PRIOR DISTRIBUTIONS IN DYNARE
(TO BE COMPLETED)

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1. Gamma function

The Gamma function is defined by the following equation:

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\mathrm{d}x \tag{1}$$

for any $x > 0$. One can easily prove that the following identities hold: $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)\Gamma(n-1)$.

2. Gamma Distributions

A positive real random variable has a gamma distribution with parameters $\alpha > 0$ (shape) and $\beta > 0$ (scale) iff its density is given by the following equation:

$$f(x) = \mathcal{C}(\alpha,\beta)^{-1} \times x^{\alpha-1}e^{-\frac{x}{\beta}} \tag{2}$$

where $\mathcal{C}(\alpha,\beta) = \Gamma(\alpha)\beta^\alpha$ is the constant of integration. We will denote $X \sim G(\alpha,\beta)$. In DYNARE this distribution may be specified as a prior in the `estimated_params` block (`GAMMA_PDF`). The user has to specify the expectation and standard deviation of the distribution... And DYNARE computes α and β from these moments by inverting the following formulas:

$$\begin{aligned} \mu &\equiv \int_0^\infty x f(x)\mathrm{d}x = \alpha\beta \\ \sigma^2 &\equiv \int_0^\infty x^2 f(x)\mathrm{d}x - \mu^2 = \alpha\beta^2 \end{aligned} \tag{3}$$

which can be done by hand in this case ($\alpha = \frac{\mu^2}{\sigma^2}$ and $\beta = \frac{\sigma^2}{\mu}$). It may be useful to know that the mode (the prior density is high around the mode) of this distribution is $(\alpha-1)\beta$ for any $\alpha \geq 1$, or, in terms of the first and second order moments, $\frac{\mu^2-\sigma^2}{\mu}$.

2.1. **Gamma type 2 & 1 distributions.** Let $X > 0$ be a real random variable with a gamma distribution parametrized by a shape $\frac{\nu}{2} > 0$ and scale $\frac{2}{s} > 0$. We will denote $X \sim G_2(\nu, s) \equiv G\left(\frac{\nu}{2}, \frac{2}{s}\right)$ and say that X has a gamma-2 distribution. Let $Y = \sqrt{X}$, we say that Y has a gamma-1 distribution and denote $Y \sim G_1(\nu, s)$. These distribution are not implemented in DYNARE, but can be easily built from the gamma distribution. The density of the gamma-2 distribution is easily obtained from density of the gamma distribution. The first two moments of the gamma-2 distribution are directly obtained from equations (3). The density of the gamma-1 distribution is defined as follows:

$$f_Y(y) = f_X(h^{-1}(y)) \times \left|\frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y)\right|$$

where $h(x) \equiv \sqrt{x}$ and f_X denotes the density of the gamma-2 distribution:

$$f_X(x) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times x^{\frac{\nu}{2}-1} e^{-\frac{sx}{2}}$$

Substituting f_X and the h^{-1} into the definition of f_Y, we get:

$$f_Y(y) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times y^{\nu-2} e^{-\frac{sy^2}{2}} \times \left|\frac{\mathrm{d}}{\mathrm{d}y} y^2\right|$$

$$\Leftrightarrow f_Y(y) = 2\mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{\nu-2} e^{-\frac{sy^2}{2}} y$$

so that:

$$f_Y(y) = \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{\nu-1} e^{-\frac{s}{2}y^2} \tag{4}$$

where $\widetilde{\mathcal{C}}(\alpha, \beta) = \mathcal{C}(\alpha, \beta)/2$ is the constant of integration. The expectation of this distribution is defined by:

$$\begin{aligned}
\mu &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \int_0^\infty y^\nu e^{-\frac{s}{2}y^2} \mathrm{d}y \\
&= \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \int_0^\infty x^{\frac{\nu}{2}-\frac{1}{2}} e^{-\frac{s}{2}x} \mathrm{d}x \\
&= \frac{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}+\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}}}
\end{aligned}$$

So that

$$\mu = \sqrt{\frac{2}{s}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \tag{5}$$

and the variance is given by:

$$\sigma^2 = \frac{\nu}{s} - \mu^2 \tag{6}$$

2.2. **Chi-squared and Exponential distributions.** A number of distributions may be defined as special cases of the Gamma distribution. A chi-squared distribution with ν degrees of freedom, $\chi^2(\nu)$, is a gamma distribution: $G\left(\frac{\nu}{2}, 2\right)$. The chi-squared prior is not implemented in DYNARE but obviously the user can obtain it by carefully choosing the expectation and variance of the gamma prior (that is, by setting $\mu = \nu$ and $\sigma^2 = 2\nu$). As long as the variance is twice the expectation, the prior is a chi-squared distribution. An exponential distribution with expectation λ^{-1}, $\xi(\lambda)$, is also a gamma distribution: $G\left(1, \frac{1}{\lambda}\right)$. Again the exponential prior is not implemented in DYNARE but is obtained from the gamma distribution as long as the prior expectation is the squared root of the prior variance. As a consequence, by using the gamma prior and setting $\mu = \sigma$ the user chooses a distribution whose mode is zero.

