 - \lambda_1 - \lambda_2) + \psi (1 - \lambda_1 - \lambda_2) \lambda_5 \frac{\exp(c-y)}{1 - \exp(c-y)}, \quad (29)$$

$$Q_1 = (\lambda_1 - 1) + \psi (1 - \lambda_1 - \lambda_2) \left[\tilde{\lambda}_3 + \lambda_5 \left(1 - \frac{\alpha}{1 - \exp(c-y)} \right) \right] + \psi \frac{\exp(c-y)}{1 - \exp(c-y)} (\lambda_4 + \lambda_5 \lambda_1) \quad (30)$$

$$Q_0 = \psi \left[\left(1 - \frac{\alpha}{1 - \exp(c-y)} \right) (\lambda_4 + \lambda_5 \lambda_1) + \tilde{\lambda}_3 \lambda_1 \right]. \quad (31)$$

Finally, we need the expression for the value function. The log return to total wealth is approximated as (ignoring intercepts):

$$r_{a,t+1} \approx \kappa_1 h_{t+1} - h_t + \Delta c_{t+1},$$

where $\kappa_1 \equiv \frac{WC^{ex}}{WC^{ex}+1} = \frac{WC-1}{WC}$, where $WC^{ex} = WC - 1$. A log-linear approximation of the wealth-

consumption ratio, where wealth is cum-dividend (consumption), yields:

$$h_t \approx \text{constant} + \frac{1}{\kappa_1} w c_t \implies w c_t \approx \text{constant} + \kappa_1 \tilde{A}_1 k_t + \kappa_1 \tilde{A}_2 z_t.$$

Since

$$\ln \frac{W_t}{C_t} = \ln \frac{1}{1-\beta} \left(\frac{V_t}{C_t} \right)^{1-1/\psi} = \ln \frac{1}{1-\beta} + (1-1/\psi) v_t - (1-1/\psi) c_t, \quad (32)$$

where v_t is the log value function, we have that (ignoring constants as usual):

$$v_t = A_1 k_t + A_2 z_t, \quad \text{where} \quad A_1 = \frac{\kappa_1 \tilde{A}_1}{1-1/\psi} + B_1 \quad \text{and} \quad A_2 = \frac{\kappa_1 \tilde{A}_2}{1-1/\psi} + B_2. \quad (33)$$

It is easy to show that if $\varphi = 1$, $B_1 = 1 - B_2$ and $A_1 = 1 - A_2$.

Proof of Corollary 2: The capital stock process: Using lag operator notation, Eq. (15) in the Appendix gives the capital stock as an AR(2) process:

$$k_{t+1} = D_1 L k_{t+1} + D_2 z_t \implies k_{t+1} = \frac{D_2}{1-D_1 L} z_t = \frac{D_2}{(1-D_1 L)(1-\varphi L)} \varepsilon_t. \quad (34)$$

The output process: Using the production function, Equation (34) in the Appendix and Equation (9) in the main text, and lag operator notation, output follows an ARMA(2,1) process:

$$y_t = (1-\alpha) z_t + \alpha k_t = (1-\alpha) \frac{1}{1-\varphi L} \varepsilon_t + \alpha \frac{D_2 L}{(1-D_1 L)(1-\varphi L)} \varepsilon_t \quad (35)$$

$$= \frac{(1-\alpha) + (\alpha D_2 - (1-\alpha) D_1) L}{(1-D_1 L)(1-\varphi L)} \varepsilon_t \quad (36)$$

The consumption process: Using lag operator notation, Equations (14) and (34) in the Appendix and Equation (9) in the main text, consumption follows an ARMA(2,1) process:

$$c_t = B_1 k_t + B_2 z_t = B_1 \frac{D_2 L}{(1-D_1 L)(1-\varphi L)} \varepsilon_t + B_2 \frac{1}{1-\varphi L} \varepsilon_t = \frac{B_2 + (B_1 D_2 - B_2 D_1) L}{(1-D_1 L)(1-\varphi L)} \varepsilon_t \quad (37)$$

Proof of Corollary 3: First, without the lag operators we have

$$\Delta c_t = D_1 \Delta c_{t-1} + (\varphi - 1) c_{t-1} - D_1 (\varphi - 1) c_{t-2} + (B_1 D_2 - B_2 D_1) \varepsilon_{t-1} + B_2 \varepsilon_t. \quad (38)$$

Thus, if technology shocks are permanent ($\varphi = 1$), consumption *growth* follows an ARMA(1,1). In this case, *expected consumption growth* follows an AR(1):

$$x_t \equiv E_t [\Delta c_{t+1}] = D_1 x_{t-1} + B_1 D_2 \varepsilon_t, \quad \text{since} \quad (39)$$

$$\Delta c_{t+1} = x_t + B_2 \varepsilon_{t+1} = D_1 x_{t-1} + B_1 D_2 \varepsilon_t + B_2 \varepsilon_{t+1} = D_1 \Delta c_t + (B_1 D_2 - B_2 D_1) \varepsilon_t + B_2 \varepsilon_{t+1}. \quad (40)$$

Proof of Proposition 4: The log-return to wealth can be written:

$$r_{a,t+1} = wc_{t+1} + \Delta c_{t+1} - \ln(e^{wc_t} - 1) \approx \text{constant} + wc_{t+1} + \Delta c_{t+1} - \frac{1}{\kappa_1} wc_t. \quad (41)$$

This expression looks slightly different from the corresponding in Bansal and Yaron (2004), because wealth here is cum consumption. Solving forward, we have:

$$wc_t = \text{constant} + \kappa_1 wc_{t+1} + \kappa_1 \Delta c_{t+1} - \kappa_1 r_{a,t+1} = \text{constant} + \sum_{j=1}^{\infty} \kappa_1^j (\Delta c_{t+j} - r_{a,t+j}). \quad (42)$$

The risk-free rate in the log-linear model is given by

$$r_{f,t+1} = \text{constant} + \frac{\theta}{\psi} E_t [\Delta c_{t+1}] - (\theta - 1) E_t (r_{a,t+1}),$$

since all shocks are homoskedastic (the precautionary savings terms are constant). Since, of the same reason, $E_t [r_{a,t+1}] = \text{constant} + r_{f,t+1}$, we have that:

$$r_{f,t+1} = \text{constant} + \frac{1}{\psi} E_t [\Delta c_{t+1}]. \quad (43)$$

Imposing the rational expectations equilibrium, we have:

$$wc_t = \text{constant} + E_t \sum_{j=1}^{\infty} \kappa_1^j (1 - 1/\psi) \Delta c_{t+j}. \quad (44)$$