New Keynesian Economics

II. A Classical Monetary Model

Dr. Michael Paetz

IWWT
Department of Economics
Hamburg University

April 2011

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Fakultät Wirtschafts- und Sozialwissenschaften

The classical model

- Perfect competition and fully flexible prices.
- Very limited role for money.
- The classical model presented in the this chapter serves as a benchmark economy.

New Keynesian Economics - II. A Classical Monetary Model

Table of contents

New Keynesian Economics - II. A Classical Monetary Model

Households

Objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right), \tag{1}$$

where C_t represents consumption, N_t hours worked, $U\left(C_t, N_t\right)$ the utility of a single period, and β describes the time preference rate.
We assume positive, but nonincreasing marginal utility of consumption, and positive, but non decreasing disutility of labor $(-U_{n,t})$:

$$U_{c,t} \equiv \frac{\partial U\left(C_t, N_t\right)}{\partial C_t} > 0, U_{cc,t} \equiv \frac{\partial^2 U\left(C_t, N_t\right)}{\partial C_t} \leq 0$$
$$U_{n,t} \equiv \frac{\partial U\left(C_t, N_t\right)}{\partial N_t} \leq 0, U_{nn,t} \equiv \frac{\partial^2 U\left(C_t, N_t\right)}{\partial N_t} \leq 0$$

New Keynesian Economics - II. A Classical Monetary Model

Households

A sequence of budget constraints (in nominal terms)

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t - T_t, \forall t \geq 0, \quad (2)$$

where P_t is the price of the consumption good, W_t is the nominal wage, B_t represents the quantity of one-period, nominal riskless discount bonds, purchased in t, paying one unit of money at maturity, Q_t is the price for these bonds, and T_t represents lump-sum transfers (taxes, dividends, etc.).

According to (2) *expenses* for consumption and purchasing of new bonds has to be covered by the *earnings* from bond holdings and wage income, adjusted for lump-sum transfers.

New Keynesian Economics - II. A Classical Monetary Model

Households

The price and the yield on a one-period bond

In equilibrium the nominal yield on a one-period bond has to fulfill $Q_t \equiv \frac{1}{1+yield_t}$:

- If the price would be higher no one would like to buy such a bond, and the price would fall,
- if the price would be lower, everyone would like to buy such a bond and the price would increase.

New Keynesian Economics - II. A Classical Monetary Model
Households

The solvency constraint

(3) $$\lim_{T\to\infty} E_t\left\{B_T\right\} \geq 0, \forall t \geq 0.$$

The solvency constraint ensures that households do not engage in a so-called **Ponzi-financing**:
Charles Ponzi was an Italian immigrant in the US, who paid investors by collecting subsequent money from new investors, without investing in any profit earning projects. As long as he finds new investors his system was working. Modern version: Bernhard Madoff...
Even though, one lesson from the financial crisis is that Ponzi schemes are still used, these schemes would create a bubble in the model and have to be excluded. Recall that we are interested in the transmission mechanism of monetary policy, not in financial bubbles.

New Keynesian Economics - II. A Classical Monetary Model

Households

Optimality conditions

In the optimum the constraints will be fulfilled with equality (no household would die with a positive amount of bonds or would have smaller expenditures than earnings).

The Lagrangian of maximizing (1) subject to (2) and (3) is given by:

$$\mathcal{L} \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U\left(C_t, N_t\right) + \lambda_t \left(P_t C_t + Q_t B_t - B_{t-1} - W_t N_t + T_t\right) + \psi_t \left(\lim_{T \to \infty} B_T \right) \right\},$$

where λ_t and ψ_t are the Lagrangian multipliers on the corresponding constraints.

Optimality conditions

Households choose C_t, N_t and B_t to maximize $\mathcal{L}$:

$$
\begin{aligned}
(4) \qquad \frac{\partial \mathcal{L}}{\partial C_t} &= U_{C_t} + \lambda_t P_t \overset{!}{=} 0 \Rightarrow \lambda_t = -\frac{U_{C_t}}{P_t} \\
(5) \qquad \frac{\partial \mathcal{L}}{\partial N_t} &= U_{N_t} - \lambda_t W_t \overset{!}{=} 0 \Rightarrow \lambda_t = \frac{U_{N_t}}{W_t} \\
(6) \qquad \frac{\partial \mathcal{L}}{\partial B_t} &= Q_t \lambda_t - \beta \lambda_{t+1} \overset{!}{=} 0 \Rightarrow \beta E_t \lambda_{t+1} = Q_t \lambda_t
\end{aligned}
$$

In addition, the partial derivatives with respect to the Lagrangian multipliers replicate the constraints.

New Keynesian Economics - II. A Classical Monetary Model

Households

Optimality conditions

Combining (4) with (5) gives:

$$-\frac{U_{N_t}}{U_{C_t}} = \frac{W_t}{P_t}. \tag{7}$$

The marginal rate of substitution between leisure and consumption is equal to the price (the opportunity costs) of leisure, the real wage. According to (7) households cannot gain utility from shifting leisure to consumption. It describes the optimal consumption-leisure decision.

New Keynesian Economics - II. A Classical Monetary Model

Households

Optimality conditions

Combining (4) with (6) gives:

$$Q_t = \beta E_t \left\{ \frac{U_{C_{t+1}}}{U_{C_t}} \frac{P_t}{P_{t+1}} \right\}$$

$$(8) \qquad \Leftrightarrow \qquad E_t \left\{ \frac{U_{C_t}}{U_{C_{t+1}}} \right\} = \beta \frac{1 + yield_t}{E_t \Pi_{t+1}},$$

where $\Pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$ represents the inflation rate. The marginal rate of substitution between present and future consumption equals the price (opportunity costs) of domestic consumption, measured by the real gross yield on saving (instead of consuming), adjusted for the time preference. According to (8) households cannot gain utility from shifting present to future consumption. It describes the optimal intertemporal consumption decision and is a so-called Euler equation.