2.3. **Shifted gamma distribution.** The support of the gamma distribution is usually the positive real line. In DYNARE the user has the possibility to shift the support of this distribution. This may be useful, for instance, if someone wants to estimate the elasticity of substitution of a CES production function with the belief that this elasticity has to be greater than one (Cobb-Douglas technology). The density is then defined with three parameters $\alpha > 0$ (shape), $\beta > 0$ (scale) and δ (location, the lower bound of the distribution's support):

$$f(x) = \mathcal{C}(\alpha, \beta)^{-1} \times (x - \delta)^{\alpha-1} e^{-\frac{x-\delta}{\beta}} \tag{7}$$

where the constant of integration is defined as before. Obviously this shift affects the first moment ($\mu = \delta + \alpha\beta$).

3. Inverted gamma distribution

Let X be a gamma distributed random variable with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. Then $Z = X^{-1}$ is said to be inverted gamma distributed, $Z \sim IG(\alpha, \beta)$, and one can show that the density of this random variable is defined by:

$$f(z) = \mathcal{C}(\alpha, \beta)^{-1} \times z^{-\alpha-1} e^{-\frac{1}{\beta z}} \tag{8}$$

where $\mathcal{C}(\alpha, \beta) = \Gamma(\alpha)\beta^{\alpha}$ is the constant of integration.

Proof. Let f_X denote the density of the gamma distribution and define $h(x) = \frac{1}{x}$. The density of the inverted gamma distribution is defined as follows:

$$\begin{aligned} f_Z(z) &= f_X(h^{-1}(z)) \times \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right| \\ &= \mathcal{C}\left(\alpha, \beta\right)^{-1} \times z^{-\alpha+1} e^{-\frac{1}{\beta z}} \times \left| \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{z} \right| \\ &= \mathcal{C}\left(\alpha, \beta\right)^{-1} \times z^{-\alpha-1} e^{-\frac{1}{\beta z}} \end{aligned}$$

□

The expectation and variance are given by :

$$\begin{aligned} \mu &= \frac{1}{\beta(\alpha-1)} \\ \sigma^2 &= \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)} \qquad \text{for any } \alpha \geq 2 \end{aligned} \tag{9}$$

This system of equations can be solved for the shape and scale parameters:

$$\begin{aligned} \alpha &= 2 + \frac{\mu^2}{\sigma^2} \\ \beta &= \frac{1}{\mu\left[1 + \frac{\mu^2}{\sigma^2}\right]} \end{aligned} \tag{10}$$

This distribution is not implemented in DYNARE. Finally the mode of this distribution is given by $\frac{1}{\beta(\alpha+1)}$ and is always strictly positive.

3.1. **Inverted gamma-2 and gamma-1 distributions.** Let $X > 0$ be a real random variable with a gamma-2 distribution, $X \sim G\left(\frac{\nu}{2}, \frac{2}{s}\right)$. $Y = X^{-1}$ is said to have an inverted gamma-2 distribution $Y \sim IG_2(\nu, s)$. From equation (8), the density is given by:

$$f(y) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times y^{-\frac{\nu}{2}-1} e^{-\frac{s}{2y}} \tag{11}$$

and from equation (9) we obtain the first two moments:

$$\begin{aligned} \mu &= \frac{s}{\nu-2} \\ \sigma^2 &= \frac{2\mu^2}{\nu-4} \end{aligned} \tag{12}$$

The inverted gamma-2 distribution is implemented in DYNARE as a prior (`INV_GAMMA2_PDF`). The user has to specify μ and σ, and DYNARE solves system (12) for the scale and shape parameters:

$$
\begin{aligned}
s &= 2\mu\left(1+\frac{\mu^2}{\sigma^2}\right)\\
\nu &= 2\left(2+\frac{\mu^2}{\sigma^2}\right)
\end{aligned}
\tag{13}
$$