New Keynesian Economics - II. A Classical Monetary Model

Households

Introducing an explicit utility function

Consider the following constant relative risk aversion (CRRA) function:

$$U\left(C_t, N_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi},$$

where σ and φ represent the coefficients of relative risk aversion with respect to consumption and leisure, respectively. The Arrow-Pratt definition of the coefficient of relative risk aversion gives:

$$R\left(C_t\right) \equiv -\frac{C_t U''\left(C_t\right)}{U'\left(C_t\right)} = \sigma, R\left(N_t\right) \equiv \frac{N_t U''\left(N_t\right)}{U'\left(N_t\right)} = \varphi$$

Moreover, σ and φ are the inverses of the intertemporal substitution elasticities between consumption (or leisure) in any two periods. The smaller σ (the larger $1/\sigma$), the more willing is the households to substitute consumption over time.

New Keynesian Economics - II. A Classical Monetary Model
Households

Optimality conditions

Using the CRRA utility function yields:

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi}, \tag{9}$$

$$Q_t = \beta E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}. \tag{10}$$

New Keynesian Economics - II. A Classical Monetary Model

Firms

Production function

A representative firms is assumed to produce output Y_t using a simple production function solely depending on labor

$$Y_t = A_t N_t^{1-\alpha}, \tag{11}$$

where A_t represents the level of technology, which is assumed to evolve exogenously, and α represents the partial elasticity of output with respect to labor.

New Keynesian Economics - II. A Classical Monetary Model
Firms

Profit maximization

The firm chooses its labor demand by maximizing profits, given by (Prices $\times$ Output $-$ Wages $\times$ Labor input)

$$P_t Y_t - W_t N_t, \tag{12}$$

subject to (11). Maximization directly gives:

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha}. \tag{13}$$

The firm hires labor up until the real wage equals the marginal product of labor ($(1-\alpha) A_t N_t^{-\alpha}$).

New Keynesian Economics - II. A Classical Monetary Model

Firms

Profit maximization

Consequently, marginal costs, given by the wage payment divided by the marginal product, have to be equal to the price:

$$\frac{W_t}{(1-\alpha)\,A_t N_t^{-\alpha}} = P_t. \tag{14}$$

This standard result under perfect competition will vanish in the New Keynesian model due to the assumption of monopolistic competition. The equations (13) and (14) can be interpreted as labor demand schedule, mapping the real wage into the labor demand.

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Why?

- Most non-linear models are difficult to solve exactly. Hence, we use an approximate solution. In a general equilibrium model, we can use a first-order Taylor-approximation around the steady state, which behaves quite good in the neighborhood of the steady state.
- Bringing models in a standardized linear form allows to solve the model and to use standardized software for estimation and simulation, and the evaluation of different policy regimes. We can now evaluate the contemporary and future responses of all endogenous variables after a change in some exogenous variable (for example an exogenous increase in inflation due to an increase in oil prices).

But: Approximations can be misleading when we leave the neighborhood of the steady state!

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Taylor approximations

A Taylor approximation of order n of a function f around a certain value a is given by

$$\begin{aligned} T_{n;f;a}(x) &\equiv f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i. \end{aligned}$$

Consequently, a first-order Taylor approximation is given by

$$T_{1;f;a}(x) \equiv f(a) + f'(a)(x-a).$$

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

A first-order Taylor approximation of a variable X_t around a steady state value $\overline{X}$:

1. Rewrite X_t:

$$X_t = \overline{X}\frac{X_t}{\overline{X}} = \overline{X}e^{\log(X_t/\overline{X})} = \overline{X}e^{\widehat{x}_t},$$

   where $\widehat{x}_t \equiv \log\left(X_t/\overline{X}\right)$ represents the **percentage deviation of X_t from its steady state value $\overline{X}$**.

2. Apply a first-order Taylor approximation for $e^{\widehat{x}_t}$ around $\widehat{x}_t = 0 \left(\Leftrightarrow X_t = \overline{X}\right)$:

$$e^{\widehat{x}_t} \approx e^0 + e^0\left(\widehat{x}_t - 0\right) = 1 + \widehat{x}_t,$$

$$\Rightarrow X_t \approx \overline{X}\left(1 + \widehat{x}_t\right)$$

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Log-linearizing a nonlinear equation: $Y_t = A_t K_t^{\alpha} N_t^{(1-\alpha)}$

1. Rewrite the equation in logs:

$$
\begin{aligned}
& \log Y_t = \log A_t + \alpha \log K_t + (1-\alpha) \log N_t \\
\Leftrightarrow \quad & \log\left(\overline{Y}\frac{Y_t}{\overline{Y}}\right) = \log\left(\overline{A}\frac{A_t}{\overline{A}}\right) + \alpha \log\left(\overline{K}\frac{K_t}{\overline{K}}\right) + (1-\alpha)\log\left(\overline{N}\frac{N_t}{\overline{N}}\right) \\
\Leftrightarrow \quad & \log\left(\overline{Y}e^{\widehat{y}_t}\right) = \log\left(\overline{A}e^{\widehat{a}_t}\right) + \alpha \log\left(\overline{K}e^{\widehat{k}_t}\right) + (1-\alpha)\log\left(\overline{N}e^{\widehat{n}_t}\right) \\
\Leftrightarrow \quad & \left(\log \overline{Y} + \widehat{y}_t\right) = \left(\log \overline{A} + \widehat{a}_t\right) + \alpha\left(\log \overline{K} + \widehat{k}_t\right) + (1-\alpha)\left(\log \overline{N} + \widehat{n}_t\right),
\end{aligned}
$$

where we used $\log\left(\overline{X}e^{\widehat{x}_t}\right) = \log \overline{X} + \widehat{x}_t$.

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Log-linearizing a nonlinear equation: $Y_t = A_t K_t^{\alpha} N_t^{(1-\alpha)}$

2. Next, subtract the steady state relationship

$$\overline{Y} = \overline{A}\overline{K}^{\alpha}\overline{N}^{(1-\alpha)} \Rightarrow \log \overline{Y} = \log \overline{A} + \alpha \log \overline{K} + (1-\alpha) \log \overline{N}$$

$$\Rightarrow \widehat{y}_t = \widehat{a}_t + \alpha \widehat{k}_t + (1-\alpha)\, \widehat{n}_t .$$

The resulting expression is now linear.

Note: We did not need any Taylor approximation so far!

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Log-linearizing a linear equation: $Y_t = C_t + I_t + G_t$

1. Rewrite the equation in percentage deviations from the steady state by using a Taylor approximation

$$\overline{Y}\left(1+\widehat{y}_t\right) = \overline{C}\left(1+\widehat{c}_t\right) + \overline{I}\left(1+\widehat{i}_t\right) + \overline{G}\left(1+\widehat{g}_t\right).$$

2. Next, subtract the steady state relationship: $\overline{Y} = \overline{C} + \overline{I} + \overline{G}$:

$$\overline{Y}\widehat{y}_t = \overline{C}\widehat{c}_t + \overline{I}\widehat{i}_t + \overline{G}\widehat{g}_t$$
$$\widehat{y}_t = \gamma_C\widehat{c}_t + \gamma I\widehat{i}_t + \gamma G\widehat{g}_t,$$

where $\gamma_C = \frac{\overline{C}}{\overline{Y}}, \gamma_I = \frac{\overline{I}}{\overline{Y}}, \gamma_G = \frac{\overline{G}}{\overline{Y}}$.

The resulting expression is still linear but measured in percentage deviations from equilibrium.

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Log-linearizing an equation including linear *and* nonlinear terms:
$C_t + B_t = R_t B_{t-1} + W_t N_t$

1. Rewrite the equation in percentage deviations from the steady state by using a Taylor approximation:

$$
\begin{aligned}
& \overline{C} e^{\widehat{c}_t} + \overline{B} e^{\widehat{b}_t} = \overline{R} e^{\widehat{r}_t} \overline{B} e^{\widehat{b}_{t-1}} + \overline{W} e^{\widehat{w}_t} \overline{N} e^{\widehat{n}_t} \\
\Leftrightarrow \quad & \overline{C} e^{\widehat{c}_t} + \overline{B} e^{\widehat{b}_t} = \overline{RB} e^{\widehat{r}_t + \widehat{b}_{t-1}} + \overline{WN} e^{\widehat{w}_t + \widehat{n}_t} \\
\Leftrightarrow \quad & \overline{C} \left(1 + \widehat{c}_t\right) + \overline{B} \left(1 + \widehat{b}_t\right) \\
& = \overline{RB} \left(1 + \widehat{r}_t + \widehat{b}_{t-1}\right) + \overline{WN} \left(1 + \widehat{w}_t + \widehat{n}_t\right).
\end{aligned}
$$

New Keynesian Economics - II. A Classical Monetary Model

Log-linear approximations

Log-linearizing an equation including linear *and* nonlinear terms:
$C_t + B_t = R_t B_{t-1} + W_t N_t$

2. Next, subtract the steady state relationship: $\overline{C} + \overline{B} = \overline{RB} + \overline{WN}$:

$$\overline{C}\widehat{c}_t + \overline{B}\widehat{b}_t = \overline{RB}\left(\widehat{r}_t + \widehat{b}_{t-1}\right) + \overline{WN}\left(\widehat{w}_t + \widehat{n}_t\right)$$

$$\Leftrightarrow \qquad \widehat{c}_t + \gamma_B \widehat{b}_t = \overline{R}\gamma_B\left(\widehat{r}_t + \widehat{b}_{t-1}\right) + \gamma_{WN}\left(\widehat{w}_t + \widehat{n}_t\right)$$

where $\gamma_B = \frac{\overline{B}}{\overline{C}}, \gamma_{WN} = \frac{\overline{WN}}{\overline{C}}$.

New Keynesian Economics - II. A Classical Monetary Model

Linearization of the benchmark model

Notation

For the rest of this course we will use the following notation (as long is it is not explicitly mentioned that a different notation is used):

- Lower case letter variables represent log-versions of their upper case counterparts: $x_t \equiv \log X_t$.
- Variables without a time index represent equilibrium values: The equilibrium of X_t is given by X.
- Variables with hats represent percentage deviations from equilibrium: $\widehat{x}_t \equiv \log\left(X_t/X\right) = \log X_t - \log X = x_t - x$.

Linearizing the demand side

In a perfect-foresight equilibrium with constant inflation Π and constant growth $\left(\frac{C_{t+1}}{C_t}\right)$ of Γ the Euler equation reads:

$$Q = \beta \frac{\Gamma^{-\sigma}}{\Pi}$$

$$\Rightarrow \quad q = -\rho - \sigma\gamma - \pi, \tag{15}$$

where $\rho \equiv -\log\beta$ can be interpreted as the household's discount rate. Defining the nominal interest rate as the as the log of the gross yield on the one-period bond, $i_t \equiv -\log Q_t = \log(1 + yield_t) \approx yield_t$, we can pin down the equilibrium interest rate of the model:

$$i = \rho + \pi + \sigma\gamma. \tag{16}$$

Linearizing the demand side

Moreover, we can rewrite the original Euler equation and apply a first-order Taylor expansion:

$$\begin{aligned}
1 &= \frac{\beta}{Q_t} E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\} \\
&= E_t \left\{ e^{i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho} \right\} \\
&\approx 1 + (i_t - i) - \sigma \left(E_t \Delta c_{t+1} - \gamma \right) - \left(\pi_{t+1} - \pi \right) \\
&= 1 + i_t - \sigma E_t \Delta c_{t+1} - \pi_{t+1} - \rho,
\end{aligned}$$

where we used the equilibrium condition, $i = \rho + \pi + \sigma\gamma$, in the last step.