The mode of this distribution is $\frac{s}{\nu+2} > 0$. Note that if someone wishes to use an inverted gamma distribution as a prior, then he should use the `INV_GAMMA2_PDF` word in the `estimated_params` block[1]. This distribution is often used as a prior for the variance of a structural shock or measurement error. In practice we instead usually define the priors over standard deviations[2], that is over the square root of the variance. This motivates the definition of the inverted gamma-1 distribution. Let Let $X > 0$ be a real random variable with a gamma-1 distribution, $X \sim G_1(\nu, s)$. $Y = X^{-1}$ is said to have an inverted gamma-1 distribution $Y \sim IG_1(\nu, s)$. The density is given by:

$$
f(y) = \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\nu-1} e^{-\frac{s}{2y^2}}
\tag{14}
$$

Proof. Let f_X denote the density of the gamma-1 distribution and define $h(x) = 1/x$. The density of the inverted gamma-1 distribution is defined as follows:

$$
\begin{aligned}
f_Y(y) &= f_X(h^{-1}(y)) \times \left|\frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y)\right|\\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\nu+1} e^{-\frac{s}{2y^2}} \frac{1}{y^2}\\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\nu-1} e^{-\frac{s}{2y^2}}
\end{aligned}
$$

□

Expectation and variance are defined by

$$
\begin{aligned}
\mu &= \sqrt{\frac{s}{2}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\\
\sigma^2 &= \frac{s}{\nu-2} - \mu^2
\end{aligned}
\tag{15}
$$

[1]The only difference between an inverted gamma distribution and the inverted gamma-2 distribution is in the definition of the shape and scale parameters. Because the distributions are defined from the first and second moments, this difference does not matter.

[2]It is the only way with DYNARE.

Proof. The first moment is defined by:

$$\begin{aligned}
\mu &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\int_0^\infty y y^{-\nu-1}e^{-\frac{s}{2y^2}}\mathrm{d}y \\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\frac{1}{2}\int_0^\infty x^{\frac{\nu}{2}-\frac{1}{2}-1}e^{-\frac{s}{2}x}\mathrm{d}x \\
&= \mathcal{C}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\int_0^\infty x^{\frac{\nu-1}{2}-1}e^{-\frac{s}{2}x}\mathrm{d}x \\
&= \frac{\Gamma\left(\frac{\nu-1}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}}} \\
&= \sqrt{\frac{s}{2}}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}
\end{aligned}$$

The second un-centered moment is defined by:

$$\begin{aligned}
\mathbb{E}[Y^2] &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\int_0^\infty y^2 y^{-\nu-1}e^{-\frac{s}{2y^2}}\mathrm{d}y \\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\int_0^\infty y^{-\nu+1}e^{-\frac{s}{2y^2}}\mathrm{d}y \\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\frac{1}{2}\int_0^\infty x^{\frac{\nu}{2}-2}e^{-\frac{s}{2}x}\mathrm{d}x \\
&= \frac{\Gamma\left(\frac{\nu}{2}-1\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}-1}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}}} \\
&= \frac{s}{2}\frac{\Gamma\left(\frac{\nu-2}{2}\right)}{\frac{\nu-2}{2}\Gamma\left(\frac{\nu-2}{2}\right)} \\
&= \frac{s}{\nu-2}
\end{aligned}$$

So that $\sigma^2 = \frac{s}{\nu-2} - \mu^2$. □

The inverted gamma-1 distribution is implemented in DYNARE as a prior (`INV_GAMMA1_PDF` or `INV_GAMMA_PDF`). The user has to specify μ and σ, and DYNARE solves system (15) for the scale and shape parameters. There is no closed form solution in this case, a numerical approach is used. DYNARE first solves for ν in the following equation:

$$2\Gamma\left(\frac{\nu}{2}\right)^2\mu^2 = (\sigma^2+\mu^2)(\nu-2)\Gamma\left(\frac{\nu-1}{2}\right)^2$$

and then computes:

$$s = (\sigma^2+\mu^2)(\nu-2)$$

This is done in the `m` file `inverse_gamma_specification.m`. Note that in the case of an infinite variance we have a closed form solution:

$$\nu = 2$$

$$s = \frac{2}{\pi}\mu^2$$

Finally the mode of this distribution is $s\sqrt{\frac{\nu}{\nu+1}} > 0$ and converges to s as ν gets larger.

☞ The inverse gamma-1(-2) is usually used as a prior for the standard deviation (resp. variance) of a structural (or measurement) shock. This is because in linear models with gaussian perturbation, the Normal (for the parameters) – Inverse Gamma (for the variance of the error) prior is conjugate. Obviously this is not true for DSGE models, there is no computational advantage in choosing the inverse gamma prior.