New Keynesian Economics - II. A Classical Monetary Model

Linearization of the benchmark model

Linearizing the demand side

Rearranging gives the log-linearized version of the Euler equation:

$$c_t = E_t\{c_{t+1}\} - \sigma^{-1}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right). \tag{17}$$

Moreover, log-linearizing the consumption-leisure decision gives:

$$w_t - p_t = \sigma c_t + \varphi n_t. \tag{18}$$

New Keynesian Economics - II. A Classical Monetary Model

Linearization of the benchmark model

Linearizing the supply side

Log-linearizing the labor demand schedule directly gives:

$$w_t - p_t = a_t - \alpha n_t + \log(1-\alpha). \tag{19}$$

Moreover, a log-linearized version of the production function is given by

$$y_t = a_t + (1-\alpha) n_t. \tag{20}$$

Market clearing

- *Goods market clearing* is characterized by $y_t = c_t$.
- *Labor market clearing* can be derived by equating the optimal labor leisure decision with the labor demand:

$$\sigma c_t + \varphi n_t = a_t - \alpha n_t + \log(1-\alpha) \tag{21}$$

- *Asset market clearing* can be derived by combining goods market clearing with the Euler equation:

$$y_t = E_t\{y_{t+1}\} - \sigma^{-1}\left(i_t - E_t\{\pi_{t+1}\} - \rho\right). \tag{22}$$

New Keynesian Economics - II. A Classical Monetary Model

Equilibrium

Dynamics

By combining the market clearing conditions with the production function, the equilibrium levels of employment and output are determined by the level of technology:

$$n_t = \psi_{na} a_t + \vartheta_n, \tag{23}$$

$$y_t = \psi_{ya} a_t + \vartheta_y, \tag{24}$$

where

$$\psi_{na} \equiv \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha}, \qquad \vartheta_n \equiv \frac{\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha}$$

$$\psi_{ya} \equiv \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}, \qquad \vartheta_y \equiv (1-\alpha)\vartheta_n.$$

Derivation of (23) and (24)

For the derivation of (23) we use the goods market and labor markets clearing conditions together with the production function:

$$\begin{aligned}
\sigma c_t + \varphi n_t &= a_t - \alpha n_t + \log(1-\alpha) \\
\Leftrightarrow (\varphi + \alpha) n_t &= \log(1-\alpha) + a_t - \sigma \left[a_t + (1-\alpha) n_t\right] \\
\Leftrightarrow (\varphi + \alpha) n_t &= \log(1-\alpha) + (1-\sigma) a_t - \sigma (1-\alpha) n_t \\
\Leftrightarrow \left[\varphi + \alpha + \sigma(1-\alpha)\right] n_t &= \log(1-\alpha) + (1-\sigma) a_t \\
\Leftrightarrow n_t &= \frac{\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha} + \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha} a_t
\end{aligned}$$

Derivation of (23) and (24)

For deriving (24) we use (23) in the production function :

$$\begin{aligned}
y_t &= a_t + (1-\alpha)\, n_t \\
&= a_t + (1-\alpha)\left[\frac{\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha} + \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha} a_t\right] \\
&= \frac{(1-\alpha)\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha} + \left[1 + (1-\alpha)\frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha}\right] a_t \\
&= \frac{(1-\alpha)\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha} + \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha} a_t
\end{aligned}$$

Dynamics

Defining the real interest rate as $r_t \equiv i_t - E_t\{\pi_{t+1}\}$ we can use the asset market clearing with the equilibrium process of output to derive:

$$\begin{aligned} r_t &= \rho + \sigma E_t\{\Delta y_{t+1}\} \\ &= \rho + \sigma \psi_{ya} E_t\{\Delta a_{t+1}\}. \end{aligned} \tag{25}$$

Finally, the real wage can be derived by combining the labor demand schedule with the equilibrium process for employment:

$$\begin{aligned} \omega_t \equiv w_t - p_t &= a_t - \alpha n_t + \log(1-\alpha) \\ &= \psi_{\omega a} a_t + \vartheta_\omega, \end{aligned} \tag{26}$$

where $\psi_{\omega a} \equiv \frac{\sigma+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}$ and $\vartheta_\omega \equiv \frac{(\sigma(1-\alpha)+\phi)\log(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha}$.

New Keynesian Economics - II. A Classical Monetary Model

Equilibrium

The neutrality of money

The equilibrium dynamics of **all** real variables (employment, output and the real interest rate) are determined independently from any monetary policy! Real variables are only affected by changes in the level of technology.

New Keynesian Economics - II. A Classical Monetary Model

Impulse response functions

Technology shock

Assuming an AR(1)-process for technology the whole model is given by:

$$\begin{aligned} n_t &= \psi_{na} a_t + \vartheta_n, \\ y_t &= \psi_{ya} a_t + \vartheta_y, \\ r_t &= \rho + \sigma \psi_{ya} E_t \left\{ \Delta a_{t+1} \right\}, \\ \omega_t &= \psi_{\omega a} a_t + \vartheta_\omega, \\ a_t &= \rho_a a_{t-1} + \varepsilon_t^a, \end{aligned}$$

where $\varepsilon_t^a \sim \mathcal{N}\left(0, \sigma_a^2\right)$.

New Keynesian Economics - II. A Classical Monetary Model

Impulse response functions

Model in deviations

In the steady state $a_t = 0$ and $n = \vartheta_n, y = \vartheta_y, \omega = \vartheta_\omega, r = \rho$. Hence we can write the model in deviations from the steady state:

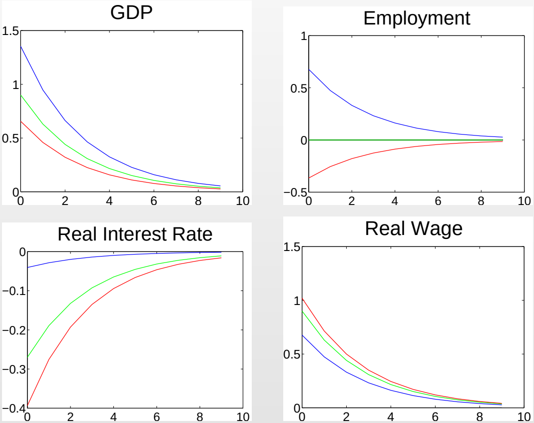
$$\begin{aligned}
\widehat{n}_t &= \psi_{na} a_t, \\
\widehat{y}_t &= \psi_{ya} a_t, \\
\widehat{r}_t &= \sigma \psi_{ya} E_t \left\{ \Delta a_{t+1} \right\}, \\
\widehat{\omega}_t &= \psi_{\omega a} a_t, \\
a_t &= \rho_a a_{t-1} + \varepsilon_t^a.
\end{aligned}$$

The following Figure shows Impulse responses of the model for $\varphi = 0.8, \alpha = 0.33, \rho_a = 0.7, \sigma_a^2 = 0.81$ and different values for the inverse of the elasticity of substitution σ.

New Keynesian Economics - II. A Classical Monetary Model

Impulse response functions

IRFs for a technology shock
(blue: $\sigma = 0.1$, green: $\sigma = 1$, red: $\sigma = 2$)



New Keynesian Economics - II. A Classical Monetary Model

Impulse response functions

Interpretation

- Output increases after a productivity shock, leading to an increase in the real wage as well.
- Since our technology shock is transitory and $E_t \{a_{t+1}\} < a_t$, the real interest rate falls in reaction to a technology shock.

New Keynesian Economics - II. A Classical Monetary Model

Impulse response functions

Interpretation

The reaction of employment depends on the value of σ, since $\widehat{n}_t = \psi_{na} a_t$, and $\psi_{na} \equiv \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha}$. σ measures the strength of the wealth effect of labor supply.

- If $\sigma < 1$, ψ_{na} is positive: The higher real wage leads to an increase in labor supply, since households substitute consumption for leisure (substitution effect).
- if $\sigma > 1$, ψ_{na} is negative: The smaller marginal utility of consumption (wealth effect) dominates the substitution effect, implying a fall in labor supply. (Households gain less from increasing consumption and decide to work less.)
- if $\sigma = 1$, ψ_{na} is zero, since both effects cancel out each other.

How to determine the reactions of the nominal variables

I opposition to the real variables, nominal variables can only be derived when the monetary policy rule is specified. In the following, we investigate the behavior of nominal variables, using different interest rate rules.
Therefore, we will use the Fisherian equation,

$$i_t = E_t\left\{\pi_{t+1}\right\} + r_t, \tag{27}$$

implying, that the nominal interest rate adjusts one by one with expected inflation, when the real interest rate is solely determined by real factors.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the nominal interest rate

The simplest assumption on the path of the interest rate would be to assume some stationary process $\{i_t\}$. Given these process, the Fisherian equation pins down the expected inflation rate.
Consequently any path for the interest rate, satisfying

$$p_{t+1} = p_t + i_t - r_t + \xi_{t+1}, \tag{28}$$

where ξ_{t+1}, $E_{t\{\xi_{t+1}\}}$ is a shock process, is consistent with equilibrium (as it solves (27)). Hence, there are an infinite number of solutions for the path of the price level and the equilibrium is called **indeterminate**. The shocks/equilibria of these type are often called **sunspot** shocks/equilibria.
$\Rightarrow$ Any inflation rate can arise and persist under these circumstances.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An inflation-based interest rate rule

Consider the following interest rate rule ($\phi_\pi \geq 0$):

$$i_t = \rho + \phi_\pi \pi_t \Rightarrow \widehat{i}_t = \phi_\pi \pi_t. \tag{29}$$

Using the Fisherian equation, it follows that

$$\phi_\pi \pi_t = E_t \left\{ \pi_{t+1} \right\} + \widehat{r}_t, \tag{30}$$

where $\widehat{r}_t \equiv r_t - \rho$.
Now, the determinacy of the price level depends on the value of ϕ_π.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An inflation-based interest rate rule

If $\phi_\pi \leq 1$, any path of inflation, satisfying

$$\pi_{t+1} = \phi_\pi \pi_t - \widehat{r}_r + \xi_{t+1}, \tag{31}$$

where $\xi_{t+1}, E_{t\{\xi_{t+1}\}}$ is a shock process, would solve (30). We end up with an infinite number of sunspot equilibria.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An inflation-based interest rate rule

If $\phi_\pi > 1$, there is a unique stationary solution for the price level. By solving (30) forward, we derive

$$\begin{aligned}
\pi_t &= \phi_\pi^{-1}\left[E_t\{\pi_{t+1}\} + \widehat{r}_t\right] \\
&= \phi_\pi^{-1}\left[E_t\left\{\phi_\pi^{-1}\left[E_t\{\pi_{t+2}\} + \widehat{r}_{t+1}\right\}\right] + \widehat{r}_t\right] \\
&= \sum_{k=0}^{\infty} \phi_\pi^{-(k+1)} E_t\{\widehat{r}_{t+k}\}. \qquad (32)
\end{aligned}$$

New Keynesian Economics - II. A Classical Monetary Model
Price-level determination

An inflation-based interest rate rule

Using the equilibrium dynamics for the real interest rate combined with $E_t\{\Delta a_{t+1}\} = (\rho_a - 1)\, a_t$, yields: $\widehat{r}_t = \sigma\psi_{ya}(\rho_a - 1)\, a_t$. This implies

$$\begin{aligned} \pi_t &= \sum_{k=0}^{\infty} \phi_\pi^{-(k+1)} \sigma\psi_{ya}(\rho_a - 1)\, E_t\{a_{t+k}\} \\ &= \sum_{k=0}^{\infty} \phi_\pi^{-(k+1)} \sigma\psi_{ya}(\rho_a - 1)\, \rho_a^k a_t \\ &= \left[\sum_{k=0}^{\infty} \left(\frac{\rho_a}{\phi_\pi}\right)^k\right] \frac{\rho_a - 1}{\phi_\pi} \sigma\psi_{ya} a_t \qquad (33) \end{aligned}$$

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An inflation-based interest rate rule

The geometric series $\sum_{k=0}^{\infty}\left(\frac{\rho_a}{\phi_\pi}\right)^k$ converges to $\frac{\phi_\pi}{\phi_\pi-\rho_a}$ as long as $\frac{\rho_a}{\phi_\pi}<1$.
Thus, equilibrium inflation is given by:

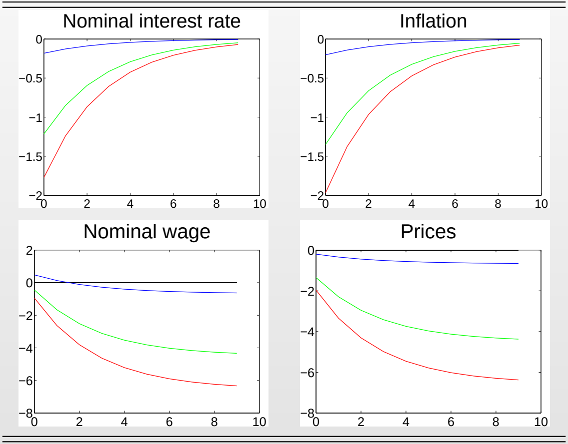
$$\pi_t = \frac{(\rho_a - 1)\,\sigma\psi_{ya}}{\phi_\pi - \rho_a}a_t. \tag{34}$$

For determinacy of the price level the central bank has to adjust the interest rate by more than one by one.
This property is known as the *Taylor principle*.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

Dynamics of nominal variables for $\phi_{pi} = 1.5$
(blue: $\sigma = 0.1$, green: $\sigma = 1$, red: $\sigma = 2$)



New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the money supply

Next, we suppose, the central bank sets an exogenous path for the money supply. Therefore, we first postulate a money demand function,

$$m_t - p_t = y_t - \eta i_t, \tag{35}$$

where η represents the *interest semi-elasticity of money demand*. Solving (35) for the nominal interest rate yields

$$i_t = \frac{1}{\eta}\left[y_t - (m_t - p_t)\right]. \tag{36}$$

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the money supply

Combining this with the Fisherian equation to eliminate the interest rate implies:

$$
\begin{aligned}
& p_t = E_t\{p_{t+1}\} + r_t - i_t \\
\Leftrightarrow \quad & p_t = E_t\{p_{t+1}\} + r_t - \frac{1}{\eta}\left[y_t - (m_t - p_t)\right] \\
\Leftrightarrow \quad & \left(1 + \frac{1}{\eta}\right) p_t = E_t\{p_{t+1}\} + r_t - \frac{1}{\eta}\left[y_t - m_t\right] \\
\Leftrightarrow \quad & p_t = \left(\frac{\eta}{1+\eta}\right) E_t\{p_{t+1}\} + \left(\frac{1}{1+\eta}\right) m_t + u_t,
\end{aligned}
$$

where $u_t \equiv \left(\frac{1}{1+\eta}\right)\left[\eta r_t - y_t\right]$ evolves independently from the money supply.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the money supply

Solving forward yields:

$$\begin{aligned} p_t &= \frac{1}{1+\eta} \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k E_t \{m_{t+k}\} + u_t' \\ &= m_t + \sum_{k=1}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k E_t \{\Delta m_{t+k}\} + u_t', \end{aligned} \tag{37}$$

with $u_t' \equiv \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k E_t \{u_{t+k}\}$ evolves independently from the money supply.

An exogenous path for the money supply determines the price level uniquely.

New Keynesian Economics - II. A Classical Monetary Model
Price-level determination

Derivation of (37):

$$
\begin{aligned}
p_t &= \frac{1}{1+\eta}\sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{m_{t+k}\right\} + u_t' \\
&= \frac{1}{1+\eta}\sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{m_{t+k}\right\} - \frac{1}{1+\eta}\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{m_{t+k-1}\right\} \\
&\quad + \frac{1}{1+\eta}\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{m_{t+k-1}\right\} + u_t' \\
&= \frac{1}{1+\eta}\left[m_t + \sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{\Delta m_{t+k}\right\}\right] \\
&\quad + \underbrace{\frac{1}{1+\eta}\sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{m_{t+k-1}\right\}}_{=\frac{\eta}{1+\eta}(p_t - u_t')} + u_t' \\
\Leftrightarrow \left(1-\frac{\eta}{1+\eta}\right)p_t &= \frac{1}{1+\eta}\left[m_t + \sum_{k=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k E_t\left\{\Delta m_{t+k}\right\}\right] + \left(1-\frac{\eta}{1+\eta}\right)u_t'
\end{aligned}
$$

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the money supply

Using the equilibrium path for the price level with (36) pins down the dynamics of the interest rate

$$i_t = \frac{1}{\eta} \sum_{k=1}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k E_t \{\Delta m_{t+k}\} + u_t'', \tag{38}$$

whith $u_t'' \equiv \frac{1}{\eta} (u_t' + y_t)$ being independently from the money supply.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the money supply

Assuming an AR(1)-process for the money growth rate (similar as for technology),

$$\Delta m_t = \rho_m \Delta m_{t-1} + \varepsilon_t^m, \tag{39}$$

where $\varepsilon_t^m \sim \mathcal{N}\left(0, \sigma_m^2\right)$, we can derive the equilibrium dynamics of the price level. In the absence of real shocks ($a_t = 0$), all real variables are constant, and can be set to zero (for convenience). The equilibrium path of the price level is then given by

$$p_t = m_t + \frac{\eta \rho_m}{1 + \eta\left(1 - \rho_m\right)} \Delta m_t. \tag{40}$$

This implies a more than one for one reaction of the price level in response to an increase in the money growth and contradicts the empirical observation of a sluggish price adjustment.

New Keynesian Economics - II. A Classical Monetary Model

Price-level determination

An exogenous path for the money supply

In addition, the response of the interest rate in this case is given by

$$i_t = \frac{\rho_m}{1 + \eta\,(1 - \rho_m)} \Delta m_t. \tag{41}$$

Hence, the model predicts an increase in the nominal interest rate in response to an increase in the money supply. This absence of a liquidity effect is at odds with empirical facts (see chapter 1).

New Keynesian Economics - II. A Classical Monetary Model

Optimal monetary policy?

Problems of the classical model

Since the real variables of the model evolve independently of the monetary policy rule, any rule that leads to stability is as good as the other, no matter whether it implies a highly volatile inflation or quite stable prices.
Moreover, the classical model cannot predict empirical observations.

New Keynesian Economics - II. A Classical Monetary Model

Money in the Utility function

Objective function

Introducing a motive for money holdings in the utility function, overcomes the problem regarding the optimality of monetary policy rules. Once real balances yield utility, the behavior of prices and the money supply changes the utility (and hence the wealth) of the households.

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Household optimization

Introducing real balances, the optimization problem is now given by

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right) \tag{42}$$

subject to

$$\begin{aligned} & P_t C_t + Q_t B_t + M_t \leq B_{t-1} + M_{t-1} + W_t N_t - T_t \\ \Leftrightarrow \quad & P_t C_t + Q_t \mathcal{A}_{t+1} + (1 - Q_t) M_t \leq \mathcal{A}_t + W_t N_t - T_t, \end{aligned} \tag{43}$$

where $\mathcal{A}_t \equiv B_{t-1} + M_{t-1}$ represents total financial wealth at the beginning of period t, and $U_{m,t} \equiv \frac{\partial U_t}{\partial (M_t / P_t)} > 0$.

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Interpretation

Financial assets yield a gross nominal return of $Q_t^{-1} = e^{-i_t}$, and agents can "purchase" the utility yielding services of money at the nominal interest rate, representing the opportunity costs of holding money instead of interest-bearing bonds.

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Optimality conditions

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}. \tag{44}$$

$$Q_t = \beta E_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} \right\} \frac{1}{E_t \Pi_{t+1}}, \tag{45}$$

$$\frac{U_{m,t}}{U_{c,t}} = 1 - e^{-i_t}. \tag{46}$$

The explicit form of the utility function determines the neutrality-properties of monetary policy:

- If utility is separable in real balances ($U_{c,m} = 0$) monetary policy stays neutral.
- If utility is non-separable in real balances ($U_{c,m} > 0$) monetary policy becomes non-neutral.

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

A CRRA utility function with separable utility

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma} + \frac{(M_t/P_t)^{1-\nu}}{1-\nu} - \frac{N_t^{1+\varphi}}{1+\varphi}. \tag{47}$$

Since neither $U_{c,t}$ not $U_{n,t}$ depend on M_t/P_t the first two optimality conditions and the equilibrium dynamics of the real variables stay the same. The third optimality condition leads to a money demand function:

$$\frac{M_t}{P_t} = C_t^{\sigma/\nu}\left(1 - e^{-i_t}\right)^{-1/\nu} \tag{48}$$

A CRRA utility function with separable utility

Ignoring constants, the money demand function can be approximated by

$$m_t - p_t = \frac{\sigma}{\nu} c_t - \eta i_t, \tag{49}$$

where $\eta \equiv \frac{1}{\nu\left(e^{i}-1\right)} \approx \frac{1}{\nu i}$ is the interest semi-elasticity of money demand.

If the central bank follows a money growth rule, (49) determines the equilibrium values of nominal variables, otherwise it determines the quantity of money needed to reach the nominal interest rate implied by the policy rule.

A CRRA utility function with non-separable utility

$$U\left(C_t, \frac{M_t}{P_t}, N_t\right) = \frac{X_t\left(C_t, M_t/P_t\right)^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}, \tag{50}$$

where

$$X_t \equiv \left[(1-\vartheta)\,C_t^{1-\nu} + \vartheta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}} \text{ for } \nu \neq 1 \tag{51}$$

$$\equiv C_t^{1-\vartheta}\left(\frac{M_t}{P_t}\right)^{1-\vartheta} \text{ for } \nu = 1, \tag{52}$$

ν represents the inverse elasticity of substitution between consumption and real money balances, and ϑ is the relative weight of real balances in utility.

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Optimality conditions

Optimization in this case leads to

$$\frac{W_t}{P_t} = N_t^{\varphi} X_t^{\sigma-\nu} C_t^{\nu} (1-\vartheta)^{-1}, \tag{53}$$

$$Q_t = \beta E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\nu} \left(\frac{X_{t+1}}{X_t} \right)^{\nu-\sigma} \frac{P_t}{P_{t+1}} \right\}, \tag{54}$$

$$\frac{M_t}{P_t} = C_t \left(1 - e^{-i_t} \right)^{-\frac{1}{\nu}} \left(\frac{\vartheta}{1-\vartheta} \right)^{\frac{1}{\nu}}. \tag{55}$$

If $\nu = \sigma$ these conditions coincide with the previous ones and the equilibrium responses of all variables stay the same (real variables evolve independently of the monetary policy).

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Optimality conditions

$$
\begin{aligned}
(56) \qquad \frac{W_t}{P_t} &= N_t^{\varphi} X_t^{\sigma-\nu} C_t^{\nu} \left(1-\vartheta\right)^{-1}, \\
(57) \qquad Q_t &= \beta E_t \left\{ \left(\frac{C_{t+1}}{C_t}\right)^{-\nu} \left(\frac{X_{t+1}}{X_t}\right)^{\nu-\sigma} \frac{P_t}{P_{t+1}} \right\}, \\
(58) \qquad \frac{M_t}{P_t} &= C_t \left(1-e^{-i_t}\right)^{-\frac{1}{\nu}} \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{1}{\nu}}.
\end{aligned}
$$

For $\nu \neq \sigma$ the labor supply and optimal consumption depend on real balances which in turn depend on the interest rate. As different monetary policy rules have different implications for the nominal interest rate, employment and output are now influenced by monetary policy!

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Optimal monetary policy

For deriving the optimal monetary policy rule, we consider the problem of a hypothetical social planner (the almighty himself):

$$\max U\left(C_t, \frac{M_t}{P_t}, N_t\right) \tag{59}$$

subject to

$$C_t = A_t N_t^{1-\alpha}. \tag{60}$$

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

The Lagrangian of the social planners problem

(61)
$$\mathcal{L} = U\left(C_t, \frac{M_t}{P_t}, N_t\right) + \lambda_t\left(C_t - A_t N_t^{1-\alpha}\right)$$

implying the following first order conditions:

(62)
$$\frac{\partial \mathcal{L}}{\partial C_t} = U_{c,t} + \lambda_t \overset{!}{=} 0 \Rightarrow \lambda_t = -U_{c,t}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = U_{n,t} - \lambda_t (1-\alpha) A_t N_t^{-\alpha} \overset{!}{=} 0$$

(63)
$$\Rightarrow \lambda_t = \frac{U_{n,t}}{(1-\alpha) A_t N_t^{-\alpha}}$$

(64)
$$\frac{\partial \mathcal{L}}{\partial (M_t/P_t)} = U_{m,t} \overset{!}{=} 0.$$

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

The social planners problem

The first two conditions can be combined to

$$-\frac{U_{n,t}}{U_{c,t}} = (1-\alpha)\, A_t N_t^{-\alpha}, \tag{65}$$

requiring that the marginal rate of substitution between between hours worked and consumption is equal to the marginal product of labor. This equation will be fulfilled in the decentralized economy independently from monetary policy, as it follows from profit maximizing firms (optimal labor demand) and utility maximizing households (optimal labor supply).

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Implementation of optimal policy

The third condition requires that the marginal utility of real balances is equal to the "social" marginal costs of producing real balances, which is zero ($U_{m,t} = 0$).
How can $U_{m,t} = 0$ be implemented by monetary policy? Recall that the optimal choice of real money balances of the optimizing households implies

$$\frac{U_{m,t}}{U_{c,t}} = 1 - e^{-i_t}. \tag{66}$$

Hence, the optimal solution can only be achieved by setting $i_t = 0$ for all t. This policy is known as the *Friedman rule*.

New Keynesian Economics - II. A Classical Monetary Model

Money in the utility function

Implementation of optimal policy

Evaluating the Fisherian equation at the steady state gives

$$\pi = -\rho \tag{67}$$

In the long-run prices decline on average by the time preference rate. The central bank can implement this interest rate by following a rule of the form

$$i_t = \phi\left(r_{t-1} + \pi_t\right) \tag{68}$$

for some $\phi > 0$. Combined with the Fisherian equation this rule implies

$$E_t\left\{i_{t+1}\right\} = \phi i_t, \tag{69}$$

which only stationary solution in $i_t = 0$ for all t, and equilibrium inflation is given by $\pi_t = -r_{t-1}$.

New Keynesian Economics - II. A Classical Monetary Model

Concluding remarks

Major drawbacks of the classical monetary model

- Money is supermeganeutral, as it has no influence on real variables at all (not even in the short-run). This contradicts empirical evidence.
- The absence of a liquidity effect implies an increase in the interest rate as reaction to an increase in the money supply, which is clearly at odds with empirical evidence.
- Introducing real balances in the utility function can solve the problem of neutrality if we assume a utility function that is non-separable in real balances and consumption, and the elasticity of substitution is different from the relative weight of real balances in utility.
- However, optimal monetary policy requires an interest-rate equal to zero, implying a permanent deflation in the steady state.

New Keynesian Economics - II. A Classical Monetary Model

Concluding remarks

The classical monetary model

The classical model will serve as a benchmark economy, showing the optimal responses in an economy under perfect markets and without any frictions. In the following lecture we will develop a New Keynesian model including different types of frictions and evaluate the model implications with respect to the benchmark economy.